On the structure of the set of k-aditive fuzzy measures

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Abstract

In this paper we present some results concerning the vertices of the set of fuzzy measures being at most k-additive. We provide an algorithm to compute them. We give some examples of the results obtained with this algorithm and give lower bounds on the number of vertices for the $(n-1)$-additive case, proving that it grows much faster than the number of vertices of the general fuzzy measures.

The results in the paper suggest that the structure of k-additive measures might be more complex than expected from their definition and, in particular, that they are more complex than general fuzzy measures.

Keywords: Fuzzy measures, k-additive measures, vertices.

1 Introduction

Fuzzy measures [28] (also known as capacities [2] or non-additive measures [7]) are a generalization of probability distributions. More concretely, they are measures in which the additivity axiom has been relaxed to a monotonicity condition. This extension is needed in many practical situations, in which additivity is too restrictive. Thus defined, fuzzy measures, together with Choquet integral [2], have proved themselves to be a powerful tool in many different fields, as Decision Theory ([10], [11]), Game Theory ([26], [9]), and many others (see for example [12] for some applications of fuzzy measures in other frameworks).

However, despite all these advantages, the practical application of fuzzy measures is limited by the increasing complexity of the measure. If we consider a finite space of cardinality n, only $n-1$

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values are needed in order to completely determine a probability, while $2^n - 2$ coefficients are needed to define a fuzzy measure on the same referential. This exponential growth is the actual *Achilles' heel* of fuzzy measures. With the aim of reducing this complexity several subfamilies have been defined. In these subfamilies some extra constraints are added in order to decrease the number of coefficients but, at the same time, keep the modelling capabilities of the measures in the subfamily as rich as possible. Examples of subfamilies include the λ-measures [29], the k-intolerant measures [17], the p-symmetric measures [21], the decomposable measures [8], etc.

In this paper we will focus on the subfamily of k-additive measures, introduced by Grabisch in [9]. The concept of k-additive measure generalizes that of probability, but it is not as general as fuzzy measures. Indeed, the subfamilies of k-additive measures for increasing values of k determine a gradation between probabilities and general fuzzy measures. Another interesting property of k-additive measures relies on the fact that the number of coefficients increases with k; thus, we can choose the value of k in terms of the desired complexity.

Let us denote by $\mathcal{FM}^k(X)$ the set of fuzzy measures on X being k'-additive for some $k' \leq k$. It can be proved that $\mathcal{FM}^k(X)$ is a convex polytope, so that it can be characterized in terms of its vertices. In this paper we deal with the problem of determining this set of vertices. Apart from the mathematical interest of this problem, determining the set of vertices of this polytope arises in the practical identification of fuzzy measures from sample data. More concretely, we have developed in [3] a procedure based on genetic algorithms (see [14] for basic properties of these algorithms) for determining the fuzzy measure that best fits a set of data. In such procedure the cross-over operator was the convex combination between individuals and, as shown in [3], it is then necessary to consider as initial population the set of vertices.

A first study about the set of vertices of $\mathcal{FM}^k(X)$ appears in [20]; in that paper, it is proved that, unexpectedly, there are vertices of $\mathcal{FM}^k(X), k \geq 3$ that are not $\{0,1\}$-valued, i.e. such that the fuzzy measure attains other values different from 0 and 1 (cf. Propositions 1 and 2 and Theorem 2). These vertices are convex combinations of $\{0,1\}$-valued measures that are not in $\mathcal{FM}^k(X)$. Another problem related to the previous one is the problem of determining the number of vertices of $\mathcal{FM}^k(X)$. For the general case, we have proved [3] that the number of vertices for the general case coincides with the n-th Dedekind number [6]. The results in this paper seem to point out that the number of vertices of $\mathcal{FM}^k(X)$ is even greater.

The paper is organized as follows: in next section we give the basic concepts and results that will be needed in the paper. In Section 3, we provide an algorithm to generate the vertices of $\mathcal{FM}^k(X)$. This algorithm allows us to compute the number of vertices of $\mathcal{FM}^k(X)$ $(k > 2)$ when $|X|$ is reduced; moreover, it seems to mean that the number of vertices of $\mathcal{FM}^k(X)$ is much larger than the corresponding number of vertices of $\mathcal{FM}(X)$. This result is proved for general $n = |X|$ and $k = n-1$ in Section 5. Section 4 deals with the special case of the algorithm for $k = n-1$. We finish with the conclusions and open problems.

2 Basic concepts and previous results

Consider a finite referential set $X = \{1, ..., n\}$ of n elements. Let us denote by $\mathcal{P}(X)$ the set of subsets of X. Subsets of X are denoted $A, B, ...$ and also by $A_1, A_2, ...$

Definition 1. *[2, 7, 28] A* **fuzzy measure**, **non-additive measure** *or* **capacity** *over X is a function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying*

- $\mu(\emptyset) = 0$, $\mu(X) = 1$ *(boundary conditions).*
- $\forall A, B \in \mathcal{P}(X)$, *if* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$ *(monotonicity).*

We will denote the set of all fuzzy measures over X by $\mathcal{FM}(X)$.

Definition 2. *Let $\mu \in \mathcal{FM}(X)$; we define the* **dual measure of** μ *as the fuzzy measure $\overline{\mu}$ given by $\overline{\mu}(A) = 1 - \mu(A^c)$.*

Notice that, as $\mathcal{FM}(X)$ is an intersection of semispaces and due to the boundary conditions, it is a bounded convex polyhedron. Consequently, any fuzzy measure μ can be put as a convex combination of the vertices, which are given in the following result.

Proposition 1. *[24] The set of $\{0, 1\}$-valued measures constitutes the set of vertices of $\mathcal{FM}(X)$.*

Despite this simple structure, the number of vertices of $\mathcal{FM}(X)$ is not simple at all. Notice that for a $\{0, 1\}$-valued measure μ, there are some non-empty subsets A satisfying the following conditions:

$$\begin{aligned} &\mu(A) = 1, \\ &\mu(B) = 1, \quad \forall B \supseteq A, \\ &\mu(C) = 0, \quad \forall C \subset A. \end{aligned}$$

We will call any subset satisfying these conditions a **minimal subset** for μ. Note that a $\{0, 1\}$-valued measure is completely defined by its minimal subsets. Note also that the collection of minimal subsets of a fuzzy measure determines an antichain in $\mathcal{P}(X)$. Therefore, the number of vertices for different values of $|X|$ coincides with the sequence of Dedekind numbers [6], whose general form is a long-standing problem in Combinatorics. The first Dedekind numbers are given in Table 1.

n	Dedekind numbers
1	1
2	4
3	18
4	166
5	7,579
6	7,828,352
7	2,414,682,040,996
8	56,130,437,228,687,557,907,786

Table 1: Number of vertices of $\mathcal{FM}(X)$

In this paper we will also need to characterize when two vertices of $\mathcal{FM}(X)$ are adjacent. This problem has been studied for $\mathcal{FM}(X)$ in [4] and later generalized in [**?**].

Remark that we can define a partial order on $\mathcal{FM}(X)$ in the following way: given two fuzzy measures μ_1, μ_2, we say that $\mu_1 \leq \mu_2$ if for every $A \subseteq X$, it holds $\mu_1(A) \leq \mu_2(A)$. Similarly, we will write $\mu_1 < \mu_2$ if $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$. With this definition it is easy to see the following:

Lemma 1. *[4] If μ_1 and μ_2 are adjacent vertices in $\mathcal{FM}(X)$, then either $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$.*

However, this condition is not necessary. In order to characterize adjacency in $\mathcal{FM}(X)$ the following definition is needed.

Definition 3. *Let $\mathbf{C}$ be a non-empty collection of non-empty subsets of X, μ a vertex of $\mathcal{FM}(X)$ and let $A_1, \ldots, A_m$ be all its minimal sets. We say that μ is* **C-decomposable** *if there exists a partition of $\{A_1, \ldots, A_m\}$ in two non-empty collections $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \not\subseteq \mathbf{C}$ and $\mathbf{B} \not\subseteq \mathbf{C}$ and if for any $A \in \mathbf{A}$ and any $B \in \mathbf{B}$, then there exists $C \in \mathbf{C}$ such that $C \subseteq A \cup B$.*

Let us denote by $\mu_{\mathbf{C}}$ the $\{0,1\}$-valued fuzzy measure whose collection of minimal subsets is $\mathbf{C}$. The following theorem characterizes adjacency on $\mathcal{FM}(X)$.

Theorem 1. *[4] If μ is an extreme point of $\mathcal{FM}(X)$ and $\mu > \mu_{\mathbf{C}}$, then μ is adjacent to $\mu_{\mathbf{C}}$ if and only if μ is not $\mathbf{C}$-decomposable.*

Example 1. *Consider the fuzzy measure $u_A, A \neq \emptyset$, whose only minimal subset is A (these measures are known as primitive measures [15] or unanimity games). In this case, it is not possible to build a $\mathbf{C}$-decomposition of $\{A\}$, no matter which is the collection $\mathbf{C}$. Applying Theorem 1, it follows that u_A is adjacent to any $\mu_{\mathbf{C}}$ such that $u_A > \mu_{\mathbf{C}}$. Moreover, two primitive measures u_A and u_B are adjacent in $\mathcal{FM}(X)$ if and only if $A \subseteq B$ or $B \subseteq A$. See [4] for more details and examples of adjacency in $\mathcal{FM}(X)$.*

It has been proved in [**?**] that from Theorem 1, the problem of whether two vertices of $\mathcal{FM}(X)$ are adjacent can be solved in quadratic time in the number of minimal subsets. This is an interesting result because in general, the problem of determining whether two vertices in a polytope are adjacent has been proved to be NP-complete [23].

To finish with general fuzzy measures, let us introduce the following operations.

Definition 4. *Consider $\sigma : X \to X$ a permutation on X. We define the* **symmetry induced by** *σ, denoted S_σ, as the transformation on $\mathcal{FM}(X)$ such that for any $\mu \in \mathcal{FM}(X)$, the fuzzy measure $S_\sigma(\mu)$ is defined by*

$$S_\sigma(\mu)(A) = \mu(\sigma(A)), \forall A \subseteq X.$$

Definition 5. *We define the* **dual transformation***, denoted D, as the transformation on $\mathcal{FM}(X)$ given by*

$$D: \begin{array}{ccc} \mathcal{FM}(X) & \to & \mathcal{FM}(X) \\ \mu & \mapsto & \overline{\mu} \end{array}.$$

It can be seen (see [19] and a generalization of these results in [5]) that these transformations are isometries on $\mathcal{FM}(X)$, i.e. bijective functions preserving distances.

As explained in the introduction, in many cases we are constrained to a subfamily of fuzzy measures. In this paper, we deal with k-additive measures, that are defined in terms of the so-called Möbius transform.

Definition 6. *[25] Let μ be a set function (not necessarily a fuzzy measure) on $\mathcal{P}(X)$ such that $\mu(\emptyset) = 0$. The* **Möbius transform (or inverse)** *of μ is another set function on X defined by*

$$m_\mu(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \ \forall A \subseteq X. \tag{1}$$

The Möbius transform given, the original set function can be recovered through the *Zeta transform* [1]:

$$\mu(A) = \sum_{B \subseteq A} m(B).$$

Thus, the Möbius transform provides an alternative representation of fuzzy measures. It is indeed the well-known *dividends* that appear in the field of Game Theory [13].

It can be seen that in the special case of probabilities, the Möbius transform can only attain non-null values for singletons. This inspires the concept of k-additive measure.

Definition 7. *[9] A fuzzy measure μ is said to be* **k-order additive** *or* **k-additive** *if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A with exactly k elements such that $m(A) \neq 0$.*

In this sense, a probability measure is just a 1-additive measure, so that the concept of k-additive measure generalizes that of probability. It provides a gradation between probabilities and general fuzzy measures; in this sense, any fuzzy measure is k-additive for a suitable choice of k.

As the Möbius transform provides an alternative representation of fuzzy measures, the number of coefficients needed to define a k-additive measure is reduced to

$$\sum_{i=1}^{k} \binom{n}{i} - 1.$$

More about k-additive measures can be found e.g. in [11]. We will denote by $\mathcal{FM}^k(X)$ the set of fuzzy measures being k'-additive, with $k' \leq k$. The set $\mathcal{FM}^1(X)$ is the set of probability distributions on X. Specially appealing is $\mathcal{FM}^2(X)$ that provides a generalization of probability allowing interactions while keeping a reduced complexity.

Notice that $\mathcal{FM}^k(X)$ is a convex polyhedron (it is the intersection of the polytope $\mathcal{FM}(X)$ and the hyperplanes $\sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B) = 0, \ \forall A \subseteq X, |A| > k$). Thus, it is characterized by its vertices. The following results have been obtained in [20].

Proposition 2. *The set of extreme points of $\mathcal{FM}^1(X)$ (resp. $\mathcal{FM}^2(X)$) are the $\{0,1\}$-valued measures that are in $\mathcal{FM}^1(X)$ (resp. $\mathcal{FM}^2(X)$). There are n vertices for $\mathcal{FM}^1(X)$ and n^2 vertices for $\mathcal{FM}^2(X)$.*

However, the general form of vertices of $\mathcal{FM}^k(X), k > 2$ is more complicated.

Theorem 2. *There are vertices of the set $\mathcal{FM}^k(X), k > 2$, that are not $\{0,1\}$-valued measures.*

The aim of this paper is to provide new results about these vertices and their number.

3 Generating the vertices of the k-additive measures

Consider the set $\mathcal{FM}(X)$. From a geometrical point of view, $\mathcal{FM}^k(X)$ is a subpolytope of $\mathcal{FM}(X)$ with the additional constraints $m(A) = 0$, $\forall |A| > k$.

Let $|A| > k$ and consider the restriction $m(A) = 0$. Then, it can be proved that the set of vertices of the polytope of fuzzy measures with this new condition consists in (see [22]):

(i) Those $\{0, 1\}$-valued measures for which $m(A) = 0$ and

(ii) The cut points of the hyper-plane $m(A) = 0$ with the edges of the polytope $\mathcal{FM}(X)$.

If we repeat the procedure for every set A with size bigger than k we will obtain all the vertices of the polytope $\mathcal{FM}^k(X)$. Thus, we obtain a characterization of the extreme points of the k-additive measures (and of all intermediate polytopes). Let us explain this in more depth; if $\mathbf{A}$ is a collection of subsets of X, let us denote by $\mathcal{FM}_{\mathbf{A}}(X)$ the set of fuzzy measures whose Möbius is zero on every set belonging to $\mathbf{A}$. Then, as a consequence of (ii), the following holds.

Corollary 1. *Let $\mathbf{A}$ be a collection of subsets of X (possibly empty), $B \in \mathcal{P}(X) \setminus \mathbf{A}$, and $\mu \in \mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ which is not a vertex of $\mathcal{FM}_{\mathbf{A}}(X)$. Then, μ is a vertex of $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ if and only if there exist μ_1 and μ_2 two adjacent vertices of $\mathcal{FM}_{\mathbf{A}}(X)$ that are not in $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ and $\lambda \in (0, 1)$ such that $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$.*

As $\mathcal{FM}^{n-1}(X)$ comes from $\mathcal{FM}(X)$, just adding the constraint $m(X) = 0$, then Proposition 1 together with the previous corollary leads us to the following result.

Corollary 2. *Let μ be a $(n-1)$-additive measure which is not $\{0, 1\}$-valued. Then μ is a vertex of $\mathcal{FM}^{n-1}(X)$ if and only if it can be written as*

$$\mu = \lambda\mu_1 + (1 - \lambda)\mu_2,$$

where $\lambda \in (0, 1)$ and μ_1 and μ_2 are two adjacent vertices of $\mathcal{FM}(X)$ that are not in $\mathcal{FM}^{n-1}(X)$.

In this particular case we can obtain all vertices of $\mathcal{FM}^{n-1}(X)$ just from adjacency relationships in $\mathcal{FM}(X)$, that are given in Theorem 1. However, this is not possible in general; for other values of k there exist non-$\{0, 1\}$-valued vertices which cannot be expressed as a convex combination of exactly two $\{0, 1\}$-valued vertices of $\mathcal{FM}(X)$, as the following example shows.

Example 2. *Consider $|X| = 5$. Let μ_1 be the fuzzy measure whose minimal subsets are the singletons $\{1\}, \ldots, \{5\}$ and μ_2 the fuzzy measure whose minimal subsets are the subsets of size 4. Also, let μ_3 be the fuzzy measure with minimal subsets $\{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}\}$.*

From Theorem 1 it follows that μ_1 and μ_2 are adjacent vertices in $\mathcal{FM}(X)$. Analogously, μ_2 and μ_3 are also adjacent vertices.

Let us define

$$\mu := \frac{4}{5}\mu_1 + \frac{1}{5}\mu_2, \mu' := \frac{4}{5}\mu_3 + \frac{1}{5}\mu_2.$$

It is easy to check that $\mu, \mu' \in \mathcal{FM}^4(X)$. By Corollary 1, these measures are extreme points of $\mathcal{FM}^4(X)$; moreover, they are not $\{0, 1\}$-valued. It can be checked that these measures are *adjacent in $\mathcal{FM}^4(X)$, as the midpoint satisfies with equality exactly $2^n - 3$ linearly independent restrictions of those defining the polytope $\mathcal{FM}^4(X)$. Moreover,*

$$\frac{1}{4}\mu + \frac{3}{4}\mu' \in \mathcal{FM}^3(X).$$

Consequently, this fuzzy measure is a vertex of $\mathcal{FM}^3(X)$. Thus, we have obtained a vertex of the 3-additive measures which requires three vertices of $\mathcal{FM}(X)$ in order to be written as a convex combination.

As an application of Corollary 1 we have computed the vertices of the polytopes $\mathcal{FM}^k(X)$ for some values of k and n. Tables 2 and 3 show the number of vertices (Table 2) and the number of vertices being $\{0,1\}$-valued (Table 3). For $k = n$ we recover the corresponding Dedekind number.

$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	4				
3	3	9	18			
4	4	16	303	166		
5	5	25	584,740	407,201	7,579	
6	6	36	?	?	232,871,070,690	7,828,352

Table 2: Number of vertices of the $\mathcal{FM}^k(X)$ polytopes.

$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	4				
3	3	9	18			
4	4	16	68	166		
5	5	25	195	1,855	7,579	
6	6	36	456	10,986	1,322,954	7,828,352

Table 3: Number of $\{0,1\}$-valued vertices of the $\mathcal{FM}^k(X)$ polytopes.

To compute these numbers for the cases with $n \leq 5$ we repeatedly use Corollary 1 to calculate the vertices of the intermediate polytopes. This process is very time-consuming and, in fact, took several weeks in a 2.1GHz PC to complete the calculations in the case $n = 5$. Indeed, when $n = 6$ the computer was not able to cope with such complexity for $k = 3$ and $k = 4$. There are two reasons for this.

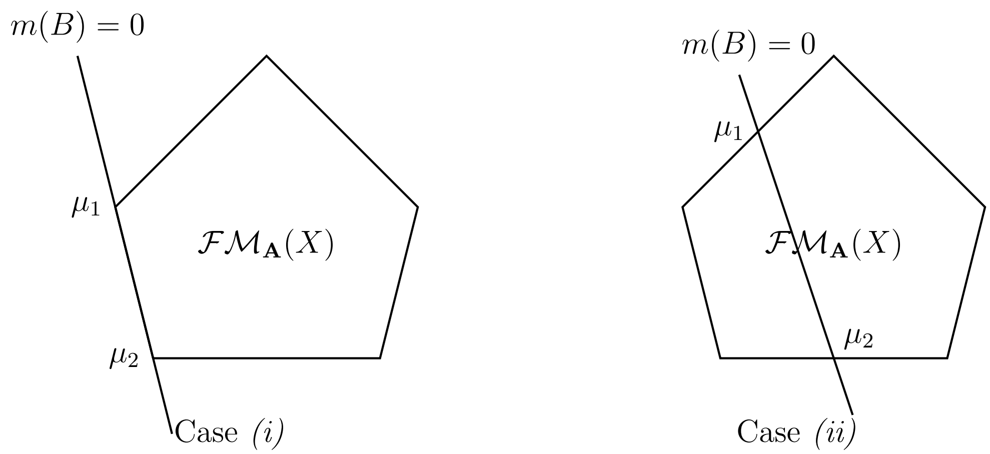
A first difficulty is that we lack criteria to quickly test adjacency (except for general fuzzy measures, where we can use Theorem 1) and thus, the complexity grows when applying *(ii)*. In our case, we are forced to use the following general result, valid for obtaining the adjacency structure of the intersection of a polytope with a hyperplane. Suppose the adjacency structure of $\mathcal{FM}_{\mathbf{A}}(X)$ is known and let us denote by r the number of variables. Then, the adjacency structure of $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ is given in next lemma.

Lemma 2. *Let μ_1, μ_2 be two vertices of $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$. Then, μ_1, μ_2 are adjacent if one of the following cases occur:*

(i) *μ_1, μ_2 are also vertices in $\mathcal{FM}_{\mathbf{A}}(X)$ and they are adjacent in $\mathcal{FM}_{\mathbf{A}}(X)$.*

(ii) *They are included in a face of dimension 2 of $\mathcal{FM}_{\mathbf{A}}(X)$, and the segment $[\mu_1, \mu_2]$ is the intersection of the face with the hyperplane $m(B) = 0$.*

The proof is straightforward and thus it is omitted. Next figure shows the idea behind the result.



Notice that, by the previous lemma, it is necessary to know the 2-dimensional faces of the polytopes. The same as for adjacency, the problem of determining the r-dimensional faces of a polytope is a NP-complete problem. The only case in which the r-dimensional faces have been characterized is the general n-additive case; in this case, it can be proved that the polytope belongs to a special class of polytopes called order polytopes [27], and a characterization of the faces can be found for example in [27].

However, the main problem appearing when applying this algorithm is that the number of extreme points grows very fast, making the number of comparisons needed grow even faster; moreover, as the number of vertices is so large, there is a problem with their storage.

All this makes unfeasible a direct approach to determine the number of vertices of $\mathcal{FM}^k(X)$ when $n \geq 6$.

The figures in Tables 2 and 3 seem to suggest that the number of non $\{0, 1\}$-valued vertices of $\mathcal{FM}^k(X)$ for $k = 3, ..., n-1$ grows much more quickly than the number of extreme points of $\mathcal{FM}(X)$. In Section 5 we will prove this fact for $k = n - 1$.

4 Improvements for the $(n-1)$-additive case

We have just pointed out the problems of applying the algorithm introduced in Section 3 to the general k-additive case. In this section, we will see that the performance can be improved when dealing with $k = n-1$, so that in this special case the procedure is very efficient.

A first result for $\mathcal{FM}^{n-1}(X)$ has been already obtained in Corollary 2. Notice also that in this case, there are not intermediate polytopes and thus, we only need to know the adjacency structure of $\mathcal{FM}(X)$, that can be checked in quadratic time. However, we can improve more the performance of the algorithm.

Observe that if σ is a permutation of X, then S_σ, the symmetry induced by σ, is an isometry such that if μ is a fuzzy measure, then $m_\mu(X)$ and $m_{S_\sigma(\mu)}(X)$ are both either positive, negative or zero, just applying Eq. (1).

On the other hand, if μ_1 and μ_2 are adjacent vertices, then it can be easily proved applying Theorem 1 that so are $S_\sigma(\mu_1)$ and $S_\sigma(\mu_2)$.

Joining these two results, if μ_1, μ_2 are two vertices of $\mathcal{FM}(X)$ such that $\lambda\mu_1 + (1-\lambda)\mu_2$ is a vertex of $\mathcal{FM}^{n-1}(X)$, (i.e. $m_{\lambda\mu_1+(1-\lambda)\mu_2} = 0$), then so is $S_\sigma(\lambda\mu_1 + (1-\lambda)\mu_2) = \lambda S_\sigma(\mu_1) + (1-\lambda)S_\sigma(\mu_2)$.

Consider then a vertex μ of $\mathcal{FM}(X)$ such that $m_\mu(X) \neq 0$, say $m_\mu(X) > 0$. Then, for any adjacent vertex μ' of μ such that $m_{\mu'}(X) < 0$, it is possible to obtain a vertex of $\mathcal{FM}^{n-1}(X)$ in the edge joining μ and μ'. We will call the set of all vertices of $\mathcal{FM}^{n-1}(X)$ obtained from μ this way the *set of generated vertices from* μ. Then, the set of generated vertices from μ and $S_\sigma(\mu)$ have the same cardinality for any permutation σ.

Consider now $D(\mu)$. Then,

$$m_{D(\mu)}(X) = \sum_{A\subseteq X}(-1)^{|X\setminus A|}D(\mu)(A) = \sum_{A\subseteq X}(-1)^{|X\setminus A|}(1-\mu(A^c)) = \sum_{A\subseteq X}(-1)^{|X\setminus A|+1}\mu(A^c).$$

If $|X| = n$ is odd, then $(-1)^{|X\setminus A|+1} = (-1)^{|X\setminus A^c|}$, so that

$$\sum_{A\subseteq X}(-1)^{|X\setminus A|+1}\mu(A^c) = \sum_{A\subseteq X}(-1)^{|X\setminus A|}\mu(A).$$

Thus, if n is odd, $m_{D(\mu)}(X) = m_\mu(X)$. We can then repeat the same process as for symmetries to conclude that for n odd, if μ is a vertex of $\mathcal{FM}(X)$ such that $m_\mu(X) \neq 0$, then the set of generated vertices from μ and $D(\mu)$ have the same cardinality. If n is even, then the set of generated vertices from μ and $D(\mu)$ have the same cardinality, but $m_{D(\mu)}(X) = -m_\mu(X)$.

Let us define on the set of vertices of $\mathcal{FM}(X)$ the following equivalence relations:

- If n is even,
$$\mu\mathcal{R}\mu' \Leftrightarrow \exists S_\sigma \mid S_\sigma(\mu) = \mu'.$$
The equivalence classes of this relation are called *orbits* under the action of the group of *symmetries*.

- If n is odd,
$$\mu\mathcal{R}\mu' \Leftrightarrow \exists S_\sigma \mid S_\sigma(\mu) = \mu' \text{ or } \mu' = D(\mu).$$

The equivalence classes of this relation are called *orbits* under the action of the group of *isometries*.

Thus, if μ and μ' are in the same orbit, then they will generate exactly the same number of vertices of $\mathcal{FM}^{n-1}(X)$. Therefore, in order to count the number of vertices of the $(n-1)$-additive measures, we can proceed as follows:

(i) List all the vertices of the general fuzzy measures.

(ii) Classify these vertices according to the sign of their Möbius transform on X.

(iii) Calculate the orbits of the vertices with positive Möbius on X.

(iv) Pick a representative of each orbit and test its adjacency to the vertices with negative Möbius on X.

(v) Multiply the number of adjacent vertices generated from this representant by the size of the orbit, and sum all the values obtained.

(vi) Add to this value the number of vertices whose Möbius transform on X is 0.

After applying this procedure in the case where $n = 6$ we obtained 1,322,954 vertices of $\mathcal{FM}(X)$ with zero Möbius transform on X, 3,252,699 with negative Möbius and also 3,252,699 with positive Möbius. These latter vertices are organized into 3403 different orbits. After 20 hours of computation on a 2.1GHz PC, the number of non $\{0, 1\}$-valued vertices of $\mathcal{FM}^5(X)$ with $|X| = 6$ was found to be 232,869,747,736, for a total of 232,871,070,690 vertices. If $n > 6$, we have the problem that the corresponding Dedekind number is too high to apply the procedure.

5 On the number of vertices of $\mathcal{FM}^{n-1}(X)$

In this section we will show that the number of non $\{0, 1\}$-valued vertices of $\mathcal{FM}^{n-1}(X)$ grows much more quickly than the number of extreme points of $\mathcal{FM}(X)$. More concretely, we will study the asymptotic behavior of the ratio of the number of vertices of $\mathcal{FM}^{n-1}(X)$ to the number of vertices of $\mathcal{FM}(X)$.

Let us denote by d_n^n the number of vertices of $\mathcal{FM}(X)$ (the n-th Dedekind number) and by d_n^{n-1} the number of vertices of $\mathcal{FM}^{n-1}(X)$.

Let us denote by $\mathcal{A}_n$ the set of vertices of $\mathcal{FM}(X)$ which are not in $\mathcal{FM}^{n-1}(X)$ and whose minimal subsets all have size at least $\lceil \frac{n-3}{2} \rceil$, where $\lceil x \rceil$ denotes the ceiling of x. The cardinality of $\mathcal{A}_n$ will be denoted by a_n.

Proposition 3. *It holds that*

$$\lim_{n \to \infty} \frac{a_n}{d_n^n} > 0.$$

Proof: In order to make the proof clearer, we will make it in several lemmas.

Suppose $n = |X|$ is even. Let us consider the following auxiliary result, whose proof can be seen in [**?**, 16].

Lemma 3. *Let us denote by $\mathcal{B}_n$ be the set of vertices μ of $\mathcal{FM}(X)$ such that:*

(i) Every minimal subset of μ has size $\frac{n}{2}-1$, $\frac{n}{2}$ or $\frac{n}{2}+1$,

(ii) $\mu(A)=1$ for every A such that $|A|>\frac{n}{2}+1$,

(iii) The number of minimal subsets of μ of size $\frac{n}{2}-1$ is at most $2^{\frac{n}{2}}$.

Let us denote the cardinality of $\mathcal{B}_n$ by b_n. Then, for even n,

$$\lim_{n\to\infty}\frac{b_n}{d_n^n}=1$$

Lemma 4. *Consider $A\subset X$ such that $|A|=\frac{n}{2}-1$ and let us define $\mathcal{C}_n(A):=\{\mu\in\mathcal{B}_n:\mu(A)=1\}$ and denote by $c_n(A)$ its cardinality. Then,*

$$c_n(A)\leq\frac{2^{\frac{n}{2}}}{\binom{n}{\frac{n}{2}-1}}b_n.$$

Proof: By symmetry, if we take $B\subset X$ such that $|B|=\frac{n}{2}-1$, it follows that $c_n(A)=c_n(B)$. Thus, as there are $\binom{n}{\frac{n}{2}-1}$ subsets B in these conditions,

$$\binom{n}{\frac{n}{2}-1}c_n(A)=\sum_{|B|=\frac{n}{2}-1}c_n(B)=\sum_{|B|=\frac{n}{2}-1}\sum_{\mu\in\mathcal{B}_n}\mu(B)=\sum_{\mu\in\mathcal{B}_n}\sum_{|B|=\frac{n}{2}-1}\mu(B)\leq\sum_{\mu\in\mathcal{B}_n}2^{\frac{n}{2}}=2^{\frac{n}{2}}b_n,$$

and hence

$$c_n(A)\leq\frac{2^{\frac{n}{2}}}{\binom{n}{\frac{n}{2}-1}}b_n.$$

This finishes the proof. ■

Take $B:=\{1,\ldots,\frac{n}{2}+1\}\subseteq X$. Consider the *shadow* of B, that is, $\nabla B:=\{A\subseteq X:|A|=\frac{n}{2}\text{ and }A\subseteq B\}$ and the shadow of its shadow, $\nabla\nabla B:=\{A\subseteq X:|A|=\frac{n}{2}-1\text{ and }A\subseteq B\}$.

Lemma 5. *Let us define*

$$\mathcal{E}_n(B):=\{\mu\in\mathcal{B}_n:\exists A\in\nabla\nabla B\ s.t.\ \mu(A)=1\}.$$

Let us denote by e_n its cardinality. Then,

$$\lim_{n\to\infty}\frac{e_n}{b_n}=0.$$

Proof: It suffices to notice that

$$e_n\leq\sum_{A\in\nabla\nabla B}c_n(A)\leq|\nabla\nabla B|\frac{2^{\frac{n}{2}}}{\binom{n}{\frac{n}{2}-1}}b_n,$$

and hence

$$\frac{e_n}{b_n} \leq |\nabla\nabla B| \frac{2^{\frac{n}{2}}}{\binom{n}{\frac{n}{2}-1}},$$

which can be easily seen to tend to 0 when n tends to infinity, using the Stirling approximation for the factorial and noticing that $|\nabla\nabla B|$ is polynomial in n. ■

Let us define

$$\mathcal{F}_n(B) := \{\mu \in \mathcal{B}_n : \mu(A) = 0, \forall\, A \in \nabla\nabla B\},$$

i.e. $\mathcal{F}_n(B) = \mathcal{B}_n \backslash \mathcal{E}_n(B)$. Denote by f_n its cardinality. By the previous lemma, we know that

$$\lim_{n\to\infty} \frac{f_n}{b_n} = 1.$$

Lemma 6. *If n is even, then*

$$\lim_{n\to\infty} \frac{a_n}{f_n} > 0.$$

Proof: On $\mathcal{F}_n(B)$ we define the following equivalence relation:

$$\mu \mathcal{R} \mu' \Leftrightarrow \mu(A) = \mu'(A) \text{ for all } A \notin \{B\} \cup \nabla B.$$

We will show that the ratio of fuzzy measures that are not $(n-1)$-additive measures in each equivalence class is at least $\frac{1}{3}$. Notice that any fuzzy measure outside $\mathcal{FM}^{n-1}(X)$ in $\mathcal{F}_n(B)$ is in $\mathcal{A}_n$; therefore, if we prove this result, the lemma will follow.

For a fixed equivalence class $\mathcal{G}$, define

$$\mathcal{S}_{\mathcal{G}}(B) := \{A \in \nabla B : \mu(A) = 1 \text{ for some } \mu \in \mathcal{G}\}.$$

Let us denote by $s_{\mathcal{G}}$ its cardinality. We have three different cases:

- If $s_{\mathcal{G}} = 0$, then $\mu(A) = 0, \forall A \in \nabla B, \forall \mu \in \mathcal{G}$. As the equivalence relation fixes the values $\mu(C), C \notin \{B\} \cup \nabla B$ for all the measures in the equivalence class $\mathcal{G}$, we conclude that we have just two measures in $\mathcal{G}$, one for which $\mu(B) = 1$ and one for which $\mu(B) = 0$. As

$$m_\mu(X) = \sum_{B\subseteq X} (-1)^{|X\backslash B|} \mu(B),$$

  by Eq. (1), and these measures differ in exactly one set, it follows that at most one of them can be in $\mathcal{FM}^{n-1}(X)$.

- If $s_{\mathcal{G}} = 1$, we have three measures in $\mathcal{G}$, namely μ_1 which is 0 on every set of $\{B\} \cup \nabla B$, μ_2 which takes value 1 on B and 0 on ∇B and μ_3 which takes value 1 on B and in exactly one set in ∇B. Applying the same argument as before, if μ_1 or μ_3 are in $\mathcal{FM}^{n-1}(X)$, then μ_2 is not, and if $\mu_2 \in \mathcal{FM}^{n-1}(X)$, then neither μ_1 nor μ_3 are.

- Finally, suppose that $s_{\mathcal{G}} > 1$. Since every $\mu \in \mathcal{G}$ is 0 on $\nabla\nabla B$ by definition of $\mathcal{F}_n(B)$, we have exactly $2^{s_{\mathcal{G}}}$ measures $\mu \in \mathcal{G}$ such that $\mu(B) = 1$, as we can freely select the values of the measures on the $s_{\mathcal{G}}$ sets on ∇B on which not every measure of E is 0, and another one measure μ^* which is 0 on B and on ∇B. Thus, there are $2^{s_{\mathcal{G}}} + 1$ fuzzy measures in the equivalence class. Consider $\mu \in \mathcal{G}$ such that $\mu \neq \mu^*$.

$$m_\mu(X) = \sum_{A \subseteq X} (-1)^{|X \setminus A|} \mu(A) = \sum_{A \subseteq X, A \notin \nabla B} (-1)^{|X \setminus A|} \mu(A) + \sum_{A \subseteq X, A \in \nabla B} (-1)^{|X \setminus A|} \mu(A).$$

  Thus, if $\mu, \mu' \in \mathcal{G} \setminus \{\mu^*\}$ are in $\mathcal{FM}^{n-1}(X)$, then the number of sets from ∇B in which they take value 1 must be equal for both, since the sets in ∇B have all the same size. Thus, the number of $(n-1)$-additive measures in $\mathcal{G}$ is at most $\binom{s_{\mathcal{G}}}{\lceil \frac{s_{\mathcal{G}}}{2} \rceil} + 1$. Hence, the ratio of fuzzy measures in $\mathcal{FM}^{n-1}(X)$ in $\mathcal{G}$ is at most $\frac{\binom{s_{\mathcal{G}}}{\lceil \frac{s_{\mathcal{G}}}{2} \rceil} + 1}{2^{s_{\mathcal{G}}} + 1}$. Since $s_{\mathcal{G}} > 1$ this fraction is at most $\frac{3}{5}$.

Therefore, the result holds. ■

Proof of Proposition 3: Joining the results of the previous lemmas,

$$\lim_{n \to \infty} \frac{a_n}{d_n^n} = \lim_n \frac{a_n}{f_n} \frac{f_n}{b_n} \frac{b_n}{d_n^n} > 0,$$

whence Proposition 3 follows for even n.

The result for odd n can be proved with an argument similar to the one used when n is even. More concretely, we consider the set $\mathcal{B}'_n$ of vertices μ of $\mathcal{FM}(X)$ such that:

(i) Every minimal subset of μ has size $\frac{n-3}{2}$, $\frac{n-1}{2}$ or $\frac{n+1}{2}$,

(ii) $\mu(A) = 1$ for every A of size greater than $\frac{n+1}{2}$,

(iii) The number of minimal subsets of μ of size $\frac{n-3}{2}$ is at most $2^{\frac{n}{2}}$.

In [?, 16] it is proven that the ratio of measures in $\mathcal{B}'_n$ for odd n tends to $\frac{1}{2}$. Then, it suffices to translate the proof for the even case. ■

We can now state the main Theorem of the section. To show the result, the following proposition, proved in [20], will be needed.

Proposition 4. *Let μ be a $\{0,1\}$-valued measure whose minimal subsets are $B_1, ..., B_r$. If we denote by m the corresponding Möbius transform of μ, then m can be computed through the following procedure:*

- **Step 0:** *Initially, set $m(A) \to 0, \forall A \subseteq X$.*
- **Step 1:** $m(B_i) \to 1, \forall i = 1, ..., r$.
- **Step 2:** *For any $i, j, i \neq j$, $m(B_i \cup B_j) \to m(B_i \cup B_j) - 1$.*

- **Step 3:** *For any different* i, j, k, $m(B_i \cup B_j \cup B_k) \to m(B_i \cup B_j \cup B_k) + 1$.
- ...
- **Step r:** $m(B_1 \cup ... \cup B_r) \to m(B_1 \cup ... \cup B_r) + (-1)^{r+1}$.

Theorem 3. *There exist* $k > 0$ *and* n_0 *such that*

$$d_n^{n-1} > k\frac{2^n}{\sqrt{n}} d_n^m, \forall n \geq n_0.$$

Proof: Let us denote $q := \lceil \frac{n-3}{2} \rceil$ and consider n such that $q > 4$. Let us define $\mathcal{P}_1$ as the collection of all the partitions π_1 of X such that

(i) Exactly one subset in π_1 has size $q-2$,

(ii) The rest of the subsets in π_1 are singletons,

and $\mathcal{P}_2$ as the collection of all the partitions π_2 of X such that

(i) Exactly one subset in π_2 has size $q-3$,

(ii) The rest of the subsets in π_2 are singletons.

Notice that, since $q > 4$ (and hence $q-2 > q-3 > 1$), the partitions in $\mathcal{P}_1 \cup \mathcal{P}_2$ are well-defined. Notice also that the number of blocks in all partitions in $\mathcal{P}_1$ is the same, so that the parity of the number of blocks of the partitions is the same for all partitions in $\mathcal{P}_1$ and the same happens for the partitions in $\mathcal{P}_2$. Moreover, the parity of the number of blocks of the partitions is different for partitions in $\mathcal{P}_1$ and partitions in $\mathcal{P}_2$.

Consider μ a vertex of $\mathcal{FM}(X)$ not in $\mathcal{FM}^{n-1}(X)$ whose minimal subsets all have size at least q, i.e. $\mu \in \mathcal{A}_n$. Consider $\pi \in \mathcal{P}_1 \cup \mathcal{P}_2$, and μ' the fuzzy measure whose collection of minimal subsets is π.

We will show that μ and μ' are adjacent. First, let us prove that $\mu' > \mu$; for this, it is enough to show that any minimal subset of μ contains a minimal subset of μ'. Take A a minimal subset of μ. By construction, the size of A is at least q. On the other hand, the biggest subset in π has size $q-2$ at most. Let us call C that biggest subset. Then $A \setminus C$ has at least 2 elements and it must contain one of the singletons in π, whence the result.

On the other hand, since the union of two subsets in π has size at most $q-1$, it is impossible to form a π-decomposition of μ, so that μ and μ' are adjacent by Theorem 1.

The number of partitions in $\mathcal{P}_1$ is $\binom{n}{q-2}$; similarly, the number of partitions in $\mathcal{P}_2$ is $\binom{n}{q-3}$. Also, the sizes of the partitions in $\mathcal{P}_1$ and of the partitions in $\mathcal{P}_2$ have different parity, so given $\pi \in \mathcal{P}_1, \pi' \in \mathcal{P}_2$, and their corresponding fuzzy measures $\mu_\pi, \mu_{\pi'}$, it follows from Proposition 4 that either $m_{\mu_\pi}(X) = 1$ and $m_{\mu_{\pi'}}(X) = -1$ or $m_{\mu_\pi}(X) = -1$ and $m_{\mu_{\pi'}}(X) = 1$ (remark that from Proposition 4 it is not possible that $m_{\mu_\pi}(X) = 0$ or $m_{\mu_{\pi'}}(X) = 0$). Since μ is not $(n-1)$-additive and it is adjacent to every μ' whose minimal subsets are in a partition of $\mathcal{P}_1 \cup \mathcal{P}_2$, then the convex combinations of μ with all such μ' generate at least $\binom{n}{q-3}$ vertices of $\mathcal{FM}^{n-1}(X)$. Consequently,

$$d_n^{n-1} \geq \binom{n}{q-3} a_n,$$

for all n such that $q > 4$.

From Proposition 3 we know that there exist $k_1 > 0$ and n_1 such that

$$\frac{a_n}{d_n^n} > k_1, \forall n \geq n_1.$$

Thus, for n big enough (such that $n \geq n_1$ and $q > 4$), it holds that

$$d_n^{n-1} \geq k_1 \binom{n}{q-3} d_n^n.$$

Define $r := 2(q-3)$. Then, if n is even, $q = \frac{n-2}{2}$ and thus,

$$r = 2(q-3) = n - 2 - 6 = n - 8.$$

Similarly, if n is odd, $r = n - 9$. Therefore, $n > r$ and $\binom{n}{q-3} \geq \binom{r}{q-3} = \binom{r}{\frac{r}{2}}$.

Applying the Stirling approximation of the factorial it follows that

$$\binom{r}{\frac{r}{2}} \approx \sqrt{\frac{2}{\pi}} \frac{2^r}{\sqrt{r}}.$$

On the other hand, since $r \geq n - 9$, then $2^r \geq \frac{2^n}{2^9}$, and since $r < n$, it is $\frac{1}{\sqrt{r}} > \frac{1}{\sqrt{n}}$. Then,

$$\sqrt{\frac{2}{\pi}} \frac{2^r}{\sqrt{r}} > \sqrt{\frac{2}{\pi}} \frac{1}{2^9} \frac{2^n}{\sqrt{n}}$$

Thus, there exists $k_2 > 0$ such that for n big enough

$$\binom{r}{\frac{r}{2}} \geq k_2 \frac{2^n}{\sqrt{n}}.$$

Hence, putting all these facts together, we conclude that there exists $k > 0$ such that for n big enough

$$d_n^{n-1} \geq k \frac{2^n}{\sqrt{n}} d_n^n.$$

This finishes the proof. ■

This Theorem shows that the growth of the number of vertices of $\mathcal{FM}^{n-1}(X)$ is much faster than that of $\mathcal{FM}(X)$. Theorem 3 together with the values on Table 2 hint that this may be also the case for other polytopes $\mathcal{FM}^k(X), k > 2$.

6 Conclusions and open problems

Determining the vertices of the polytope $\mathcal{FM}^k(X)$ is an interesting problem from a mathematical point of view and it also appears in the practical identification of fuzzy measures from sample data. In this paper we have studied this set.

We have proposed an algorithm to obtain the vertices of $\mathcal{FM}^k(X)$. We have applied this algorithm for small values of $n = |X|$ and obtained the number of vertices of $\mathcal{FM}^k(X)$ in these cases. These figures suggest that the number of vertices of these polytopes increase more quickly than the corresponding Dedekind number. In the paper we prove that this is indeed the case for $\mathcal{FM}^{n-1}(X)$. Similar results for other values of k seem plausible and deeper investigation of the asymptotical behavior on those cases might provide further insights on the polytopes $\mathcal{FM}^k(X), k > 2$. This suggests that, despite their simple interpretation, k-additive measures have a complex structure and the choice of the value of k should be made with care. In this sense, the proof provided in Section 5 is quite technical. It could be the case that other approaches could shed light on the problem and simplify the proof. For example, in [18] two generalizations of the concept of k-additivity are proposed; however, we have not seen a simple way to apply these results in our case and more research about these extensions is needed.

The algorithm proposed in Section 3 has the drawback that it requires the knowledge of a wide number of intermediate polytopes in order to compute the vertices of the $\mathcal{FM}^k(X)$, making this computation very time consuming. More concretely, it requires the knowledge of the adjacency structure of the intermediate polytopes, and this is a NP-complete problem, thus affecting the performance of the procedure. A study of particular conditions for the adjacency in the polytope $\mathcal{FM}^k(X)$ (and in the intermediate ones) could help to decrease this computation time.

However, as the adjacency structure of $\mathcal{FM}(X)$ is known and can be computed in quadratic time, the performance for $k = n - 1$ is very efficient. Moreover, the efficiency can be improved in this case if we use the concept of orbits, as shown in Section 4.

We also intend to investigate subfamilies of $\mathcal{FM}^k(X)$ which retain the modelling power of these measures but have a reduced number of vertices. For instance, it could be interesting to restrict the vertices to those which are $\{0, 1\}$-valued and study the resulting polytope.

Related to this problem, in order to obtain a suitable initial population (for measure identification with a genetic algorithm), it could be useful to study methods for the random generation of k-additive measures.

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