On the polytopes of belief and plausibility functions

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Abstract

In this paper we study some properties of the polytope of belief functions on a finite referential. These properties can be used in the problem of identification of a belief function from sample data. More concretely, we study the set of isometries, the set of invariant measures and the adjacency structure. From these results, we prove that the polytope of belief functions is not an order polytope if the referential has more than two elements. Similar results are obtained for plausibility functions.

Keywords: Belief functions, plausibility functions, isometries, invariant measures, adjacency, order polytope.

1 Introduction

Fuzzy measures are a generalization of probabilities in which additivity turns into monotonicity. They have been proved to be an interesting tool in many different problems in which probabilities are too restrictive (see [1]).

This ability to model many situations has to be paid with an increment in the complexity of the measure. In order to solve this problem, some additional constraints are added with the aim of reducing the complexity while trying to keep the modeling capabilities as good as possible. In this sense, many subfamilies have appeared in the literature, as k-additive measures [2], p-symmetric measures [3] and many others.

Among these subfamilies, some of them, as p-symmetric measures, have similar properties to general fuzzy measures [4]. On the other hand, other subfamilies, as for example k-additive measures, behave completely different [5]. It can be proved that the reason for this is that some subfamilies have a structure of order polytopes [6], while others do not [4]. For the first group, we can study order polytopes and derive properties that apply to any subfamily being an order polytope. For example, it can be proved that the vertices of these subfamilies are $\{0,1\}$-valued measures, we can study adjacency of the vertices in quadratic time, and many other properties.

For subfamilies in the second group, these properties have to be studied independently. In this paper we aim to study the subfamily of belief functions (and consequently, the subfamily of plausibility functions), a problem that, to our knowledge, has not been addressed yet. While these functions can be considered as a subfamily of fuzzy measures, they are interesting on their own, as their are the base of the Theory of Evidence, developed by Dempster and Shafer.

The paper is organized as follows: In next section we introduce the basic concepts and facts about fuzzy measures and the Theory of Evidence. Then, in Section 3, we study the polytope of belief functions. Section 4 deals with the corresponding results for the polytope of plausibility functions.

2 Basic concepts

Consider a finite referential set $X = \{1, ..., n\}$ of n elements. Let us denote by $\mathcal{P}(X)$ the set of subsets of X. Subsets of X are denoted $A, B, ...$

Definition 1. *[7] A* **fuzzy measure** *over X is a function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying*

- $\mu(\emptyset) = 0$, $\mu(X) = 1$.
- $\forall A, B \in \mathcal{P}(X)$, *if* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$.

We will denote the set of all fuzzy measures over X by $\mathcal{FM}(X)$. Remark that $\mathcal{FM}(X)$ is a bounded convex polyhedron in $\mathbb{R}^{2^n-2}$ (or $\mathbb{R}^{2^n}$ if we include coordinates for X and $\emptyset$).

Definition 2. *Let μ be a fuzzy measure over X; we define the* **dual measure of** μ *as the fuzzy measure $\overline{\mu}$ given by $\overline{\mu}(A) = 1 - \mu(A^c)$.*

Observe that the dual application is an internal operation on $\mathcal{FM}(X)$ and $\overline{\overline{\mu}} = \mu$.

Definition 3. *[8] Let μ be a set function (not necessarily a fuzzy measure) on X. The* **Möbius transform** *of μ is another set function on X defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \; \forall A \subseteq X. \tag{1}$$

The Möbius transform given, the original set function can be recovered through the *Zeta transform* [9]:

$$\mu(A) = \sum_{B \subseteq A} m(B). \tag{2}$$

Thus, the Möbius transform provides an alternative representation of fuzzy measures. Remark that the Möbius transform can attain negative values.

The monotonicity and boundary conditions of fuzzy measures can be written in terms of the Möbius transform as follows:

Proposition 1. *[9] A set of 2^n coefficients $m(A), A \subset X$ corresponds to the Möbius representation of a fuzzy measure if and only if*

(i) $m(\emptyset) = 0, \; \sum_{A \subseteq X} m(A) = 1,$

(ii) $\sum_{i \in B \subseteq A} m(B) \geq 0$, *for all* $A \subseteq X$, *for all* $i \in A$.

An example of fuzzy measures that we will need below is the so-called *unanimity game* or *primitive measure* [10] on $A \neq \emptyset$, given by

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

For u_A, the corresponding Möbius transform is given by

$$m_A(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Let us now turn to the Theory of Evidence, developed by Dempster [11] and Shafer [12]. Suppose an experiment is carried on and we aim to find out which is the result. The whole theory spins around the so-called basic belief assignment.

Definition 4. *Let X be a referential set. A function $m : \mathcal{P}(X) \rightarrow [0,1]$ is a* **basic belief assignment** *if it satisfies the following conditions:*

- $m(\emptyset) = 0$.
- $\sum_{A \subseteq X} m(A) = 1$.

The basic belief assignment represents the proportion of all relevant and available evidence that supports the claim that the result of the experiment belongs to A but to no particular subset of A. In other words, the value of $m(A)$ does not make any additional claims about any subsets of A, each of which have, by definition, their own mass.

From the mass assignment, we can define the minimal and maximal support that we have in order to give a probability value to the event $A \equiv$ the result of the experiment belongs to A. This is done through the corresponding belief and plausibility functions.

Definition 5. *Given a basic belief assignment m, the* **belief function** $bel_m : \mathcal{P}(X) \rightarrow [0,1]$ *associated to m is a map given by*

$$bel_m(A) := \sum_{B \subseteq A} m(B), \ \forall A \subseteq X. \tag{5}$$

Similarly, the **plausibility function** *associated to m is a map $pl_m : \mathcal{P}(X) \rightarrow [0,1]$ defined by*

$$pl_m(A) := \sum_{B \cap A \neq \emptyset} m(B), \ \forall A \subseteq X. \tag{6}$$

It is easy to see that for a given basic belief assignment m,

$$bel_m(A) = 1 - pl_m(A^c), \ bel_m(A) \leq pl_m(A) \ \forall A \subseteq X. \tag{7}$$

Therefore, belief and plausibility functions are dual to each other. We will denote by $\mathcal{BEL}(X)$ (resp. $\mathcal{PL}(X)$) the set of all belief (resp. plausibility) functions on X.

It can be proved that for a probability distribution on X, the Möbius transform vanishes for any A such that $|A| > 1$. By Proposition 1, the Möbius transform on singletons is non-negative, whence probabilities are special cases of belief functions. On the other hand, they are also selfdual measures, whence they are also plausibilities. Indeed, they are the only fuzzy measures that are simultaneously belief and plausibility functions.

Observe that the basic belief assignment satisfies the conditions of Proposition 1, so that it can be considered as the Möbius transform of a fuzzy measure. Moreover, from Eq. (2), we conclude that this fuzzy measure is the corresponding belief function. Thus, $\mathcal{BEL}(X) \subseteq \mathcal{FM}(X)$. Indeed, $\mathcal{BEL}(X)$ is the set of fuzzy measures whose Möbius transform is nonnegative. Consequently, primitive measures are belief functions. In the Theory of Evidence, these belief functions are called *categorical belief functions* [13]. As plausibility functions are the dual measures of belief functions, we conclude that $\mathcal{PL}(X) \subseteq \mathcal{FM}(X)$.

Consider a linear order on $\mathcal{P}(X)\backslash\{X, \emptyset\}$. We will denote by A_i the i-th subset in this order. Then, we can identify a fuzzy measure with a vector in $[0,1]^{2^n-2}$, whose i-th coordinate corresponds to the value of the fuzzy measure on A_i.

Lemma 1. *[14] $\mathcal{BEL}(X)$ and $\mathcal{PL}(X)$ are convex polytopes.*

In next sections we are going to study the vertices of these polytopes, their group of isometries, their invariant measures and their adjacency structure. See [5] for the interest of these properties.

Recall that an **isometry** on a set $\mathcal{F}$ is a bijective function $h : \mathcal{F} \rightarrow \mathcal{F}$ preserving distances, i.e. we look for functions h satisfying

$$d(a, b) = d(h(a), h(b)) \ \forall a, b \in \mathcal{F}. \tag{8}$$

We will consider the Euclidean distance, although the results presented in the paper also hold if, instead of the Euclidean distance, a p-norm-metric ($1 \leq p < \infty$) is used.

It can be proved [5] that if $\mathcal{F}$ is a convex set and h is an isometry on $\mathcal{F}$, then

$$h(\lambda\mu_1 + (1 - \lambda)\mu_2) = \lambda h(\mu_1) + (1 - \lambda)h(\mu_2) \tag{9}$$

for all $\mu_1, \mu_2 \in \mathcal{F}$ and $\lambda \in [0, 1]$. As a consequence, if $\mathcal{F}$ is a polytope, h maps vertices into vertices. Therefore, any isometry on a polytope is defined by the images of the vertices.

Finally, given a poset $(P, \preceq)$ with m elements, the **order polytope** of P (cf. [6]), denoted by $O(P)$, is formed by the m-tuples f of real numbers indexed by the elements of P satisfying

- $0 \leq f(x) \leq 1$ for every x in P
- $f(x) \leq f(y)$ whenever $x \preceq y$ in P.

3 The polytope $\mathcal{BEL}(X)$

3.1 Vertices of $\mathcal{BEL}(X)$

The set of vertices of $\mathcal{BEL}(X)$ is a well-known result in the Theory of Evidence.

Lemma 2. *[14] The vertices of $\mathcal{BEL}(X)$ are the primitive measures $u_A, A \neq \emptyset$.*

Consequently,

Corollary 1. *The number of vertices of $\mathcal{BEL}(X)$ is $2^n - 1$.*

We will denote the number of vertices of $\mathcal{BEL}(X)$ when $|X| = n$ by U_n. This result is interesting because it shows that the number of vertices is not very large compared to the case of general fuzzy measures. Indeed, for the general case of fuzzy measures, it has been shown in [15] that the number of vertices coincides with the corresponding Dedekind number, whose general form is not known and whose number increases very quickly, as it can be seen in Table **??**.

Table 1: Comparison D_n-U_n.

n	Dedekind numbers (D_n)	U_n	$\frac{U_n}{D_n}$
1	1	1	1
2	4	3	0.75
3	18	7	0.3888
4	166	15	0.09036
5	7579	31	0.00409
6	7828352	63	8.04767e-6
7	2414682040996	127	5.25949e-11
8	56130437228687557907786	255	4.54299e-21

3.2 Isometries on $\mathcal{BEL}(X)$

Definition 6. *Consider $\sigma : X \to X$ a permutation on X. We define the* **symmetry induced by** *σ, denoted S_σ, as the transformation on $\mathcal{BEL}(X)$ such that for any $\mu \in \mathcal{BEL}(X)$, the fuzzy measure $S_\sigma(\mu)$ is defined by*

$$S_\sigma(\mu)(A) = \mu(\sigma(A)), \forall A \subseteq X.$$

with $\sigma(A) = \{\sigma(x_1), ..., \sigma(x_r)\}$ if $A = \{x_1, ..., x_r\}$.

Lemma 3. *S_σ is an isometry on $\mathcal{BEL}(X)$ for any permutation σ on X.*

Proof: It has been proved in [5] that S_σ is an isometry on $\mathcal{FM}(X)$. Thus, it suffices to show that it maps belief functions on belief functions. Let μ be a belief function and m_μ its Möbius transform. Then, the Möbius transform $m_{S_\sigma(\mu)}$ of $S_\sigma(\mu)$ is given by

$$m_{S_\sigma(\mu)}(A) = m_\mu(\sigma^{-1}(A)), \tag{10}$$

whence $S_\sigma(\mu) \in \mathcal{BEL}(X)$. As the inverse of a permutation is another permutation, we conclude that $S_\sigma(\mathcal{BEL}(X)) = \mathcal{BEL}(X)$. □

Remark 1. *It has been proved in [5] that the only isometries on $\mathcal{FM}(X)$ are permutations and compositions of permutations with the dual application. Observe however that the dual application is not an isometry on $\mathcal{BEL}(X)$, because the dual of a belief function is a plausibility function and thus it is not a belief function except for the special case of probabilities.*

Let us prove that symmetries are indeed the only isometries on $\mathcal{BEL}(X)$.

Lemma 4. *Consider u_A, u_B two primitive measures. Then,*

$$d(u_A, u_B)^2 = 2^{n-|A|} + 2^{n-|B|} - 2^{n-|A\cup B|+1}. \tag{11}$$

Proof: Let us define

$$n_1 := |\{C \subseteq X \mid A \subseteq C, B \not\subseteq C\}|.$$
$$n_2 := |\{C \subseteq X \mid B \subseteq C, A \not\subseteq C\}|.$$
$$n_3 := |\{C \subseteq X \mid A \subseteq C, B \subseteq C\}|.$$

Then, $d(u_A, u_B)^2 = n_1 + n_2$. On the other hand,

$$|\{C \subseteq X \mid u_A(C) = 1\}| = 2^{n-|A|} = n_1 + n_3$$

$$|\{C \subseteq X \mid u_A(C) = 1, u_B(C) = 1\}| = |\{C \subseteq X \mid A \cup B \subseteq C\}| = 2^{n-|A\cup B|} = n_3,$$

whence $n_1 = 2^{n-|A|} - 2^{n-|A\cup B|}$. Similarly, $n_2 = 2^{n-|B|} - 2^{n-|A\cup B|}$. Joining all these facts, we obtain the result. □

Remark 2. *Applying the previous lemma, it follows that $d(u_A, u_B)^2$ is always an even number if neither $A = X$ nor $B = X$.*

Lemma 5. *If h is an isometry on $\mathcal{BEL}(X)$, then $h(u_X) = u_X$.*

Proof: Suppose $h(u_X) \neq u_X$. Take $A \neq X$ such that $h(u_A) \neq u_X$. This is always possible if $n \geq 2$. Applying the previous remark, $d(u_X, u_A)^2$ is an odd number and $d(h(u_X), h(u_A))^2$ is an even number, contradicting the fact that h preserves distances. □

Lemma 6. *Let h be an isometry on $\mathcal{BEL}(X)$. Then, $h(u_A) = u_B$, where B is such that $|B| = |A|$.*

Proof: It suffices to note that $d(u_X, u_A)^2 = 2^{n-|A|} + 1 - 2 = 2^{n-|A|} - 1$ by Lemma 4. As $h(u_X) = u_X$, we have $d(h(u_X), h(u_A))^2 = 2^{n-|B|} - 1$. As h preserves distances, the result holds. □

Theorem 1. *Let h be an isometry on $\mathcal{BEL}(X)$. Then, h is a symmetry induced by a permutation.*

Proof: By the previous lemma, $h(u_{\{i\}}) = u_{\{j\}}$. Let us define $\sigma : X \to X$ such that $\sigma(i) = j$ if $h(u_{\{i\}}) = u_{\{j\}}$. As h is an isometry, σ is a permutation on X. We will prove that $h = S_\sigma$. For this, we will show by induction on $|A|$ that $h(u_A) = S_\sigma(u_A)$.

If $|A| = 1$, the result holds by construction. Thus, assume the result holds for $|A| = 1, 2, ..., i-1$. Suppose $|A| = i$. Take $j \in A$. Then, if $h(u_A) = u_B$, applying Lemma 4, we have

$$d(u_A, u_{A\backslash\{j\}})^2 = 2^{n-|A|} + 2^{n-|A|+1} - 2^{n-|A|+1} = 2^{n-|A|}. \tag{12}$$

As h is an isometry and by induction,

$$d(h(u_A), h(u_{A\backslash\{j\}}))^2 = d(u_B, S_\sigma(u_{A\backslash\{j\}}))^2, \tag{13}$$

where $|B| = |A|$ by the previous lemma.

Suppose $\sigma(A\backslash\{j\}) \not\subset B$. Then,

$$d(u_B, S_\sigma(u_{A\backslash\{j\}}))^2 = 2^{n-|A|} + 2^{n-|A|+1} - 2^{n-|\sigma(A\backslash\{j\})\cup B|+1}$$

$$\geq 2^{n-|A|+1} > 2^{n-|A|},$$

as $|\sigma(A\backslash\{j\}) \cup B| \geq |B| + 1 = |A| + 1$. But then h does not preserve distances, a contradiction.

We conclude that $\sigma(A\backslash\{j\}) \subset B$, $\forall j \in A$, whence $B = \sigma(A)$. □

Then, we have proved the following result:

Corollary 2. *The group of isometries on $\mathcal{BEL}(X)$ is isomorphic to the symmetric group of order n.*

3.3 Invariant belief functions

Once the group of isometries for $\mathcal{BEL}(X)$ has been obtained, we can study which are the belief functions that remain invariant under the action of any isometry. In other words, we look for the set of belief functions bel satisfying

$$h(bel) = bel, \forall h \text{ isometry} .$$

As h is induced by a permutation, this is equivalent to look for belief functions such that

$$bel(A) = bel(\sigma(A)),$$

where σ is a permutation on X. The set of invariant belief functions under the group of isometries is given in next proposition.

Proposition 2. *The set of invariant measures under the group of all isometries for $\mathcal{BEL}(X)$ is a polytope whose vertices are $\mu_1, ..., \mu_n$, where μ_i is given by the following Möbius transform:*

$$m_i(A) := \begin{cases} \frac{1}{\binom{n}{i}} & \textit{if } |A| = i \\ 0 & \textit{otherwise} \end{cases} \tag{14}$$

Proof: Consider μ, μ' two invariant belief functions under any isometry. Then,

$$h(\alpha\mu + (1-\alpha)\mu') = \alpha h(\mu) + (1-\alpha)h(\mu') = \alpha\mu + (1-\alpha)\mu', \forall h \text{ isometry}, \tag{15}$$

whence we conclude that this set is a convex set.

Note that, by Theorem 1, the set of invariant belief functions is formed by the symmetric belief functions, i.e. those satisfying $bel(A) = bel(B)$ if $|A| = |B|$. Therefore, $\mu_1, ..., \mu_n$ are invariant measures.

Suppose μ is a symmetric belief function. This implies that its Möbius transform can be written as

$$m(A) = r(|A|), \forall A \subseteq X, r(1), ..., r(n) \geq 0. \tag{16}$$

On the other hand, as μ is also a fuzzy measure, by Proposition 1, it is

$$1 = \sum_{A \subseteq X} m(A) = \sum_{i=1}^{n} \binom{n}{i} r(i). \tag{17}$$

Therefore,

$$m = \sum_{i=1}^{n} \binom{n}{i} r(i) m_i, \tag{18}$$

whence m is a convex combination of $m_1, ..., m_n$.

Finally, we need to show that $\mu_1, ..., \mu_n$ are vertices. Suppose there exists bel_1, bel_2 two symmetric belief functions such that $\mu_i = \alpha bel_1 + (1-\alpha)bel_2$, for some $\alpha \in (0,1)$. This implies that $m_i = \alpha m_{bel_1} + (1-\alpha)m_{bel_2}$. As $bel_1, bel_2 \in \mathcal{BEL}(X)$ and $m_i(A) = 0$ when $|A| \neq i$, it follows that $m_{bel_1}(A) = 0 = m_{bel_2}(A)$ if $|A| \neq i$. As bel_1, bel_2 are symmetric and $\sum_{A \subseteq X} m_{bel_1}(A) = \sum_{A \subseteq X} m_{bel_2}(A) = 1$, we conclude that $m_{bel_1}(A) = \frac{1}{\binom{n}{i}} = m_{bel_2}(A)$ if $|A| = i$, whence $m_i = m_{bel_1} = m_{bel_2}$. □

3.4 Adjacency in $\mathcal{BEL}(X)$.

The structure of facets of $\mathcal{BEL}(X)$ has been obtained in [14]. We adapt the results in that paper to our case. Consider the set of primitive measures $\{u_A : A \neq \emptyset\}$. If we write them in terms of the Möbius transform, it is easy to see that they constitute a set of affine independent points in $\mathbb{R}^{2^n-1}$. Then, the convex closure of $\{u_A : A \neq \emptyset\}$ is a simplex and thus, $\mathcal{BEL}(X)$ is a simplex.

The facet structure of a simplex is very simple, as a facet of dimension k is just the convex closure of $k+1$ vertices. Thus, edges of the polytope are exactly the convex combinations of two primitive measures, whence we can conclude the following:

Lemma 7. *Consider two primitive measures $u_A, u_B, A, B \neq \emptyset$. Then, they are adjacent vertices in $\mathcal{BEL}(X)$.*

3.5 $\mathcal{BEL}(X)$ is not an order polytope if $|X| > 2$.

From the previous results in this section, it can be seen that $\mathcal{BEL}(X)$ is not an order polytope for $|X| > 2$. Indeed, suppose that there exist a poset $(P, \preceq)$ such that $\mathcal{BEL}(X) = O(P)$ and $|X| > 2$.

A *filter* F is a subset of P such that if $x \in F$ and $x \preceq y$, then $y \in P$. It has been proved in [16] that the vertices of an order polytope can be identified with the (characteristic function of) filters of P. In other words, the vertices of $O(P)$ are $\{f_F : F \text{ filter }\}$, where f_F is given by

$$f_F(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases}$$

From now on and in order to avoid heavy notation, we will identify f_F with F. The adjacency structure of an order polytope can be stated in terms of the subjacent poset P and is given by the following results:

Lemma 8. *[17] A necessary condition for F_1 and F_2 to be adjacent vertices in $O(P)$ is that either $F_1 \subset F_2$ or $F_2 \subset F_1$.*

Theorem 2. *[17] If F_1 and F_2 are filters of P and $F_1 \subset F_2$, then F_1 and F_2 are adjacent vertices in $O(P)$ if and only if $F_2 \setminus F_1$ is a connected subposet of P.*

On the other hand, we have proved in Lemma 7 that all vertices in $\mathcal{BEL}(X)$ are adjacent to each other. For order polytopes such that all vertices are adjacent to each other, the following has been proved (see [4], Proposition 2):

Proposition 3. *In an order polytope, all vertices are adjacent to each other if and only if P is a chain.*

If P is a chain, the following holds:

Proposition 4. *[18] Suppose P is a chain $1 \prec 2 ... \prec n$. Then, there are only two isometries on $O(P)$, namely the identity map and the reverse-order application.*

But we have seen in Corollary 2 that there are $n!$ isometries, a contradiction if $n > 2$. Then, we conclude that $\mathcal{BEL}(X)$ is not an order polytope if $|X| > 2$.

Let us now deal with the case $|X| = 2$. In this case, we have just three vertices for $\mathcal{BEL}(X)$, namely $u_X, u_{\{1\}}$ and $u_{\{2\}}$. Notice that each belief function over a referential of two elements can be

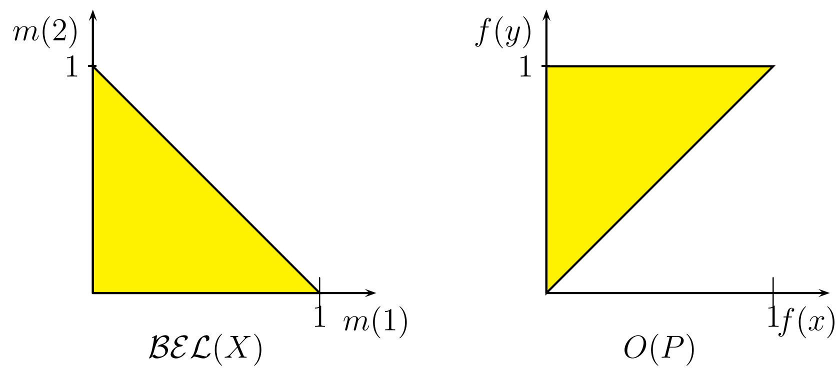

Figure 1: The sets $\mathcal{BEL}(X)$ when $|X| = 2$ and $O(P)$ for a chain of two elements

identified with a pair $(m(1), m(2))$ such that $m(1), m(2) \geq 0$ and $m(1) + m(2) \leq 1$ (the value $m(X)$ given by $1 - m(1) - m(2)$). See Figure 1 left.

Consider the poset $(P, \preceq)$ given by $P = \{x, y\}$ and $x \preceq y$. In this case, we have three filters, namely $F_1 = \emptyset, F_2 = \{x, y\}, F_3 = \{y\}$. And $O(P)$ is given by the pairs $(f(x), f(y))$ such that $0 \leq f(x) \leq f(y) \leq 1$. See Figure 1 right.

Let us define the map:

$$h : \mathcal{BEL}(X) \mapsto O(P)$$

given by $h(m(1), m(2)) = (m(1), m(1) + m(2))$. It is easy to see that h is a bijective linear map from $\mathcal{BEL}(X)$ to $O(P)$. Therefore, in this case $\mathcal{BEL}(X)$ is an order polytope.

4 The polytope $\mathcal{PL}(X)$

In this section we deal with the polytope of plausibility functions. This polytope is the dual polytope of $\mathcal{BEL}(X)$ and consequently, all the results obtained in the previous section can be translated to this case. We prove the first result as an example.

Lemma 9. *Let μ be a plausibility function. Then, μ is a vertex of $\mathcal{PL}(X)$ if and only if $\overline{\mu}$ is a primitive measure.*

Proof: Suppose $\overline{u}_A$ is not a vertex of $\mathcal{PL}(X)$. Then, there exist pl_1, pl_2 plausibilities different to $\overline{u}_A$ such that

$$\overline{u}_A = \alpha pl_1 + (1 - \alpha) pl_2, \; \alpha \in (0, 1). \tag{19}$$

But then,

$$u_A = \alpha \overline{pl}_1 + (1 - \alpha) \overline{pl}_2, \; \alpha \in (0, 1). \tag{20}$$

As $\overline{pl}_1, \overline{pl}_2 \in \mathcal{BEL}(X)$, we conclude that u_A is not a vertex of $\mathcal{BEL}(X)$, a contradiction.

On the other hand, suppose pl is a vertex of $\mathcal{PL}(X)$. If $\overline{pl}$ is not a primitive measure, there exist bel_1, bel_2 two belief functions such that

$$\overline{pl} = \alpha bel_1 + (1 - \alpha) bel_2, \; \alpha \in (0, 1). \tag{21}$$

But as before, taking the dual measures, this implies that pl is not a vertex of $\mathcal{PL}(X)$, a contradiction. $\square$

Thus, the vertices of $\mathcal{PL}(X)$ are given by the plausibility functions $\{pl_A \mid A \neq X\}$, duals of u_A, defined by

$$pl_A(B) := \begin{cases} 1 & \text{if } B \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

Similarly, we can prove the following results:

Lemma 10. *The isometries on $\mathcal{PL}(X)$ are the symmetries.*

Lemma 11. *The set of invariant plausibilities for any isometry on $\mathcal{PL}(X)$ is a convex polytope whose vertices are $\overline{\mu}_1, ..., \overline{\mu}_n$, where $\mu_1, ..., \mu_n$ are the vertices of the polytope of invariant belief functions.*

Lemma 12. *Given pl_A, pl_B two vertices of $\mathcal{PL}(X)$, then they are adjacent.*

Lemma 13. *The set $\mathcal{PL}(X)$ is not an order polytope for $|X| > 2$ and it is an order polytope for $|X| = 2$.*

5 Conclusions

In this paper we have studied the polytopes of belief and plausibility functions. Although we have treated them as subfamilies of fuzzy measures, these functions are important on their own as a tool to model uncertainty.

In this paper we have studied some properties of these polytopes. Besides its interest from a mathematical point of view, they are also interesting from a practical point of view; more concretely, these problems arises in the practical identification of these functions through genetic algorithms when sample information is available (see [19, 5]).

The main result of the paper is that these functions are not order polytopes; this is an unexpected result, as belief and plausibility functions share many of the properties of order polytopes; for example, their vertices are 0/1-valued, adjacency is simple, symmetries are isometries, ...

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