

# Empirical phi-divergence test statistics for testing simple null hypotheses based on exponentially tilted empirical likelihood estimators\*

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## Abstract

In Econometrics, imposing restrictions without assuming underlying distributions to modelize complex realities is a valuable methodological tool. However, if a subset of restrictions were not correctly specified, the usual test-statistics for correctly specified models tend to reject erroneously a simple null hypothesis. In this setting, we may say that the model suffers from misspecification. We study the behavior of empirical phi-divergence test-statistics, introduced in *Balakrishnan et al. (2015)*, by using the exponential tilted empirical likelihood estimators of *Schennach (2007)*, as a good compromise between the efficiency of the significance level for small sample sizes and the robustness under misspecification.

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## 1 Introduction

Let  $\mathbf{X}_1, \dots, \mathbf{X}_n$  be i.i.d. observations on a data vector  $\mathbf{X}$  with unknown distribution function  $F$  having a finite expectation, a non-singular variance-covariance matrix and a  $p$ -dimensional parameter of interest,  $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^p$ . All the information about  $F$  and  $\boldsymbol{\theta}$  is available in the form of  $r \geq p$  estimating functions of the data observation  $\mathbf{X}$  and the parameter  $\boldsymbol{\theta}$

$$\mathbf{g}(\mathbf{X}, \boldsymbol{\theta}) = (g_1(\mathbf{X}, \boldsymbol{\theta}), \dots, g_r(\mathbf{X}, \boldsymbol{\theta}))^T. \quad (1)$$

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The model has a true parameter  $\theta_0$  satisfying the moment condition

$$E_F [g(\mathbf{X}, \theta_0)] = \mathbf{0}_r, \quad (2)$$

where  $E_F [\cdot]$  denotes expectation taken with respect to the distribution of  $F$  of  $\mathbf{X}$ . The parameter  $\theta$  has been traditionally estimated using two-step efficient generalized method of moments estimators (GMM). This method of estimation was introduced by Hansen (1982). In Hayashi (2000), for instance, all the estimation techniques are presented and discussed in the GMM framework. A GMM estimator for  $\theta_0$  is  $\hat{\theta}_{GMM}$ , defined by

$$\hat{\theta}_{GMM} = \arg \min_{\theta \in \Theta} \bar{g}_n^T(\mathbf{X}, \theta) \mathbf{W}_n^{-1}(\theta) \bar{g}_n(\mathbf{X}, \theta),$$

where

$$\bar{g}_n(\mathbf{X}, \theta) = \frac{1}{n} \sum_{i=1}^n g(\mathbf{X}_i, \theta) \quad (3)$$

and  $\mathbf{W}_n$  is a positive semidefinite matrix. Under some regularity conditions  $\hat{\theta}_{GMM}$  is consistent for  $\theta_0$  but in general it is not efficient if  $r > p$ . The  $\hat{\theta}_{GMM}$  will be asymptotically efficient if the limit of the matrix  $\mathbf{W}_n$  is the matrix

$$\mathbf{S}_{11}(\theta_0) = E_F [g(\mathbf{X}, \theta_0) g^T(\mathbf{X}, \theta_0)]. \quad (4)$$

A feasible version of this efficient procedure is based on obtaining an initial consistent estimator  $\hat{\theta}$  of  $\theta_0$  by,

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \bar{g}_n^T(\mathbf{X}, \theta) \bar{g}_n(\mathbf{X}, \theta)$$

and then to consider

$$\hat{\theta}_{GMM} = \arg \min_{\theta \in \Theta} \bar{g}_n^T(\mathbf{X}, \theta) \hat{\mathbf{S}}_{11}^{-1}(\hat{\theta}) \bar{g}_n(\mathbf{X}, \theta),$$

with

$$\hat{\mathbf{S}}_{11}(\theta) = \frac{1}{n} \sum_{i=1}^n g(\mathbf{X}_i, \theta) g^T(\mathbf{X}_i, \theta). \quad (5)$$

An alternative to the GMM estimator is the (CU) continuous updating estimator obtained by

$$\hat{\theta}_{CU} = \arg \min_{\theta \in \Theta} \bar{g}_n^T(\mathbf{X}, \theta) \hat{\mathbf{S}}_{11}^{-1}(\theta) \bar{g}_n(\mathbf{X}, \theta).$$

The GMM estimators have nice asymptotic properties (see Gallant and White (1988), Newey and McFadden (1990)). They are consistent, asymptotically normal and asymptotically efficient under some regularity assumptions. However, several authors report that the two-step GMM estimator suffers from a substantial amount of bias in finite samples (see Altonji and Segal (1996), Andersen and Sørensen (1996) and Hansen, Heaton and Yaron (1996)). This encourages the increasing literature on alternatives to the GMM. Maybe the most known alternative estimators to the GMM are: the continuously updated (CU) estimator of Hansen, Heaton and Yaron (1996), the empirical likelihood estimator (EL) of Owen (1988, 1990), Qin and Lawless (1994), and Imbens (1997), the exponential tilting (ET) estimator of Kitamura and Stutzer (1997) and Imbens, Spady and Johnson (1998), the minimum Hellinger distance estimator of Kitamura, Otsa and Evdokimov (2013) and the generalized empirical likelihood (GEL) estimators of Newey and Smith (2004). Although EL estimator is preferable to the previous estimators in higher-order asymptotic properties, these properties hold only under correct

specification of the moment condition, and the asymptotic behavior of EL estimator becomes problematic under misspecification. The ET estimator is inferior to the EL estimator in relation to higher-order asymptotic properties, but remain well behaved in presence of misspecification under relative weak regularity conditions. To overcome this problem, Schennach (2007) suggests the exponentially tilted empirical likelihood (ETEL) that shares the same higher-order property with EL under correct specification while maintaining usual asymptotic properties such as  $\sqrt{n}$ -consistency and asymptotic normality under misspecification.

Qin and Lawless (1994) studied the empirical likelihood ratio statistic for testing simple null hypotheses based on the EL estimators. Later Balakrishnan et al. (2015), using EL, considered some families of test statistics based on  $\phi$ -divergence measures: empirical  $\phi$ -divergence test statistics, which contain the empirical likelihood ratio test as a particular case. Some members of this family have a better behavior for small sample sizes in the sense of the size and power of the test. The contribution of the current paper is to extend the empirical  $\phi$ -divergence test statistics replacing the EL estimators by the ET and ETEL estimators to study their robustness, in particular under misspecification, which is their major advantage with respect to the previous ones.

In Section 2 we introduce the ETEL estimator given by Schennach (2007) which is obtained as a combination of EL and ET procedures to deliver an estimator and we present its asymptotic properties. Section 3 is devoted to introduce the empirical  $\phi$ -divergence statistics for testing simple null hypotheses on the basis of the ETEL estimator and we present their asymptotic distribution. Based on it, power approximations of the empirical  $\phi$ -divergence test statistics are derived. A rigorous study of the robustness of the empirical  $\phi$ -divergence test statistics is derived in Section 4 and the asymptotic distribution of the empirical  $\phi$ -divergence is developed under misspecified alternative hypotheses. In Section 5 a simulation study is presented and finally, in Section 6 some conclusions are given.

## 2 Exponentially tilted empirical likelihood

Let  $\mathbf{x}_1, \dots, \mathbf{x}_n$  be a realization of  $\mathbf{X}_1, \dots, \mathbf{X}_n$ . The empirical likelihood function is given by

$$\mathcal{L}_{F_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \prod_{i=1}^n dF(\mathbf{x}_i) = \prod_{i=1}^n p_i,$$

where  $p_i = dF(\mathbf{x}_i) = P(\mathbf{X} = \mathbf{x}_i)$ . Only distributions with an atom of probability at each  $\mathbf{x}_i$  have non-zero likelihood, and without consideration of estimating functions, the empirical likelihood function  $\mathcal{L}_{F_n}$  is seen to be maximized, at  $\mathbf{X}_1 = \mathbf{x}_1, \dots, \mathbf{X}_n = \mathbf{x}_n$ , by the empirical distribution function

$$F_n(\mathbf{x}) = \sum_{i=1}^n u_i I(\mathbf{X}_i \leq \mathbf{x}),$$

which is associated with the  $n$ -dimensional discrete uniform distribution

$$\mathbf{u} = (u_1, \dots, u_n)^T = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)^T.$$

Let

$$F_{n,\theta}(\mathbf{x}) = \sum_{i=1}^n p_i(\theta) I(\mathbf{X}_i \leq \mathbf{x}),$$

be an empirical distribution function associated with the probability vector

$$\mathbf{p}(\boldsymbol{\theta}) = (p_1(\boldsymbol{\theta}), \dots, p_n(\boldsymbol{\theta}))^T, \quad p_i(\boldsymbol{\theta}) > 0, \quad \sum_{i=1}^n p_i(\boldsymbol{\theta}) = 1, \quad (6)$$

and

$$\ell_{EL}(\boldsymbol{\theta}) = \sum_{i=1}^n \log p_i(\boldsymbol{\theta}) \quad (7)$$

the kernel of the empirical log-likelihood function. The moment conditions given in (2) can be expressed from an empirical point of view as

$$E_{F_{n,\boldsymbol{\theta}}}[\mathbf{g}(\mathbf{X}, \boldsymbol{\theta})] = \sum_{i=1}^n p_i(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) = \mathbf{0}_r, \quad (8)$$

which are the so-called estimating equations. If we are interested in maximizing (7) subject to (8), by applying the Lagrange multipliers method it is possible to reduce the dimension of the probability vector ( $n$ ), to the number of estimating functions ( $r$ ), since

$$p_{EL,i}(\boldsymbol{\theta}) = \frac{1}{n} \frac{1}{1 + \mathbf{t}_{EL}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})}, \quad i = 1, \dots, n, \quad (9)$$

where  $\mathbf{t}_{EL}(\boldsymbol{\theta})$  is an  $r$ -dimensional vector to be determined by solving the non-linear system of  $r$  equations,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \mathbf{t}_{EL}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) &= \mathbf{0}_r, \\ \text{s.t. } \mathbf{t}_{EL}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) &> \frac{1-n}{n}. \end{aligned} \quad (10)$$

Maximizing expression (7) is equivalent to minimize the expression

$$-\frac{1}{n} \sum_{i=1}^n \log(np_i(\boldsymbol{\theta}))$$

and this expression can be written as the Kullback–Leibler divergence measure between the probability vectors  $\mathbf{u}$  and  $\mathbf{p}(\boldsymbol{\theta})$ , i.e.,

$$D_{\text{Kull}}(\mathbf{u}, \mathbf{p}(\boldsymbol{\theta})) = \sum_{i=1}^n u_i \log \frac{u_i}{p_i(\boldsymbol{\theta})}.$$

Therefore,

$$\hat{\boldsymbol{\theta}}_{EL} = \arg \min_{\boldsymbol{\theta} \in \Theta} D_{\text{Kull}}(\mathbf{u}, \mathbf{p}_{EL}(\boldsymbol{\theta}))$$

subject to the restrictions given in (8).

If we consider  $D_{\text{Kull}}(\mathbf{p}(\boldsymbol{\theta}), \mathbf{u})$ , rather than  $D_{\text{Kull}}(\mathbf{u}, \mathbf{p}(\boldsymbol{\theta}))$ , we get the empirical exponential tilting (ET) estimator, considered for instance in Kitamura and Stutzer (1997). In that case

$$\hat{\boldsymbol{\theta}}_{ET} = \arg \min_{\boldsymbol{\theta} \in \Theta} D_{\text{Kull}}(\mathbf{p}(\boldsymbol{\theta}), \mathbf{u}).$$

where

$$D_{\text{Kull}}(\mathbf{p}(\boldsymbol{\theta}), \mathbf{u}) = \sum_{i=1}^n p_i(\boldsymbol{\theta}) \log(np_i(\boldsymbol{\theta})) \quad (11)$$

and

$$p_{ET,i}(\boldsymbol{\theta}) = \frac{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta})\mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}}{\sum_{j=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta})\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})\}}, \quad i = 1, \dots, n, \quad (12)$$

where  $\mathbf{t}_{ET}(\boldsymbol{\theta})$  is an  $r$ -dimensional vector to be determined by solving the non-linear system of  $r$  equations

$$\frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta})\mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) = \mathbf{0}_r. \quad (13)$$

The exponentially tilted empirical likelihood (ETEL) introduced by Schennach (2007) combines EL and ET procedures to deliver an estimator. The ETEL estimator is defined as

$$\hat{\boldsymbol{\theta}}_{ETEL} = \arg \min_{\boldsymbol{\theta} \in \Theta} D_{\text{Kull}}(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})), \quad (14)$$

where

$$\mathbf{p}_{ET}(\boldsymbol{\theta}) = (p_{ET,1}(\boldsymbol{\theta}), \dots, p_{ET,n}(\boldsymbol{\theta}))^T, \quad (15)$$

and  $p_{ET,i}(\boldsymbol{\theta})$  is given by (12). Theorem 1 in Schennach establishes that the ETEL estimator of  $\boldsymbol{\theta}$  maximizes the kernel of the empirical log-likelihood function given by

$$\ell_{ETEL}(\boldsymbol{\theta}) = \sum_{i=1}^n \log p_{ET,i}(\boldsymbol{\theta}) = -\log \left( \frac{1}{n} \sum_{i=1}^n \exp \{ \mathbf{t}_{ET}^T(\boldsymbol{\theta}) [\mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) - \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta})] \} \right), \quad (16)$$

where  $\mathbf{t}_{ET}(\boldsymbol{\theta})$  is obtained by solving (13) and  $\bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta})$  was defined in (3). In Schennach (2007, page 659) the following important relation for this paper is presented,

$$\begin{pmatrix} \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) \\ \mathbf{0}_p \end{pmatrix} + \begin{pmatrix} \mathbf{S}_{11}(\boldsymbol{\theta}_0) & \mathbf{S}_{12}(\boldsymbol{\theta}_0) \\ \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) & \mathbf{0}_{p \times p} \end{pmatrix} \begin{pmatrix} \mathbf{t}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL}) \\ \hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0 \end{pmatrix} = o_p(n^{-1/2}), \quad (17)$$

with  $\mathbf{S}_{11}(\boldsymbol{\theta}_0)$  given in (4), and

$$\mathbf{S}_{12}(\boldsymbol{\theta}) = \mathbb{E}_F[\mathbf{G}_\mathbf{X}(\boldsymbol{\theta})], \quad (18)$$

$$\mathbf{G}_\mathbf{X}(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}^T} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}). \quad (19)$$

Based on (17), we have

$$\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0 = \mathbf{V}(\boldsymbol{\theta}_0) \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(n^{-1/2}),$$

where

$$\mathbf{V}(\boldsymbol{\theta}_0) = \left( \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \mathbf{S}_{12}(\boldsymbol{\theta}_0) \right)^{-1}, \quad (20)$$

and

$$\mathbf{t}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL}) = -\mathbf{R}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(n^{-1/2}). \quad (21)$$

where

$$\mathbf{R}(\boldsymbol{\theta}_0) = \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) - \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \mathbf{S}_{12}(\boldsymbol{\theta}_0) \mathbf{V}(\boldsymbol{\theta}_0) \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0).$$

Expression (21) is obtained from (17). Hence,

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_p, \mathbf{V}(\boldsymbol{\theta}_0)),$$

and

$$\sqrt{nt_{ET}}(\widehat{\boldsymbol{\theta}}_{ETEL}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_r, \mathbf{R}(\boldsymbol{\theta}_0)).$$

In the following section we propose a new family of empirical test statistics for testing a simple null hypothesis, when the unknown parameters are estimated using the ETEL estimator defined in (14) and then derive their asymptotic distribution.

### 3 New family of empirical phi-divergence test statistics

The empirical likelihood ratio statistic for testing

$$H_0: \boldsymbol{\theta} = \boldsymbol{\theta}_0 \text{ vs. } H_1: \boldsymbol{\theta} \neq \boldsymbol{\theta}_0 \quad (22)$$

based on the ETEL estimator has the expression

$$\begin{aligned} G_n^2(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) &= 2 \sum_{i=1}^n \log p_{ET,i}(\widehat{\boldsymbol{\theta}}_{ETEL}) - 2 \sum_{i=1}^n \log p_{ET,i}(\boldsymbol{\theta}_0) \\ &= -2n \left( \ell_{ETEL}(\boldsymbol{\theta}_0) - \ell_{ETEL}(\widehat{\boldsymbol{\theta}}_{ETEL}) \right), \end{aligned} \quad (23)$$

where  $\ell_{ETEL}(\bullet)$  was defined in (16). Schennach (2007) established that under  $H_0$

$$G_n^2(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \chi_p^2.$$

It is clear that the empirical likelihood ratio test statistic given in (23) can be expressed as

$$G_n^2(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) = 2n \left( D_{\text{Kull}}(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) - D_{\text{Kull}}(\mathbf{u}, \mathbf{p}_{ET}(\widehat{\boldsymbol{\theta}}_{ETEL})) \right),$$

where  $\mathbf{p}_{ET}(\boldsymbol{\theta})$  is (15).

We shall denote by  $\Phi^*$  the class of all convex functions  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that at  $x = 1$ ,  $\phi(1) = 0$ ,  $\phi''(1) > 0$ , and at  $x = 0$ ,  $0\phi(0/0) = 0$  and  $0\phi(p/0) = p \lim_{u \rightarrow \infty} \frac{\phi(u)}{u}$ . If instead of considering the Kullback-Leibler divergence measure, we consider a general function  $\phi \in \Phi^*$  to define the  $\phi$ -divergence measure between the probability vectors  $\mathbf{u}$  and  $\mathbf{p}(\boldsymbol{\theta})$  as

$$D_\phi(\mathbf{u}, \mathbf{p}(\boldsymbol{\theta})) = \sum_{i=1}^n p_i(\boldsymbol{\theta}) \phi\left(\frac{u_i}{p_i(\boldsymbol{\theta})}\right), \quad \phi \in \Phi^*, \quad (24)$$

we obtain a new family of empirical test statistics for testing (22) given by

$$T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) = \frac{2n}{\phi''(1)} \left( D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) - D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\widehat{\boldsymbol{\theta}}_{ETEL})) \right), \quad (25)$$

i.e.,

$$T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) = \frac{2}{\phi''(1)} \left( \sum_{i=1}^n n p_{ET,i}(\boldsymbol{\theta}_0) \phi\left(\frac{1}{n p_{ET,i}(\boldsymbol{\theta}_0)}\right) - \sum_{i=1}^n n p_{ET,i}(\widehat{\boldsymbol{\theta}}_{ETEL}) \phi\left(\frac{1}{n p_{ET,i}(\widehat{\boldsymbol{\theta}}_{ETEL})}\right) \right).$$

Moreover, the empirical likelihood ratio test statistic falls inside this new family since  $G_n^2(\widehat{\boldsymbol{\theta}}_{ETEL,n}, \boldsymbol{\theta}_0) = T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ , with  $\phi(x) = x \log x - x + 1$ .

It is well-known that the family of test statistics based on  $\phi$ -divergence measures has some nice and optimal properties for different inferential problems in relation to efficiency, but especially in relation to robustness; see Pardo (2006) and Basu et al. (2011).

For every  $\phi \in \Phi^*$  differentiable at  $x = 1$ , the function  $\varphi(x) \equiv \phi(x) - (x - 1)\phi'(1)$  also belongs to  $\Phi^*$ . Then, we have

$$T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) = T_n^\varphi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$$

and  $\varphi$  has the additional property that  $\varphi'(1) = 0$ . Since the two divergence measures are equivalent, without any loss of generality we can consider the set  $\Phi = \Phi^* \cap \{\phi : \phi'(1) = 0\}$ . In what follows, we shall assume that  $\phi \in \Phi$ .

Another family of statistics for testing the hypotheses in (22) based only on the  $\phi$ -divergence measure between  $\mathbf{p}_{ET}(\widehat{\boldsymbol{\theta}}_{ETEL})$  and  $\mathbf{p}_{ET}(\boldsymbol{\theta}_0)$ , namely,  $D_\phi(\mathbf{p}_{ET}(\widehat{\boldsymbol{\theta}}_{ETEL}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0))$ , is given by

$$\begin{aligned} S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) &= \frac{2n}{\phi''(1)} D_\phi(\mathbf{p}_{ET}(\widehat{\boldsymbol{\theta}}_{ETEL}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) \\ &= \frac{2n}{\phi''(1)} \sum_{i=1}^n p_{ET,i}(\boldsymbol{\theta}_0) \phi\left(\frac{p_{ET,i}(\widehat{\boldsymbol{\theta}}_{ETEL})}{p_{ET,i}(\boldsymbol{\theta}_0)}\right), \end{aligned} \quad (26)$$

where  $\phi$  is a function satisfying the same conditions as function  $\phi$  used to construct  $T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ .

We shall refer to both families of test statistics as empirical  $\phi$ -divergence test statistics. The first family has been applied for the first time in Broniatowski and Keziou (2012) but using the EL estimator rather than the ETEL estimator and only in the case that the parameter dimension is equal to the number of estimating equations ( $p = r$ ). Both families were applied in Balakrishnan et al. (2015) only with the EL estimator.

**Condition 1** Let  $\|\cdot\|$  denote any vector or matrix norm. We shall assume the following regularity conditions (Theorem 1 in Qin and Lawless, 1994):

- i)  $\mathbf{S}_{11}(\boldsymbol{\theta}_0)$  in (4) is positive definite, and for  $\mathbf{S}_{12}(\boldsymbol{\theta}_0)$  in (18),  $\text{rank}(\mathbf{S}_{12}(\boldsymbol{\theta}_0)) = p$ ;
- ii) There exists a neighborhood of  $\boldsymbol{\theta}_0$  in which  $\|\mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\|^3$  is bounded by some integrable function of  $\mathbf{X}$ ;
- iii) There exists a neighborhood of  $\boldsymbol{\theta}_0$  in which  $\mathbf{G}_\mathbf{X}(\boldsymbol{\theta})$ , given in (19), is continuous and  $\|\mathbf{G}_\mathbf{X}(\boldsymbol{\theta})\|$  is bounded by some integrable function of  $\mathbf{X}$ ;
- iv) There exists a neighborhood of  $\boldsymbol{\theta}_0$  in which  $\frac{\partial \mathbf{G}_\mathbf{X}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$  is continuous and  $\left\| \frac{\partial \mathbf{G}_\mathbf{X}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\|$  is bounded by some integrable function of  $\mathbf{X}$ .

The asymptotic distribution of the empirical  $\phi$ -divergence test statistics,  $T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  and  $S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ , is given in the following theorem.

**Theorem 2** Under Condition 1 and under the null hypothesis given in (22),

$$T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0), \quad S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \chi_p^2.$$

**Proof.** We shall prove the result for  $S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ . In a similar way can be established the result for  $T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ .

Let us consider

$$\hat{\mathbf{t}}_{ETEL} = \mathbf{t}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL}) \quad \text{and} \quad \mathbf{t}_0 = \mathbf{t}(\boldsymbol{\theta}_0).$$

We rename  $D_\phi(\mathbf{p}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)$  as a function of  $\hat{\mathbf{t}}_{ETEL}$  and  $\mathbf{t}_0$ , i.e.

$$d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0) = \sum_{i=1}^n \frac{\exp\{\mathbf{t}_0^T \mathbf{g}(\mathbf{x}_i, \boldsymbol{\theta}_0)\}}{\sum_{j=1}^n \exp\{\mathbf{t}_0^T \mathbf{g}(\mathbf{x}_j, \boldsymbol{\theta}_0)\}} \phi \left( \frac{\exp\{\hat{\mathbf{t}}_{ETEL}^T \mathbf{g}(\mathbf{x}_i, \hat{\boldsymbol{\theta}}_{ETEL})\}}{\sum_{j=1}^n \exp\{\hat{\mathbf{t}}_{ETEL}^T \mathbf{g}(\mathbf{x}_j, \hat{\boldsymbol{\theta}}_{ETEL})\}} \middle/ \frac{\exp\{\mathbf{t}_0^T \mathbf{g}(\mathbf{x}_i, \boldsymbol{\theta}_0)\}}{\sum_{j=1}^n \exp\{\mathbf{t}_0^T \mathbf{g}(\mathbf{x}_j, \boldsymbol{\theta}_0)\}} \right).$$

A second-order Taylor expansion of  $d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)$  around  $(\mathbf{0}_r, \mathbf{0}_r)$  gives

$$\begin{aligned} d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0) &= d_\phi(\mathbf{0}_r, \mathbf{0}_r) + \frac{\partial d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_1^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} \hat{\mathbf{t}}_{ETEL} + \frac{\partial d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_2^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} \mathbf{t}_0 \\ &\quad + \frac{1}{2} \hat{\mathbf{t}}_{ETEL}^T \frac{\partial^2 d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_1 \partial \mathbf{t}_1^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} \hat{\mathbf{t}}_{ETEL} + \frac{1}{2} \mathbf{t}_0^T \frac{\partial^2 d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_2 \partial \mathbf{t}_2^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} \mathbf{t}_0 \\ &\quad + \hat{\mathbf{t}}_{ETEL}^T \frac{\partial^2 d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_2 \partial \mathbf{t}_1^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} \mathbf{t}_0 + o(\|\hat{\mathbf{t}}_{ETEL}\|^2) + o(\|\mathbf{t}_0\|^2). \end{aligned}$$

It is easy to show that

$$\begin{aligned} d_\phi(\mathbf{0}_r, \mathbf{0}_r) &= 0, \quad \frac{\partial d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_1^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} = \frac{\partial d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_2^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} = \mathbf{0}_r^T, \\ \frac{\partial^2 d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_1 \partial \mathbf{t}_1^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} &= \frac{\partial^2 d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_2 \partial \mathbf{t}_2^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} = \phi''(1) \hat{\mathbf{S}}_{11}(\boldsymbol{\theta}_0) = \phi''(1) \mathbf{S}_{11}(\boldsymbol{\theta}_0) + o_p(\mathbf{1}_{r \times r}), \\ \frac{\partial^2 d_\phi(\mathbf{t}_1, \mathbf{t}_2)}{\partial \mathbf{t}_2 \partial \mathbf{t}_1^T} \bigg|_{\mathbf{t}_1=\mathbf{t}_2=\mathbf{0}_r} &= -\phi''(1) \hat{\mathbf{S}}_{11}(\boldsymbol{\theta}_0) = -\phi''(1) \mathbf{S}_{11}(\boldsymbol{\theta}_0) + o_p(\mathbf{1}_{r \times r}). \end{aligned}$$

Then, we have

$$\begin{aligned} S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) &= \frac{2nd_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)}{\phi''(1)} \\ &= n\hat{\mathbf{t}}_{ETEL}^T \mathbf{S}_{11}(\boldsymbol{\theta}_0) \hat{\mathbf{t}}_{ETEL} + n\mathbf{t}_0^T \mathbf{S}_{11}(\boldsymbol{\theta}_0) \mathbf{t}_0 - 2n\hat{\mathbf{t}}_{ETEL}^T \mathbf{S}_{11}(\boldsymbol{\theta}_0) \mathbf{t}_0 + o(n\|\hat{\mathbf{t}}_{ETEL}\|^2) + o(n\|\mathbf{t}_0\|^2). \end{aligned}$$

Denoting

$$\mathbf{h}(\mathbf{t}(\boldsymbol{\theta})) = \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}),$$

from (13) the Taylor expansion of  $\mathbf{h}(\mathbf{t}_0)$  around  $\mathbf{t}_0 = \mathbf{0}_r$  is equal to

$$\mathbf{0}_r = \mathbf{h}(\mathbf{0}_r) + \left( \frac{\partial}{\partial \mathbf{t}_0^T} \mathbf{h}(\mathbf{t}_0) \big|_{\mathbf{t}_0=\mathbf{0}_r} \right) \mathbf{t}_0 + o(\|\mathbf{t}_0\| \mathbf{1}_r),$$

where  $\mathbf{h}(\mathbf{0}_r) = \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0)$ ,  $\frac{\partial}{\partial \mathbf{t}_0^T} \mathbf{h}(\mathbf{t}_0) \big|_{\mathbf{t}_0=\mathbf{0}_r} = \hat{\mathbf{S}}_{11}(\boldsymbol{\theta}_0) = \mathbf{S}_{11}(\boldsymbol{\theta}_0) + o_p(\mathbf{1}_{r \times r})$ , and from it the following relation is obtained

$$n^{1/2} \mathbf{t}_0 = -\mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) n^{1/2} \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(\mathbf{1}_r), \quad (27)$$



Taking into account (20), (21) and (27), it holds

$$\begin{aligned} n\hat{\mathbf{t}}_{ETEL}^T \mathbf{S}_{11}(\boldsymbol{\theta}_0) \hat{\mathbf{t}}_{ETEL} &= n\bar{\mathbf{g}}_n^T(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{R}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(1), \\ n\mathbf{t}_0^T \mathbf{S}_{11}(\boldsymbol{\theta}_0) \mathbf{t}_0 &= n\bar{\mathbf{g}}_n^T(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(1), \\ n\hat{\mathbf{t}}_{ETEL}^T \mathbf{S}_{11}(\boldsymbol{\theta}_0) \mathbf{t}_0 &= n\bar{\mathbf{g}}_n^T(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{R}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(1), \end{aligned}$$

and consequently

$$\begin{aligned} S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) &= \frac{2nd_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)}{\phi''(1)} \\ &= n\bar{\mathbf{g}}_n^T(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \mathbf{S}_{12}(\boldsymbol{\theta}_0) \mathbf{V}(\boldsymbol{\theta}_0) \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(1) \\ &= n\bar{\mathbf{g}}_n^T(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \mathbf{S}_{12}(\boldsymbol{\theta}_0) \mathbf{V}(\boldsymbol{\theta}_0) \mathbf{V}^{-1}(\boldsymbol{\theta}_0) \mathbf{V}(\boldsymbol{\theta}_0) \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}_0) + o_p(1) \\ &= \sqrt{n}(\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0)^T \mathbf{V}^{-1}(\boldsymbol{\theta}_0) \sqrt{n}(\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) + o_p(1) \\ &= \left( \sqrt{n} \mathbf{V}^{-1/2}(\boldsymbol{\theta}_0) (\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) \right)^T \sqrt{n} \mathbf{V}^{-1/2}(\boldsymbol{\theta}_0) (\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) + o_p(1). \end{aligned}$$

It is clear that

$$\sqrt{n} \mathbf{V}^{-1/2}(\boldsymbol{\theta}_0) (\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}, \mathbf{I}_p),$$

where  $\mathbf{I}_p$  is the  $p \times p$  identity matrix. Now, applying Lemma 3 of Ferguson (1996), we readily obtain the desired asymptotic distribution. ■

Based on the asymptotic null distribution presented in Theorem 2, we reject the null hypothesis in (22), with significance level  $\alpha$ , in favour of the alternative hypothesis, if  $S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) > \chi_{p,\alpha}^2$  (or if  $T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) > \chi_{p,\alpha}^2$ ), where  $\chi_{p,\alpha}^2$  is the  $(1 - \alpha)$ -th quantile of the chi-squared distribution with  $p$  degrees of freedom. In most cases, the power function of this test procedure cannot be derived explicitly. In the following theorem, we present an asymptotic result, which provides an approximation of the power of the empirical  $\phi$ -divergence test statistics described previously.

**Theorem 3** Under the assumption that  $\boldsymbol{\theta}^* \neq \boldsymbol{\theta}_0$  is the true parameter value

$$\frac{n^{1/2}}{\sqrt{s_{T_n^\phi}^T(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \mathbf{M}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) s_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)}} \left( \frac{\phi''(1) T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

where

$$s_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) = E_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}] E_{F_{\boldsymbol{\theta}^*}} \left[ \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \psi \left( \frac{E_{F_{\boldsymbol{\theta}^*}} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \right) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \right], \quad (28)$$

$\boldsymbol{\tau}$  is the solution of

$$E_{F_{\boldsymbol{\theta}^*}} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)] = \mathbf{0}_r,$$

$$\psi(x) = \phi(x) - x\phi'(x), \quad (29)$$

$$\begin{aligned} \mathbf{M}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) &= E_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] E_{F_{\boldsymbol{\theta}^*}} [\exp\{2\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] \\ &\quad \times E_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)], \end{aligned} \quad (30)$$

and

$$\mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) = \mathbb{E}_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}] \mathbb{E}_{F_{\boldsymbol{\theta}^*}} \left[ \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \phi \left( \frac{\mathbb{E} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \right) \right]. \quad (31)$$

**Proof.** We rename  $D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})) = d_\phi(\mathbf{u}, \mathbf{t}(\boldsymbol{\theta}))$  as a function of  $\mathbf{u}$  and  $\mathbf{t}(\boldsymbol{\theta})$ , i.e.

$$d_\phi(\mathbf{u}, \mathbf{t}(\boldsymbol{\theta})) = \left( \sum_{j=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})\} \right)^{-1} \sum_{i=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \phi \left( \frac{\sum_{j=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})\}}{n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}} \right),$$

and in particular for  $\boldsymbol{\theta} = \boldsymbol{\theta}_0$  and  $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}_{ETEL}$ ,  $D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = d_\phi(\mathbf{u}, \mathbf{t}_0)$  and  $D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL})) = d_\phi(\mathbf{u}, \hat{\mathbf{t}}_{ETEL})$ . Since  $\mathbf{t}_0 \xrightarrow[n \rightarrow \infty]{P} \boldsymbol{\tau}$ , we shall consider, on one hand, the first order Taylor expansion of  $d_\phi(\mathbf{u}, \mathbf{t}_0)$  around  $\mathbf{t}_0 = \boldsymbol{\tau}$

$$d_\phi(\mathbf{u}, \mathbf{t}_0) = d_\phi(\mathbf{u}, \boldsymbol{\tau}) + \left. \frac{\partial d_\phi(\mathbf{u}, \mathbf{t}_0)}{\partial \mathbf{t}_0^T} \right|_{\mathbf{t}_0 = \boldsymbol{\tau}} (\mathbf{t}_0 - \boldsymbol{\tau}) + o(\|\mathbf{t}_0 - \boldsymbol{\tau}\|),$$

where

$$\frac{\partial d_\phi(\mathbf{u}, \mathbf{t}(\boldsymbol{\theta}))}{\partial \mathbf{t}(\boldsymbol{\theta})} = \left( \sum_{j=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})\} \right)^{-1} \sum_{i=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \psi \left( \frac{\sum_{j=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})\}}{n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}} \right) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}),$$

and since  $\hat{\mathbf{t}}_{ETEL} \xrightarrow[n \rightarrow \infty]{P} \mathbf{0}_r$ , we shall consider, on the other hand, the first order Taylor expansion of  $d_\phi(\mathbf{u}, \hat{\mathbf{t}}_{ETEL})$  around  $\hat{\mathbf{t}}_{ETEL} = \mathbf{0}_r$

$$d_\phi(\mathbf{u}, \hat{\mathbf{t}}_{ETEL}) = o(\|\hat{\mathbf{t}}_{ETEL}\|).$$

Then,

$$d_\phi(\mathbf{u}, \mathbf{t}_0) - d_\phi(\mathbf{u}, \hat{\mathbf{t}}_{ETEL}) = d_\phi(\mathbf{u}, \boldsymbol{\tau}) + \mathbf{s}_{T_n^\phi}^T(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)(\mathbf{t}_0 - \boldsymbol{\tau}) + o(\|\mathbf{t}_0 - \boldsymbol{\tau}\|) + o(\|\hat{\mathbf{t}}_{ETEL}\|), \quad (32)$$

where  $\mathbf{s}_{T_n^\phi}$ , given by (28), is such that

$$\left. \frac{\partial d_\phi(\mathbf{u}, \mathbf{t}_0)}{\partial \mathbf{t}_0} \right|_{\mathbf{t}_0 = \boldsymbol{\tau}} \xrightarrow[n \rightarrow \infty]{P} \mathbf{s}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*).$$

Denoting

$$\mathbf{h}(\mathbf{t}(\boldsymbol{\theta})) = \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}),$$

the Taylor expansion of  $\mathbf{h}(\mathbf{t}_0)$  around  $\mathbf{t}_0 = \boldsymbol{\tau}$  is equal to

$$\mathbf{0}_r = \mathbf{h}(\boldsymbol{\tau}) + \left( \frac{\partial}{\partial \mathbf{t}_0^T} \mathbf{h}(\mathbf{t}_0) \Big|_{\mathbf{t}_0 = \boldsymbol{\tau}} \right) (\mathbf{t}_0 - \boldsymbol{\tau}) + o(\|\mathbf{t}_0 - \boldsymbol{\tau}\|_{1_r}),$$

where

$$\begin{aligned} \mathbf{h}(\boldsymbol{\tau}) &= \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0), \\ \frac{\partial}{\partial \mathbf{t}_0^T} \mathbf{h}(\mathbf{t}_0) \Big|_{\mathbf{t}_0 = \boldsymbol{\tau}} &= \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}_0) \\ &= \mathbb{E}_{F_{\boldsymbol{\theta}^*}} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] + o_p(\mathbf{1}_{r \times r}), \end{aligned}$$

and from it the following relation is obtained

$$\mathbf{t}_0 - \boldsymbol{\tau} = -\mathbf{E}_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] \left( \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \right) + o_p(\mathbf{1}_r).$$

We obtain in virtue of the Central Limit Theorem

$$\sqrt{n}(\mathbf{t}_0 - \boldsymbol{\tau}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_r, \mathbf{M}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)),$$

where  $\mathbf{M}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)$  is (30), since

$$\mathbf{E}_{F_{\boldsymbol{\theta}^*}}[\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)] = \mathbf{0}_r,$$

and

$$\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_r, \mathbf{E}_{F_{\boldsymbol{\theta}^*}}[\exp\{2\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)]),$$

On the other hand, since

$$d_\phi(\mathbf{u}, \boldsymbol{\tau}) = \left( \sum_{j=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}_0)\} \right)^{-1} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \phi \left( \frac{\sum_{j=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}_0)\}}{n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\}} \right),$$

it holds

$$d_\phi(\mathbf{u}, \mathbf{t}_0) \xrightarrow[n \rightarrow \infty]{P} \mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*),$$

where  $\mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)$  is (31). Hence, from (32) it follows

$$\sqrt{n} \left( \frac{d_\phi(\mathbf{u}, \mathbf{t}_0) - d_\phi(\mathbf{u}, \hat{\mathbf{t}}_{ETEL}) - \mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)}{\sqrt{\mathbf{s}_{T_n^\phi}^T(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \mathbf{M}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \mathbf{s}_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)}} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

which is equivalent to the enunciated result. ■

**Theorem 4** Under the assumption that  $\boldsymbol{\theta}^* \neq \boldsymbol{\theta}_0$  is the true parameter value

$$\frac{n^{1/2}}{\sqrt{\mathbf{s}_{S_n^\phi}^T(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \mathbf{M}_{S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \mathbf{s}_{S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)}} \left( \frac{\phi''(1) S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

where

$$\begin{aligned} \mathbf{s}_{S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) &= \begin{pmatrix} \mathbf{s}_{1, S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \\ \mathbf{s}_{2, S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \end{pmatrix}, \\ \mathbf{s}_{1, S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) &= -\mathbf{R}(\boldsymbol{\theta}^*) \mathbf{E}_{F_{\boldsymbol{\theta}^*}} \left[ \phi' \left( \frac{\mathbf{E}_{F_{\boldsymbol{\theta}^*}}[\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \right) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}^*) \right], \\ \mathbf{s}_{2, S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) &= -\mathbf{E}_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}] \mathbf{E}_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] \\ &\quad \times \mathbf{E}_{F_{\boldsymbol{\theta}^*}} \left[ \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \psi \left( \frac{\mathbf{E}_{F_{\boldsymbol{\theta}^*}}[\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \right) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \right], \end{aligned} \tag{33}$$

$$M_{S_n^\phi}(\theta_0, \theta^*) = \begin{pmatrix} \mathbf{S}_{11}(\theta^*) & \Sigma_{12}(\theta^*, \theta_0) \\ \Sigma_{12}^T(\theta^*, \theta_0) & \Sigma_{22}(\theta^*, \theta_0) \end{pmatrix}, \quad (34)$$

$$\Sigma_{12}(\theta_0, \theta^*) = \mathbb{E}_{F_{\theta^*}} [\exp\{\tau^T g(\mathbf{X}, \theta_0)\} g(\mathbf{X}, \theta^*) g^T(\mathbf{X}, \theta_0)],$$

$$\Sigma_{22}(\theta_0, \theta^*) = \mathbb{E}_{F_{\theta^*}} [\exp\{2\tau^T g(\mathbf{X}, \theta_0)\} g(\mathbf{X}, \theta_0) g^T(\mathbf{X}, \theta_0)].$$

$\tau$ ,  $\psi$  and  $\mu_\phi(\theta_0, \theta^*)$  as in Theorem 3.

**Proof.** Since  $\hat{\mathbf{t}}_{ETEL} \xrightarrow[n \rightarrow \infty]{P} \mathbf{0}_r$  and  $\mathbf{t}_0 \xrightarrow[n \rightarrow \infty]{P} \tau$ , we shall consider the first order Taylor expansion of  $d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)$  around  $(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0) = (\mathbf{0}_r, \tau)$ ,

$$\begin{aligned} d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0) &= d_\phi(\mathbf{0}_r, \tau) + \frac{\partial d_\phi(\hat{\mathbf{t}}_{ETEL}, \tau)}{\partial \hat{\mathbf{t}}_{ETEL}^T} \bigg|_{\hat{\mathbf{t}}_{ETEL} = \mathbf{0}_r} \hat{\mathbf{t}}_{ETEL} + \frac{\partial d_\phi(\mathbf{0}_r, \mathbf{t}_0)}{\partial \mathbf{t}_0^T} \bigg|_{\mathbf{t}_0 = \tau} (\mathbf{t}_0 - \tau) \\ &\quad + o(\|\hat{\mathbf{t}}_{ETEL}\|) + o(\|\mathbf{t}_0 - \tau\|), \end{aligned}$$

where

$$d_\phi(\mathbf{0}_r, \tau) = \left( \sum_{j=1}^n \exp\{\tau^T g(\mathbf{X}_j, \theta_0)\} \right)^{-1} \sum_{i=1}^n \exp\{\tau^T g(\mathbf{X}_i, \theta_0)\} \phi \left( \frac{\sum_{j=1}^n \exp\{\tau^T g(\mathbf{X}_j, \theta_0)\}}{n \exp\{\tau^T g(\mathbf{X}_i, \theta_0)\}} \right),$$

$$\begin{aligned} \frac{\partial d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)}{\partial \hat{\mathbf{t}}_{ETEL}} &= \left( \sum_{j=1}^n \exp\{\hat{\mathbf{t}}_{ETEL}^T g(\mathbf{X}_j, \hat{\theta}_{ETEL})\} \right)^{-1} \sum_{i=1}^n \exp\{\hat{\mathbf{t}}_{ETEL}^T g(\mathbf{X}_i, \hat{\theta}_{ETEL})\} \\ &\quad \times \phi' \left( \frac{\sum_{j=1}^n \exp\{\mathbf{t}_0^T g(\mathbf{X}_j, \theta_0)\}}{\exp\{\mathbf{t}_0^T g(\mathbf{X}_i, \theta_0)\}} \frac{\exp\{\hat{\mathbf{t}}_{ETEL}^T g(\mathbf{X}_i, \hat{\theta}_{ETEL})\}}{\sum_{j=1}^n \exp\{\hat{\mathbf{t}}_{ETEL}^T g(\mathbf{X}_j, \hat{\theta}_{ETEL})\}} \right) g(\mathbf{X}_i, \hat{\theta}_{ETEL}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)}{\partial \mathbf{t}_0} &= \left( \sum_{j=1}^n \exp\{\mathbf{t}_0^T g(\mathbf{X}_j, \theta_0)\} \right)^{-1} \sum_{i=1}^n \exp\{\mathbf{t}_0^T g(\mathbf{X}_i, \theta_0)\} \\ &\quad \times \psi \left( \frac{\sum_{j=1}^n \exp\{\mathbf{t}_0^T g(\mathbf{X}_j, \theta_0)\}}{\exp\{\mathbf{t}_0^T g(\mathbf{X}_i, \theta_0)\}} \frac{\exp\{\hat{\mathbf{t}}_{ETEL}^T g(\mathbf{X}_i, \hat{\theta}_{ETEL})\}}{\sum_{j=1}^n \exp\{\hat{\mathbf{t}}_{ETEL}^T g(\mathbf{X}_j, \hat{\theta}_{ETEL})\}} \right) g(\mathbf{X}_i, \theta_0), \end{aligned}$$

with  $\psi(x)$  given by (29). Then,

$$d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0) = \mu_\phi(\theta_0, \theta^*) + \bar{s}_{1, S_n^\phi}^T(\theta_0, \theta^*) \hat{\mathbf{t}}_{ETEL} + \bar{s}_{2, S_n^\phi}^T(\theta_0, \theta^*) (\mathbf{t}_0 - \tau) + o(\|\hat{\mathbf{t}}_{ETEL}\|) + o(\|\mathbf{t}_0 - \tau\|), \quad (35)$$

where

$$\bar{s}_{1, S_n^\phi}(\theta^*, \theta_0) = \mathbb{E}_{F_{\theta^*}} \left[ \phi' \left( \frac{\mathbb{E}_{F_{\theta^*}} [\exp\{\tau^T g(\mathbf{X}, \theta_0)\}]}{\exp\{\tau^T g(\mathbf{X}, \theta_0)\}} \right) g(\mathbf{X}, \theta^*) \right], \quad (36)$$

$$\bar{s}_{2, S_n^\phi}(\theta^*, \theta_0) = \mathbb{E}_{F_{\theta^*}}^{-1} [\exp\{\tau^T g(\mathbf{X}, \theta_0)\}] \mathbb{E}_{F_{\theta^*}} \left[ \exp\{\tau^T g(\mathbf{X}, \theta_0)\} \psi \left( \frac{\mathbb{E}_{F_{\theta^*}} [\exp\{\tau^T g(\mathbf{X}, \theta_0)\}]}{\exp\{\tau^T g(\mathbf{X}, \theta_0)\}} \right) g(\mathbf{X}, \theta_0) \right],$$

are such that

$$\begin{aligned}
d_\phi(\mathbf{0}_r, \boldsymbol{\tau}) &\xrightarrow[n \rightarrow \infty]{P} \mu_\phi(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*), \\
\left. \frac{\partial d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)}{\partial \hat{\mathbf{t}}_{ETEL}} \right|_{\hat{\mathbf{t}}_{ETEL} = \mathbf{0}_r} &\xrightarrow[n \rightarrow \infty]{P} \bar{\mathbf{s}}_{1, S_n^\phi}(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0), \\
\left. \frac{\partial d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0)}{\partial \mathbf{t}_0} \right|_{\mathbf{t}_0 = \boldsymbol{\tau}} &\xrightarrow[n \rightarrow \infty]{P} \bar{\mathbf{s}}_{2, S_n^\phi}(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0).
\end{aligned}$$

Denoting

$$\mathbf{h}(\mathbf{t}(\boldsymbol{\theta})) = \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}),$$

the Taylor expansion of  $\mathbf{h}(\mathbf{t}_0)$  around  $\mathbf{t}_0 = \boldsymbol{\tau}$  is equal to

$$\mathbf{0}_r = \mathbf{h}(\boldsymbol{\tau}) + \left( \frac{\partial}{\partial \mathbf{t}_0^T} \mathbf{h}(\mathbf{t}_0) \Big|_{\mathbf{t}_0 = \boldsymbol{\tau}} \right) (\mathbf{t}_0 - \boldsymbol{\tau}) + o(\|\mathbf{t}_0 - \boldsymbol{\tau}\| \mathbf{1}_r),$$

where

$$\begin{aligned}
\mathbf{h}(\boldsymbol{\tau}) &= \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0), \\
\frac{\partial}{\partial \mathbf{t}_0^T} \mathbf{h}(\mathbf{t}_0) \Big|_{\mathbf{t}_0 = \boldsymbol{\tau}} &= \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}_0) \\
&= \mathbb{E}_{F_{\boldsymbol{\theta}^*}} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] + o_p(\mathbf{1}_{r \times r}),
\end{aligned}$$

and from it the following relation is obtained

$$\mathbf{t}_0 - \boldsymbol{\tau} = -\mathbb{E}_{F_{\boldsymbol{\theta}^*}}^{-1} [\exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}_0)] \left( \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \right) + o_p(\mathbf{1}_r).$$

From (21) it follows

$$\hat{\mathbf{t}}_{ETEL} = -\mathbf{R} \bar{\mathbf{g}}_n(\mathbf{X}, \boldsymbol{\theta}^*) + o_p(n^{-1/2}),$$

and then

$$\begin{aligned}
&\bar{\mathbf{s}}_{1, S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) \hat{\mathbf{t}}_{ETEL} + \bar{\mathbf{s}}_{2, S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) (\mathbf{t}_0 - \boldsymbol{\tau}) \\
&= \mathbf{s}_{1, S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) \bar{\mathbf{g}}_n(\mathbf{X}, \hat{\boldsymbol{\theta}}_{ETEL}) + \mathbf{s}_{2, S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) \left( \frac{1}{n} \sum_{i=1}^n \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \right) \\
&= \frac{1}{n} \sum_{i=1}^n \mathbf{s}_{1, S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}_i, \hat{\boldsymbol{\theta}}_{ETEL}) + \mathbf{s}_{2, S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \\
&= \frac{1}{n} \sum_{i=1}^n \mathbf{s}_{S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0) \tilde{\mathbf{g}}(\mathbf{X}_i, \hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0),
\end{aligned}$$

where  $\mathbf{s}_{S_n^\phi}^T(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0)$  is (33),

$$\tilde{\mathbf{g}}(\mathbf{X}_i, \boldsymbol{\theta}^*, \boldsymbol{\theta}_0) = \begin{pmatrix} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}^*) \\ \exp\{\boldsymbol{\tau}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0) \end{pmatrix},$$

and taking into account that

$$\mathbb{E}_{F_{\theta^*}} \left[ \mathbf{s}_{S_n^\phi}^T(\theta^*, \theta_0) \tilde{\mathbf{g}}(\mathbf{X}, \hat{\theta}_{ETEL}, \theta_0) \right] = \mathbf{s}_{S_n^\phi}^T(\theta^*, \theta_0) \mathbb{E}_{F_{\theta^*}} \left[ \tilde{\mathbf{g}}(\mathbf{X}, \hat{\theta}_{ETEL}, \theta_0) \right] = 0,$$

we obtain in virtue of the Central Limit Theorem

$$\frac{\sqrt{n}}{\sqrt{\mathbf{s}_{S_n^\phi}^T(\theta^*, \theta_0) \text{Var}_{F_{\theta^*}} [\tilde{\mathbf{g}}(\mathbf{X}_i, \theta^*, \theta_0)] \mathbf{s}_{S_n^\phi}(\theta^*, \theta_0)}} \left( d_\phi(\hat{\mathbf{t}}_{ETEL}, \mathbf{t}_0) - \mu_\phi(\theta_0, \theta^*) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

which is equivalent to the theorems result. ■

Let

$$\begin{aligned} \beta_{T_n^\phi}(\theta^*) &= P(T_n^\phi(\hat{\theta}_{ETEL}, \theta_0) > \chi_{p, \alpha}^2 | \theta^*), \\ \beta_{S_n^\phi}(\theta^*) &= P(S_n^\phi(\hat{\theta}_{ETEL}, \theta_0) > \chi_{p, \alpha}^2 | \theta^*), \end{aligned}$$

be the exact power functions of  $T_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  and  $S_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  respectively, with respect to the asymptotic critical value of the test, at  $\theta^* \neq \theta_0$ , for a significance level  $\alpha$ . Notice that in practice, since the exact distributions of  $T_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  and  $S_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  are unknown,  $\beta_{T_n^\phi}^*(\theta^*)$  and  $\beta_{S_n^\phi}^*(\theta^*)$  are also unknown. The following result provides an approximation for  $\beta_{T_n^\phi}(\theta^*)$  and  $\beta_{S_n^\phi}(\theta^*)$ .

**Remark 5** From Theorem 3, we can present the approximation to the asymptotic power  $\beta_{T_n^\phi}(\theta^*)$ , at  $\theta^* \neq \theta_0$ , of the empirical  $\phi$ -divergence test  $T_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  for a significance level  $\alpha$ , as

$$\beta_{T_n^\phi}^*(\theta^*) = 1 - \Phi \left( \nu_{T_n^\phi}(\theta^*, \theta_0) \right) \simeq \beta_{T_n^\phi}(\theta), \quad (37)$$

where  $\Phi(\cdot)$  is the standard normal distribution function and

$$\nu_{T_n^\phi}(\theta^*, \theta_0) = \frac{n^{1/2}}{\sqrt{\mathbf{s}_{T_n^\phi}^T(\theta_0, \theta^*) \mathbf{M}_{T_n^\phi}(\theta_0, \theta^*) \mathbf{s}_{T_n^\phi}(\theta_0, \theta^*)}} \left( \frac{\phi''(1) \chi_{p, \alpha}^2}{2n} - \mu_\phi(\theta_0, \theta^*) \right).$$

If some alternative  $\theta^* \neq \theta_0$  is the true parameter, then the probability of rejecting (22) with the rejection rule  $T_n^\phi(\hat{\theta}_{ETEL}, \theta_0) > \chi_{p, \alpha}^2$ , for fixed significance level  $\alpha$ , tends to one as  $n \rightarrow \infty$ . Thus, the test is consistent in the sense of Fraser (1957). In a similar way, an approximation to the asymptotic power function  $\beta_{S_n^\phi}(\theta^*)$ , at  $\theta^* \neq \theta_0$ , for the empirical  $\phi$ -divergence test  $S_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  can be obtained as

$$\beta_{S_n^\phi}^*(\theta^*) = 1 - \Phi \left( \nu_{S_n^\phi}(\theta^*, \theta_0) \right), \quad (38)$$

where

$$\nu_{S_n^\phi}(\theta^*, \theta_0) = \frac{n^{1/2}}{\sqrt{\mathbf{s}_{S_n^\phi}^T(\theta_0, \theta^*) \mathbf{M}_{S_n^\phi}(\theta_0, \theta^*) \mathbf{s}_{S_n^\phi}(\theta_0, \theta^*)}} \left( \frac{\phi''(1) \chi_{p, \alpha}^2}{2n} - \mu_\phi(\theta_0, \theta^*) \right).$$

From the parametric statistical inference,  $\beta_{T_n^\phi}^*(\theta^*)$  and  $\beta_{S_n^\phi}^*(\theta^*)$  are known to be good approximations of  $\beta_{T_n^\phi}(\theta^*)$  and  $\beta_{S_n^\phi}(\theta^*)$  respectively (see for instance, Menéndez et al. (1998)). Notice that in practice, since  $F$  is unknown,  $\beta_{T_n^\phi}^*(\theta^*)$  and  $\beta_{S_n^\phi}^*(\theta^*)$  are also unknown. However, in practice  $\beta_{T_n^\phi}^*(\theta^*)$  and  $\beta_{S_n^\phi}^*(\theta^*)$  are consistently estimated, by replacing expectations by sample means.

To produce some less trivial asymptotic powers that are not all equal to 1, we can use a Pitman-type local analysis, as developed by Le Cam (1960), by confining attention to  $n^{1/2}$ -neighborhoods of the true parameter values. A key tool to get the asymptotic distribution of the statistic  $T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  (or  $S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ ) under such a contiguous hypothesis is Le Cam's third lemma, as presented in Hájek and Sidák (1967). Instead of relying on these results, we present in the following theorem a proof which is easy and direct to follow. This proof is based on the results of Morales and Pardo (2001). Specifically, we consider the power at contiguous alternative hypotheses of the form

$$H_{1,n} : \boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + n^{-1/2} \boldsymbol{\Delta}, \quad (39)$$

where  $\boldsymbol{\Delta}$  is a fixed vector in  $\mathbb{R}^p$  such that  $\boldsymbol{\theta}_n \in \Theta \subset \mathbb{R}^p$ .

**Theorem 6** *Under Condition 1 and  $H_{1,n}$  in (39), the asymptotic distribution of the empirical  $\phi$ -divergence test statistics  $S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  and  $T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  is a non-central chi-squared with  $p$  degrees of freedom and non-centrality parameter*

$$\delta(\boldsymbol{\theta}_0) = \boldsymbol{\Delta}^T \mathbf{V}^{-1}(\boldsymbol{\theta}_0) \boldsymbol{\Delta}. \quad (40)$$

i.e.

$$S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) \text{ (or } T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \chi_p^2(\delta(\boldsymbol{\theta}_0)),$$

where  $\mathbf{V}(\boldsymbol{\theta}_0)$  was defined in (20).

**Proof.** We can write

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) = \sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_n) + \sqrt{n}(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0) = \sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_n) + \boldsymbol{\Delta}.$$

Under  $H_{1,n}$ , we have

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_n) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}, \mathbf{V}(\boldsymbol{\theta}_0))$$

and

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\boldsymbol{\Delta}, \mathbf{V}(\boldsymbol{\theta}_0)).$$

In Theorem 2, it has been shown that

$$S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) = \left( \mathbf{V}(\boldsymbol{\theta}_0)^{-1/2} \sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) \right)^T \mathbf{V}(\boldsymbol{\theta}_0)^{-1/2} \sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) + o_p(1).$$

On the other hand, we have

$$\mathbf{V}(\boldsymbol{\theta}_0)^{-1/2} \sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{V}(\boldsymbol{\theta}_0)^{-1/2} \boldsymbol{\Delta}, \mathbf{I}_p).$$

We thus obtain

$$S_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \chi_p^2(\delta(\boldsymbol{\theta}_0)),$$

with  $\delta(\boldsymbol{\theta}_0)$  as in (40). A similar procedure can be followed for the proof of  $T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ . ■

## 4 Robustness of empirical $\phi$ -divergence test statistics

In Robust Statistics, two concepts of robustness can be distinguished, robustness with respect to contamination and robustness with respect to model misspecification. We shall understand misspecification in the sense that (2) is not verified for any  $\theta \in \Theta$ , in particular there is misspecification for the null hypothesis in (22) if

$$\|E_F[g(\mathbf{X}, \theta_0)]\| > 0.$$

For brevity, in the sequel  $E_F[\cdot]$  is denoted by  $E[\cdot]$ .

It is well-known (see Imbens et al. (1998)) that the estimating equation with respect to  $\theta$  for the EL and ET estimators are given by

$$\sum_{i=1}^n \rho_\ell(\mathbf{x}_i, \hat{\theta}_\ell, \mathbf{t}_\ell(\hat{\theta}_\ell)) = 0, \quad \ell \in \{EL, ET\},$$

with

$$\rho_{EL}(\mathbf{x}, \theta, \mathbf{t}_{EL}(\theta)) = \frac{\mathbf{t}_{EL}^T(\theta) \mathbf{G}_\mathbf{x}(\theta)}{1 + \mathbf{t}_{EL}^T(\theta) \mathbf{g}(\mathbf{x}, \theta)}, \quad (41)$$

$$\rho_{ET}(\mathbf{x}, \theta, \mathbf{t}_{ET}(\theta)) = \mathbf{t}_{ET}^T(\theta) \mathbf{G}_\mathbf{x}(\theta) \exp\{\mathbf{t}_{ET}^T(\theta) \mathbf{g}(\mathbf{x}, \theta)\}. \quad (42)$$

In relation to the ETEL estimators, from Theorem 2 of Schennach (2007) the following estimating equation with respect to  $\theta$  is obtained

$$\sum_{i=1}^n \rho_{ETEL}(\mathbf{x}_i, \hat{\theta}_{ETEL}, \mathbf{t}_{ET}(\hat{\theta}_{ETEL})) = 0,$$

with

$$\begin{aligned} \rho_{ETEL}(\mathbf{x}, \theta, \mathbf{t}_{ET}(\theta)) &= \mathbf{t}_{ET}^T(\theta) \mathbf{G}_\mathbf{x}(\theta) (\exp\{\mathbf{t}_{ET}^T(\theta) \mathbf{g}(\mathbf{x}, \theta)\} - \overline{\exp}_{ET}(\theta)), \\ \overline{\exp}_{ET}(\theta) &= \frac{1}{n} \sum_{j=1}^n \exp\{\mathbf{t}_{ET}^T(\theta) \mathbf{g}(\mathbf{x}_j, \theta)\}, \quad \mathbf{x} \in \{\mathbf{x}_j\}_{j=1}^n. \end{aligned} \quad (43)$$

The influence functions for the three types of estimators, EL, ET, ETEL, are proportional to the  $\rho_\ell(\mathbf{x}, \theta, \mathbf{t}_\ell(\theta))$  function, for  $\ell \in \{EL, ET, ETEL\}$ , respectively, given in (41)-(43),

$$\mathcal{IF}(\mathbf{x}, \hat{\theta}_\ell, F_{n,\theta}) \propto \rho_\ell(\mathbf{x}, \hat{\theta}_\ell, \mathbf{t}_\ell(\hat{\theta}_\ell)),$$

where  $\mathbf{t}_{ETEL}(\theta) = \mathbf{t}_{ET}(\theta)$ . Evaluating  $\rho_{EL}(\mathbf{x}, \hat{\theta}_{EL}, \mathbf{t}_{EL}(\hat{\theta}_{EL}))$  at perturbations of  $\mathbf{t}_{EL}(\hat{\theta}_{EL}) \neq \mathbf{0}_r$ , it can become unbounded even if  $\mathbf{g}(\mathbf{x}, \theta)$  is bounded, i.e. the influence function of  $\hat{\theta}_{EL}$  can be unbounded. This is in contrast with the influence function of  $\hat{\theta}_{ET}$  and  $\hat{\theta}_{ETEL}$ , since  $\rho_{ET}(\mathbf{x}, \hat{\theta}_{EL}, \mathbf{t}_{EL}(\hat{\theta}_{ET}))$  and  $\rho_{ETEL}(\mathbf{x}, \hat{\theta}_{EL}, \mathbf{t}_{EL}(\hat{\theta}_{ETEL}))$  are affected to a much less extent by perturbations of  $\mathbf{t}_{ET}(\hat{\theta}_\ell)$ ,  $\ell \in \{ET, ETEL\}$ , respectively. At the limiting values of the estimators,  $\hat{\theta}_\ell \xrightarrow[n \rightarrow \infty]{P} \theta_0$ ,  $\mathbf{t}_\ell(\hat{\theta}_\ell) \xrightarrow[n \rightarrow \infty]{P} \mathbf{0}_r$ , for  $\ell \in \{EL, ET, ETEL\}$ , respectively, the influence functions for the three types of estimators, are identical,

$$\mathcal{IF}(\mathbf{x}, \hat{\theta}_\ell, F_{n,\theta_0}) = \mathbf{V}(\theta_0) \mathbf{S}_{12}^T(\theta_0) \mathbf{S}_{11}^{-1}(\theta_0) \mathbf{g}(\mathbf{x}, \theta_0),$$

reflecting the first order equivalence of the estimators (for a detailed proof see Lemma 1 in Balakrishnan et al. (2015)).



Let  $\mathbf{T}(\bullet)$  be the functional associated the ETEL estimator of  $\boldsymbol{\theta}$ , i.e.

$$\mathbf{T}(F_{n,\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}_{ETEL}, \quad \mathbf{T}(F_{n,\boldsymbol{\theta}_0}) = \boldsymbol{\theta}_0,$$

and the test-statistic  $S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$ , given in (26), defined now through its functional

$$S_n^\phi(F_{n,\boldsymbol{\theta}}) = \frac{2n}{\phi''(1)} D_\phi(\mathbf{p}_{ET}(\mathbf{T}(F_{n,\boldsymbol{\theta}})), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = \frac{2n}{\phi''(1)} \sum_{i=1}^n p_{ET,i}(\boldsymbol{\theta}_0) \phi\left(\frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)}\right).$$

**Theorem 7** *The first and second order influence functions of  $S_n^\phi(F_{n,\boldsymbol{\theta}})$  are*

$$\mathcal{IF}(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}}) = \frac{\partial}{\partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\mathbf{T}(F_{n,\boldsymbol{\theta}})} \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}),$$

and

$$\begin{aligned} \mathcal{IF}_2(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}}) &= \frac{2n}{\phi''(1)} \mathcal{IF}^T(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}) \sum_{i=1}^n \phi''\left(\frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)}\right) \frac{1}{p_{ET,i}(\boldsymbol{\theta}_0)} \\ &\times \left(\frac{\partial}{\partial \mathbf{T}(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))\right) \left(\frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))\right) \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}) \\ &+ \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi'\left(\frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)}\right) \left(\frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))\right) \mathcal{IF}_2(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}). \end{aligned}$$

**Proof.** Let

$$F_{n,\varepsilon,\boldsymbol{\theta}} = (1 - \varepsilon)F_{n,\boldsymbol{\theta}} + \varepsilon\delta_{\mathbf{x}}, \quad \delta_{\mathbf{x}}(\mathbf{s}) = \begin{cases} 0, & \mathbf{s} < \mathbf{x}, \\ 1, & \mathbf{s} \geq \mathbf{x}. \end{cases},$$

the  $\varepsilon$ -perturbation of  $F_{n,\boldsymbol{\theta}}$  at  $\mathbf{x}$ . The first and second order influence functions of  $S_n^\phi(F_{n,\boldsymbol{\theta}})$  are defined as

$$\begin{aligned} \mathcal{IF}(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}}) &= \frac{\partial}{\partial \varepsilon} S_n^\phi(F_{n,\varepsilon,\boldsymbol{\theta}}) \Big|_{\varepsilon=0} \\ &= \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi'\left(\frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)}\right) \frac{\partial}{\partial \varepsilon} p_{ET,i}(\mathbf{T}(F_{n,\varepsilon,\boldsymbol{\theta}})) \Big|_{\varepsilon=0} \\ &= \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi'\left(\frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)}\right) \left(\frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))\right) \left(\frac{\partial}{\partial \varepsilon} \mathbf{T}(F_{n,\varepsilon,\boldsymbol{\theta}}) \Big|_{\varepsilon=0}\right) \\ &= \frac{\partial}{\partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\mathbf{T}(F_{n,\boldsymbol{\theta}})} \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}), \end{aligned}$$

and

$$\begin{aligned}
\mathcal{IF}_2(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}}) &= \frac{\partial^2}{\partial \varepsilon^2} S_n^\phi(F_{n,\varepsilon,\boldsymbol{\theta}}) \Big|_{\varepsilon=0} \\
&= \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi'' \left( \frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \frac{\left( \frac{\partial}{\partial \varepsilon} p_{ET,i}(\mathbf{T}(F_{n,\varepsilon,\boldsymbol{\theta}})) \Big|_{\varepsilon=0} \right)^2}{p_{ET,i}(\boldsymbol{\theta}_0)} \\
&\quad + \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi' \left( \frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \frac{\partial^2}{\partial \varepsilon^2} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \Big|_{\varepsilon=0} \\
&= \frac{2n}{\phi''(1)} \sum_{i=1}^n \frac{1}{p_{ET,i}(\boldsymbol{\theta}_0)} \phi'' \left( \frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \left( \frac{\partial}{\partial \varepsilon} \mathbf{T}^T(F_{n,\varepsilon,\boldsymbol{\theta}}) \Big|_{\varepsilon=0} \right) \\
&\quad \times \left( \frac{\partial}{\partial \mathbf{T}(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \right) \left( \frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \right) \left( \frac{\partial}{\partial \varepsilon} \mathbf{T}(F_{n,\varepsilon,\boldsymbol{\theta}}) \Big|_{\varepsilon=0} \right) \\
&\quad + \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi' \left( \frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \left( \frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \right) \left( \frac{\partial^2}{\partial \varepsilon^2} \mathbf{T}(F_{n,\varepsilon,\boldsymbol{\theta}}) \Big|_{\varepsilon=0} \right) \\
&= \frac{2n}{\phi''(1)} \mathcal{IF}^T(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}) \sum_{i=1}^n \phi'' \left( \frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \frac{1}{p_{ET,i}(\boldsymbol{\theta}_0)} \\
&\quad \times \left( \frac{\partial}{\partial \mathbf{T}(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \right) \left( \frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \right) \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}) \\
&\quad + \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi' \left( \frac{p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}}))}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \left( \frac{\partial}{\partial \mathbf{T}^T(F_{n,\boldsymbol{\theta}})} p_{ET,i}(\mathbf{T}(F_{n,\boldsymbol{\theta}})) \right) \mathcal{IF}_2(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}}),
\end{aligned}$$

■

**Corollary 8** Under the null hypothesis of the test (22), the first and second order influence functions of the test-statistic  $S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  are given by

$$\begin{aligned}
\mathcal{IF}(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}_0}) &= \frac{\partial}{\partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) = 0, \\
\mathcal{IF}_2(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}_0}) &= \mathcal{IF}^T(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}).
\end{aligned}$$

In particular, for large samples

$$\begin{aligned}
\mathcal{IF}_2(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}_0}) &= \mathcal{IF}^T(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \mathbf{V}^{-1}(\boldsymbol{\theta}_0) \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \\
&= \mathbf{g}^T(\mathbf{x}, \boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \mathbf{S}_{12}(\boldsymbol{\theta}_0) \mathbf{V}(\boldsymbol{\theta}_0) \mathbf{S}_{12}^T(\boldsymbol{\theta}_0) \mathbf{S}_{11}^{-1}(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}_0).
\end{aligned} \tag{44}$$

**Proof.** Both equalities are obtained taking into account

$$\frac{\partial}{\partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \frac{2n\phi'(1)}{\phi''(1)} \sum_{i=1}^n \frac{\partial}{\partial \boldsymbol{\theta}^T} p_{ET,i}(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\mathbf{T}(F_{n,\boldsymbol{\theta}_0})=\boldsymbol{\theta}_0} = \mathbf{0}_p^T,$$

since  $\phi'(1) = 0$ , and

$$\begin{aligned}
&\mathcal{IF}^T(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\mathbf{T}(F_{n,\boldsymbol{\theta}_0})=\boldsymbol{\theta}_0} \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \\
&= \mathcal{IF}^T(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi''(1) \frac{1}{p_{ET,i}(\boldsymbol{\theta}_0)} \frac{\partial}{\partial \boldsymbol{\theta}} p_{ET,i}(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \frac{\partial}{\partial \boldsymbol{\theta}^T} p_{ET,i}(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \mathcal{IF}(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}) \\
&\quad + \frac{2n}{\phi''(1)} \sum_{i=1}^n \phi'(1) \frac{\partial}{\partial \boldsymbol{\theta}^T} p_{ET,i}(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \mathcal{IF}_2(\mathbf{x}, \hat{\boldsymbol{\theta}}_{ETEL}, F_{n,\boldsymbol{\theta}_0}).
\end{aligned}$$

Since

$$\begin{aligned} \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} &= 2n \sum_{i=1}^n \frac{\partial}{\partial \boldsymbol{\theta}} \log p_{ET,i}(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} p_{ET,i}(\boldsymbol{\theta}_0) \frac{\partial}{\partial \boldsymbol{\theta}^T} \log p_{ET,i}(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \\ &= 2\mathbf{V}^{-1}(\boldsymbol{\theta}_0) + o_p(1), \end{aligned}$$

an alternative expression for the second order influence function, for large sample sizes, is (44). ■

Notice that  $\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} S_n^\phi(F_{n,\boldsymbol{\theta}}) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$  is the same for any  $\phi$  function and plugging any estimator into  $S_n^\phi$ , either EL, ET or ETEL,  $\mathcal{IF}_2(\mathbf{x}, S_n^\phi, F_{n,\boldsymbol{\theta}_0})$  remains unchanged.

A similar results of Theorem 7 and Corollary 8 can be enuntiated for the other family of test-statistics,  $T_n^\phi(F_{n,\boldsymbol{\theta}})$ .

Let  $\boldsymbol{\theta}_{*,ETEL}$  denote the ETEL's pseudo-true value associated with the misspecified model, i.e.

$$\begin{aligned} \boldsymbol{\theta}_{*,ETEL} &= \arg \min \log E \left[ \exp \left\{ \mathbf{t}^T(\boldsymbol{\theta}) (g(\mathbf{X}, \boldsymbol{\theta}) - E[g(\mathbf{X}, \boldsymbol{\theta})]) \right\} \right], \\ \text{s.t. } E \left[ \exp \left\{ \mathbf{t}^T(\boldsymbol{\theta}) g(\mathbf{X}, \boldsymbol{\theta}) \right\} g(\mathbf{X}, \boldsymbol{\theta}) \right] &= \mathbf{0}_r. \end{aligned}$$

The ETEL's pseudo-true value can be interpreted as the best approximation to the true value, according to the ETEL's estimation method.

**Condition 9** We shall assume the following regularity conditions (Schennach, 2007):

- i) There exists a neighborhood of  $\boldsymbol{\theta}_{*,ETEL}$  in which  $\frac{\partial \mathbf{G}_\mathbf{X}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$  is continuous and  $\left\| \frac{\partial \mathbf{G}_\mathbf{X}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\|$  is bounded by some integrable function of  $\mathbf{X}$ ;
- ii)  $E \left[ \sup_{\boldsymbol{\theta} \in \Theta} \exp \left\{ \mathbf{t}^T(\boldsymbol{\theta}) g(\mathbf{X}, \boldsymbol{\theta}) \right\} \right] < \infty$  s.t.  $E \left[ \exp \left\{ \mathbf{t}^T(\boldsymbol{\theta}) g(\mathbf{X}, \boldsymbol{\theta}) \right\} g(\mathbf{X}, \boldsymbol{\theta}) \right] = \mathbf{0}_r$ ;
- iii) There exists a function of  $\mathbf{X}$ ,  $f(\mathbf{X})$ , such that  $\|\mathbf{G}_\mathbf{X}(\boldsymbol{\theta})\|, \left\| \frac{\partial \mathbf{G}_\mathbf{X}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\|$  are bounded by  $f(\mathbf{X})$  and  $E \left[ \sup_{\boldsymbol{\theta} \in \Theta} \exp \left\{ k_1 \mathbf{t}^T(\boldsymbol{\theta}) g(\mathbf{X}, \boldsymbol{\theta}) \right\} f^{k_2}(\mathbf{X}) \right] < \infty$ ,  $k_2 = 1, 2, k_2 = 0, 1, 2, 3, 4$ , s.t.  $E \left[ \exp \left\{ \mathbf{t}^T(\boldsymbol{\theta}) g(\mathbf{X}, \boldsymbol{\theta}) \right\} g(\mathbf{X}, \boldsymbol{\theta}) \right] = \mathbf{0}_r$ .

The ETEL estimator of  $\boldsymbol{\theta}_{*,ETEL}$ ,  $\hat{\boldsymbol{\theta}}_{ETEL}$ , associated with the misspecified model, is obtained in the same manner done for the true model, in fact in practice it is not possible to know when the model is misspecified. By following Lemma 9 of Schennach (2007), it is convenient to study, apart from the vector of parameters of interest  $\boldsymbol{\theta}$  and the Lagrange multipliers vector  $\mathbf{t}$ , two additional auxiliary variables  $\boldsymbol{\kappa} \in \mathbb{R}^r$  and  $\tau \in \mathbb{R}$  in a joint vector

$$\boldsymbol{\beta} = (\boldsymbol{\theta}^T, \mathbf{t}^T, \boldsymbol{\kappa}^T, \tau)^T.$$

According to Theorem 10 of Schennach (2007), by calculating first the asymptotic distribution of  $\hat{\boldsymbol{\beta}}_{ETEL} = (\hat{\boldsymbol{\theta}}_{ETEL}^T, \hat{\mathbf{t}}_{ETEL}^T, \hat{\boldsymbol{\kappa}}_{ETEL}^T, \hat{\tau}_{ETEL})^T$ , and subtracting thereafter the marginal distribution of  $\hat{\boldsymbol{\theta}}_{ETEL}$ , the procedure to calculate the asymptotic distribution of  $\hat{\boldsymbol{\theta}}_{ETEL}^T$  is simplified, under misspecification. The following auxiliary function

$$\boldsymbol{\varphi}(\mathbf{X}, \boldsymbol{\beta}) = (\boldsymbol{\varphi}_1^T(\mathbf{X}, \boldsymbol{\beta}), \boldsymbol{\varphi}_2^T(\mathbf{X}, \boldsymbol{\beta}), \boldsymbol{\varphi}_3^T(\mathbf{X}, \boldsymbol{\beta}), \boldsymbol{\varphi}_4(\mathbf{X}, \boldsymbol{\beta}))^T,$$

with

$$\begin{aligned}
\varphi_1(\mathbf{X}, \beta) &= \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \theta)\} \mathbf{G}_{\mathbf{X}}^T(\theta) (\boldsymbol{\kappa} + \mathbf{t} \mathbf{g}^T(\mathbf{X}, \theta) \boldsymbol{\kappa} - \mathbf{t}) + \tau \mathbf{G}_{\mathbf{X}}^T(\theta) \mathbf{t}, \\
\varphi_2(\mathbf{X}, \beta) &= (\tau - \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \theta)\}) \mathbf{g}(\mathbf{X}, \theta) + \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \theta)\} \mathbf{g}(\mathbf{X}, \theta) \mathbf{g}^T(\mathbf{X}, \theta) \boldsymbol{\kappa}, \\
\varphi_3(\mathbf{X}, \beta) &= \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \theta)\} \mathbf{g}(\mathbf{X}, \theta), \\
\varphi_4(\mathbf{X}, \beta) &= \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \theta)\} - \tau,
\end{aligned}$$

defines  $\widehat{\boldsymbol{\beta}}_{ETEL}$ , as the solution of  $\frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{X}_i, \beta) = \mathbf{0}_{p+2r+1}$ , and the pseudo-true value

$$\boldsymbol{\beta}_{*,ETEL} = (\boldsymbol{\theta}_{*,ETEL}^T, \mathbf{t}_{*,ETEL}^T, \boldsymbol{\kappa}_{*,ETEL}^T, \tau_{*,ETEL})^T,$$

as the solution of  $\mathbb{E}[\varphi(\mathbf{X}, \beta)] = \mathbf{0}_{p+2r+1}$ . Under Condition 9, the asymptotic distribution of  $\widehat{\boldsymbol{\beta}}_{ETEL}$  is given by

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_{ETEL} - \boldsymbol{\beta}_{*,ETEL}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}\left(\mathbf{0}_{p+2r+1}, \Gamma^{-1}(\boldsymbol{\beta}_{*,ETEL}) \Phi(\boldsymbol{\beta}_{*,ETEL}) (\Gamma^{-1}(\boldsymbol{\beta}_{*,ETEL}))^T\right),$$

with

$$\begin{aligned}
\Gamma(\boldsymbol{\beta}_{*,ETEL}) &= \mathbb{E} \left[ \frac{\partial}{\partial \boldsymbol{\beta}} \varphi(\mathbf{X}, \beta) |_{\boldsymbol{\beta}=\boldsymbol{\beta}_{*,ETEL}} \right], \\
\Phi(\boldsymbol{\beta}_{*,ETEL}) &= \mathbb{E} [\varphi(\mathbf{X}, \boldsymbol{\beta}_{*,ETEL}) \varphi^T(\mathbf{X}, \boldsymbol{\beta}_{*,ETEL})],
\end{aligned}$$

assuming that  $\Gamma(\boldsymbol{\beta}_{*,ETEL})$  is nonsingular. Based on this result,

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL}) \xrightarrow[n \rightarrow \infty]{} \mathcal{N}\left(\mathbf{0}_p, \boldsymbol{\Sigma}_{\sqrt{n}\widehat{\boldsymbol{\theta}}_{ETEL}}\right), \quad (45)$$

with

$$\boldsymbol{\Sigma}_{\sqrt{n}\widehat{\boldsymbol{\theta}}_{ETEL}} = \begin{pmatrix} \mathbf{I}_p & \mathbf{0}_{(2r+1) \times (2r+1)} \end{pmatrix} \Gamma^{-1}(\boldsymbol{\beta}_{*,ETEL}) \Phi(\boldsymbol{\beta}_{*,ETEL}) (\Gamma^{-1}(\boldsymbol{\beta}_{*,ETEL}))^T \begin{pmatrix} \mathbf{I}_p \\ \mathbf{0}_{(2r+1) \times (2r+1)} \end{pmatrix}. \quad (46)$$

**Lemma 10** *The first derivative of (12) is given by*

$$\frac{\partial}{\partial \boldsymbol{\theta}} p_{ET,i}(\boldsymbol{\theta}) = p_{ET,i}(\boldsymbol{\theta}) \left[ \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) - \overline{\text{exp}}_{ET}^{-1}(\boldsymbol{\theta}) \overline{\text{exp}_{ET} \mathbf{G}^T(\boldsymbol{\theta})} \mathbf{t}_{ET}(\boldsymbol{\theta}) - \widehat{\mathbf{K}}(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \right],$$

where  $\overline{\text{exp}}_{ET}(\boldsymbol{\theta})$  was defined in (43),

$$\begin{aligned}
\overline{\text{exp}_{ET} \mathbf{G}^T(\boldsymbol{\theta})} &= \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}), \\
\widehat{\mathbf{K}}(\boldsymbol{\theta}) &= \left( \overline{\text{exp}_{ET} \mathbf{G}^T \mathbf{t}_{ET} \mathbf{g}^T(\boldsymbol{\theta})} + \overline{\text{exp}_{ET} \mathbf{G}^T(\boldsymbol{\theta})} \right) \overline{\text{exp}_{ET} \mathbf{g} \mathbf{g}^T}^{-1}(\boldsymbol{\theta}), \\
\overline{\text{exp}_{ET} \mathbf{G}^T \mathbf{t}_{ET} \mathbf{g}^T(\boldsymbol{\theta})} &= \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}), \\
\overline{\text{exp}_{ET} \mathbf{g} \mathbf{g}^T(\boldsymbol{\theta})} &= \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}).
\end{aligned}$$

(For the proof see Appendix)

**Lemma 11** *The first derivative of  $D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}))$  is given by*

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})) &= \overline{\text{exp}}_{ET}^{-1}(\boldsymbol{\theta}) \left[ \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \psi \left( \frac{\overline{\text{exp}}_{ET}(\boldsymbol{\theta})}{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}} \right) \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) \right. \\ &\quad - \overline{\text{exp}}_{ET}^{-1}(\boldsymbol{\theta}) \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \psi \left( \frac{\overline{\text{exp}}_{ET}(\boldsymbol{\theta})}{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}} \right) \overline{\text{exp}}_{ET} \mathbf{G}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) \\ &\quad \left. - \widehat{\mathbf{K}}(\boldsymbol{\theta}) \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \psi \left( \frac{\overline{\text{exp}}_{ET}(\boldsymbol{\theta})}{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}} \right) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \right], \end{aligned} \quad (47)$$

and

$$\frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})) \xrightarrow[n \rightarrow \infty]{P} \mathbf{r}_{T_n^\phi}(\boldsymbol{\theta}),$$

with  $\psi(x)$  given by (29),

$$\mathbf{r}_{T_n^\phi}(\boldsymbol{\theta}) = \mathbf{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \right] \{ \mathbf{r}_1(\boldsymbol{\theta}) - \mathbf{r}_2(\boldsymbol{\theta}) - \mathbf{r}_3(\boldsymbol{\theta}) \}, \quad (48)$$

$$\begin{aligned} \mathbf{r}_1(\boldsymbol{\theta}) &= \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \psi \left( \frac{\mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}} \right) \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}) \right] \bar{\mathbf{t}}_{ET}(\boldsymbol{\theta}), \\ \mathbf{r}_2(\boldsymbol{\theta}) &= \mathbf{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \right] \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \psi \left( \frac{\mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}} \right) \right] \\ &\quad \times \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}) \right] \bar{\mathbf{t}}_{ET}(\boldsymbol{\theta}), \\ \mathbf{r}_3(\boldsymbol{\theta}) &= \mathbf{K}(\boldsymbol{\theta}) \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \psi \left( \frac{\mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}} \right) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}) \right], \\ \mathbf{K}(\boldsymbol{\theta}) &= \left\{ \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}) \bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}) \right] + \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}) \right] \right\} \\ &\quad \times \mathbf{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}, \boldsymbol{\theta}) \right], \end{aligned} \quad (49)$$

$\bar{\mathbf{t}}_{ET}(\boldsymbol{\theta})$  is the solution in  $\mathbf{t}$  of  $\mathbf{E} \left[ \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}) \right] = \mathbf{0}_r$ .

Let  $\widehat{\mathbf{S}}_{12}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^n \mathbf{G}_{\mathbf{X}_i}(\boldsymbol{\theta})$  be a consistent estimator of  $\mathbf{S}_{12}(\boldsymbol{\theta})$  given in (18). It is interesting that according to formula (42) of Schennach (2007),

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\theta}} D_{\text{Kull}}(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})) &= -\frac{\partial}{\partial \boldsymbol{\theta}} \ell_{ETEL}(\boldsymbol{\theta}) \\ &= \widehat{\mathbf{K}}(\boldsymbol{\theta}) \left( \frac{1}{n} \sum_{i=1}^n \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \right) + \left( \sum_{i=1}^n p_{ET,i}(\boldsymbol{\theta}) \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) - \widehat{\mathbf{S}}_{12}^T(\boldsymbol{\theta}) \right) \mathbf{t}_{ET}(\boldsymbol{\theta}) \\ &= \widehat{\mathbf{K}}(\boldsymbol{\theta}) \left( \frac{1}{n} \sum_{i=1}^n \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \right) + \left[ \overline{\text{exp}}_{ET}^{-1}(\boldsymbol{\theta}) \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \right) - \widehat{\mathbf{S}}_{12}^T(\boldsymbol{\theta}) \right] \mathbf{t}_{ET}(\boldsymbol{\theta}), \end{aligned}$$

which matches (47) with  $\phi(x) = x \log x - x + 1$ .

(For the proof see Appendix)

**Lemma 12** *The first derivative of  $D_\phi(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0))$  is given by*

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) &= \overline{\text{exp}}_{ET}^{-1}(\boldsymbol{\theta}) \frac{1}{n} \sum_{i=1}^n \phi' \left( \frac{p_{ET,i}(\boldsymbol{\theta})}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \overline{\text{exp}}_{ET} \mathbf{G}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) \\ &+ \widehat{\mathbf{K}}(\boldsymbol{\theta}) \frac{1}{n} \sum_{i=1}^n \phi' \left( \frac{p_{ET,i}(\boldsymbol{\theta})}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) - \frac{1}{n} \sum_{i=1}^n \phi' \left( \frac{p_{ET,i}(\boldsymbol{\theta})}{p_{ET,i}(\boldsymbol{\theta}_0)} \right) \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}), \end{aligned} \quad (50)$$

where

$$\frac{p_{ET,i}(\boldsymbol{\theta})}{p_{ET,i}(\boldsymbol{\theta}_0)} = \frac{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}}{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}_0)\}} \frac{\overline{\text{exp}}_{ET}(\boldsymbol{\theta}_0)}{\overline{\text{exp}}_{ET}(\boldsymbol{\theta})},$$

and

$$\frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) \xrightarrow[n \rightarrow \infty]{P} \mathbf{q}_{S_n^\phi}(\boldsymbol{\theta}, \boldsymbol{\theta}_0),$$

with

$$\mathbf{q}_{S_n^\phi}(\boldsymbol{\theta}, \boldsymbol{\theta}_0) = \mathbf{q}_1(\boldsymbol{\theta}, \boldsymbol{\theta}_0) + \mathbf{q}_2(\boldsymbol{\theta}, \boldsymbol{\theta}_0) - \mathbf{q}_3(\boldsymbol{\theta}, \boldsymbol{\theta}_0), \quad (51)$$

$$\begin{aligned} \mathbf{q}_1(\boldsymbol{\theta}, \boldsymbol{\theta}_0) &= \mathbf{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \right] \mathbf{E} \left[ \phi' \left( \frac{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \frac{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}]} \right) \right] \\ &\times \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}) \right] \bar{\mathbf{t}}_{ET}(\boldsymbol{\theta}), \\ \mathbf{q}_2(\boldsymbol{\theta}, \boldsymbol{\theta}_0) &= \mathbf{K}(\boldsymbol{\theta}) \mathbf{E} \left[ \phi' \left( \frac{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \frac{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}]} \right) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}) \right], \\ \mathbf{q}_3(\boldsymbol{\theta}, \boldsymbol{\theta}_0) &= \mathbf{E} \left[ \phi' \left( \frac{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \frac{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\}]} \right) \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}) \right] \bar{\mathbf{t}}_{ET}(\boldsymbol{\theta}), \end{aligned}$$

$\bar{\mathbf{t}}_{ET}(\boldsymbol{\theta})$  is the solution in  $\mathbf{t}$  of  $\mathbf{E} [\exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})] = \mathbf{0}_r$ .

The following two theorems evaluate the effect of a misspecified alternative hypothesis on the asymptotic distribution of the empirical  $\phi$ -divergence test-statistics.

**Theorem 13** *Under the assumption that the pseudo-true parameter value  $\boldsymbol{\theta}_{*,ETEL}$  is different from  $\boldsymbol{\theta}_0$*

$$\frac{n^{1/2}}{\sqrt{\mathbf{r}_{T_n^\phi}^T(\boldsymbol{\theta}_{*,ETEL}) \boldsymbol{\Sigma}_{\sqrt{n}\widehat{\boldsymbol{\theta}}_{ETEL}} \mathbf{r}_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL})}} \left( \frac{\phi''(1) T_n^\phi(\widehat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

where  $\boldsymbol{\Sigma}_{\sqrt{n}\widehat{\boldsymbol{\theta}}_{ETEL}}$  is given by (46),  $\mathbf{r}_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL})$  by (48) and

$$\begin{aligned} \mu_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) &= \mathbf{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right] \\ &\times \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \phi \left( \frac{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}} \right) \right] \\ &- \mathbf{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \right] \\ &\times \mathbf{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \phi \left( \frac{\mathbf{E} [\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \right) \right], \end{aligned}$$

with  $\bar{\mathbf{t}}_{ET}(\boldsymbol{\theta})$  being the solution in  $\mathbf{t}$  of  $\mathbf{E} [\exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})] = \mathbf{0}_r$ .

**Proof.** The first order Taylor expansion of  $D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}))$  around  $\boldsymbol{\theta}_{*,ETEL}$  is

$$D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})) = D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_{*,ETEL})) + \frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}))|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{*,ETEL}} (\boldsymbol{\theta} - \boldsymbol{\theta}_{*,ETEL}) + o(\|\boldsymbol{\theta} - \boldsymbol{\theta}_{*,ETEL}\|).$$

In particular, for  $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}_{ETEL}$

$$\begin{aligned} D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL})) &= D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_{*,ETEL})) + \frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}))|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{*,ETEL}} (\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL}) \\ &\quad + o(\|\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL}\|). \end{aligned}$$

According to Theorem 11  $\frac{\partial}{\partial \boldsymbol{\theta}} D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}))$  converges in probability to a fixed vector, and so

$$D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL})) = D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_{*,ETEL})) + \mathbf{r}_{T_n^\phi}^T(\boldsymbol{\theta}_{*,ETEL}) (\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL}) + o(\|\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL}\|).$$

From (45) it holds  $\sqrt{n} o(\|\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL}\|) = o_p(1)$ . Thus, the random variables

$$\sqrt{n} \left( D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL})) - D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_{*,ETEL})) \right) \text{ and } \mathbf{r}_{T_n^\phi}^T(\boldsymbol{\theta}^*) \sqrt{n} (\hat{\boldsymbol{\theta}}_{ETEL} - \boldsymbol{\theta}_{*,ETEL})$$

have the same asymptotic distribution, and since

$$\begin{aligned} \frac{\phi''(1)}{2\sqrt{n}} \left( T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0) - T_n^\phi(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) \right) &= \sqrt{n} \left( D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\hat{\boldsymbol{\theta}}_{ETEL})) - D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}_{*,ETEL})) \right), \\ E_{F_{\boldsymbol{\theta}_{*,ETEL}}} \left[ \frac{\phi''(1) T_n^\phi(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0)}{2n} \right] &= \mu_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}), \end{aligned}$$

the desired result is obtained. ■

**Theorem 14** Under the assumption that the pseudo-true parameter value  $\boldsymbol{\theta}_{*,ETEL}$  is different from  $\boldsymbol{\theta}_0$

$$\frac{n^{1/2}}{\sqrt{\mathbf{q}_{S_n^\phi}^T(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) \boldsymbol{\Sigma}_{\sqrt{n}\hat{\boldsymbol{\theta}}_{ETEL}} \mathbf{q}_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0)}} \left( \frac{\phi''(1) S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_{S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

where  $\boldsymbol{\Sigma}_{\sqrt{n}\hat{\boldsymbol{\theta}}_{ETEL}}$  is given by (46),  $\mathbf{q}_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0)$  by (51) and

$$\begin{aligned} \mu_{S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) &= \mathbb{E}^{-1} \left[ \exp\{\tilde{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \right] \\ &\times \mathbb{E} \left[ \exp\{\tilde{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \phi \left( \frac{\exp\{\tilde{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}}{\exp\{\tilde{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\}} \frac{\mathbb{E} \left[ \exp\{\tilde{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_0)\} \right]}{\mathbb{E} \left[ \exp\{\tilde{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]} \right) \right]. \end{aligned}$$

with  $\tilde{\mathbf{t}}_{ET}(\boldsymbol{\theta})$  being the solution in  $\mathbf{t}$  of  $\mathbb{E} \left[ \exp\{\mathbf{t}^T \mathbf{g}(\mathbf{X}, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}, \boldsymbol{\theta}) \right] = \mathbf{0}_r$ .

**Proof.** It is omitted since similar steps of the proof for Theorem 13 are needed. ■

**Corollary 15** Under the assumption that the pseudo-true parameter value  $\boldsymbol{\theta}_{*,ETEL}$  is different from  $\boldsymbol{\theta}_0$ , the asymptotic distribution of the likelihood ratio test-statistics is given by

$$\frac{n^{1/2}}{\sqrt{\mathbf{r}_{G^2}^T(\boldsymbol{\theta}_{*,ETEL}) \boldsymbol{\Sigma}_{\sqrt{n}\hat{\boldsymbol{\theta}}_{ETEL}} \mathbf{r}_{G^2}(\boldsymbol{\theta}_{*,ETEL})}} \left( \frac{\phi''(1) G_n^2(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_{G^2}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1),$$

where

$$\mathbf{r}_{G^2}(\boldsymbol{\theta}_{*,ETEL}) = \mathbf{r}_{G,2}(\boldsymbol{\theta}_{*,ETEL}) - \mathbf{r}_{G,3}(\boldsymbol{\theta}_{*,ETEL}), \quad (52)$$

with

$$\begin{aligned} \mathbf{r}_{G,2}(\boldsymbol{\theta}_{*,ETEL}) &= \mathbf{K}(\boldsymbol{\theta}_{*,ETEL}) \mathbb{E} [g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})], \\ \mathbf{r}_{G,3}(\boldsymbol{\theta}_{*,ETEL}) &= \left\{ \mathbb{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right] \right. \\ &\quad \times \mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}_{*,ETEL}) \right] - \mathbf{S}_{12}^T(\boldsymbol{\theta}_{*,ETEL}) \left. \right\} \bar{\mathbf{t}}_{ET}(\boldsymbol{\theta}_{*,ETEL}), \end{aligned}$$

$\mathbf{K}(\boldsymbol{\theta}_{*,ETEL})$  is given by (49) and

$$\begin{aligned} \mu_{G^2}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) &= \log \frac{E_{F_{\boldsymbol{\theta}_{*,ETEL}}} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]}{E_{F_{\boldsymbol{\theta}_0,ETEL}} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) g(\mathbf{X}, \boldsymbol{\theta}_0)\} \right]} \\ &\quad - E_{F_{\boldsymbol{\theta}_{*,ETEL}}} \left[ \bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL}) - \bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_0) g(\mathbf{X}, \boldsymbol{\theta}_0) \right]. \end{aligned} \quad (53)$$

**Proof.** With  $\phi(x) = x \log x - x + 1$  plugged into (29)

$$\psi \left( \frac{\mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}} \right) = \frac{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}}{\mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]} - 1,$$

is obtained, and then according to Theorem 13, plugging

$$\begin{aligned} &\mathbb{E} \left[ \psi \left( \frac{\mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}} \right) \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}_{*,ETEL}) \right] \\ &= \mathbb{E}^{-1} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \mathbf{G}_{\mathbf{X}}^T(\boldsymbol{\theta}_{*,ETEL}) \right] - \mathbf{S}_{12}^T(\boldsymbol{\theta}_{*,ETEL}), \right. \\ &\quad \mathbb{E} \left[ \psi \left( \frac{\mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}} \right) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL}) \right] = -\mathbb{E} [g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})], \\ &\quad \left. \mathbb{E} \left[ \psi \left( \frac{\mathbb{E} \left[ \exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\} \right]}{\exp\{\bar{\mathbf{t}}_{ET}^T(\boldsymbol{\theta}_{*,ETEL}) g(\mathbf{X}, \boldsymbol{\theta}_{*,ETEL})\}} \right) \right] = 0, \right. \end{aligned}$$

into  $\mathbf{r}_1^\phi(\boldsymbol{\theta}_{*,ETEL})$ ,  $\mathbf{r}_2^\phi(\boldsymbol{\theta}_{*,ETEL})$ ,  $\mathbf{r}_3^\phi(\boldsymbol{\theta}_{*,ETEL})$  of Theorem 13 respectively, the desired result is obtained. The expression of (53) is a particular case of  $\mu_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL})$  with  $\phi(x) = x \log x - x + 1$ . ■

**Remark 16** From the previous two theorems, we can present an approximation of the power function under misspecification  $\beta_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL})$ , at  $\boldsymbol{\theta}_{*,ETEL} \neq \boldsymbol{\theta}_0$ , of the empirical  $\phi$ -divergence test  $T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  for a significance level  $\alpha$ , as

$$\beta_{T_n^\phi}^*(\boldsymbol{\theta}_{*,ETEL}) = 1 - \Phi \left( \nu_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) \right) \simeq \beta_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL}),$$

where

$$\nu_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) = \frac{n^{1/2}}{\sqrt{\mathbf{r}_{T_n^\phi}^T(\boldsymbol{\theta}_{*,ETEL}) \boldsymbol{\Sigma}_{\sqrt{n}\hat{\boldsymbol{\theta}}_{ETEL}} \mathbf{r}_{T_n^\phi}(\boldsymbol{\theta}_{*,ETEL})}} \left( \frac{\phi''(1) T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_{T_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) \right).$$

In a similar way, an approximation to the asymptotic power function under misspecification  $\beta_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL})$ , at  $\boldsymbol{\theta}_{*,ETEL} \neq \boldsymbol{\theta}_0$ , for the empirical  $\phi$ -divergence test  $T_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)$  can be obtained as

$$\beta_{S_n^\phi}^*(\boldsymbol{\theta}_{*,ETEL}) = 1 - \Phi \left( \nu_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) \right) \simeq \beta_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL}),$$



where

$$\nu_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) = \frac{n^{1/2}}{\sqrt{\mathbf{q}_{S_n^\phi}^T(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0) \boldsymbol{\Sigma}_{\sqrt{n}\hat{\boldsymbol{\theta}}_{ETEL}} \mathbf{q}_{S_n^\phi}(\boldsymbol{\theta}_{*,ETEL}, \boldsymbol{\theta}_0)}} \left( \frac{\phi''(1) S_n^\phi(\hat{\boldsymbol{\theta}}_{ETEL}, \boldsymbol{\theta}_0)}{2n} - \mu_{S_n^\phi}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_{*,ETEL}) \right).$$

In practice,  $\beta_{T_n^\phi}^*(\boldsymbol{\theta}_{*,ETEL})$  and  $\beta_{S_n^\phi}^*(\boldsymbol{\theta}_{*,ETEL})$  are unknown but their consistent estimators are obtained by replacing the population mean by the sample mean.

**Remark 17** The class of  $\phi$ -divergence measures is a wide family of divergence measures but unfortunately there are some classical divergence measures that are not included in this family of  $\phi$ -divergence measures such as the Rényi's divergence or the Sharma and Mittal's divergence. The expression of Rényi's divergence is given by

$$D_{Rényi}^a(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = \frac{1}{a(a-1)} \log \sum_{i=1}^n \mathbf{p}_{ET}(\boldsymbol{\theta})^a \mathbf{p}_{ET}(\boldsymbol{\theta}_0)^{1-a}, \text{ if } a \neq 0, 1, \quad (54)$$

with

$$D_{Rényi}^0(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = \lim_{a \rightarrow 0} D_{Rényi}(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = D_{\text{Kull}}(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0))$$

and

$$D_{Rényi}^1(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = \lim_{a \rightarrow 1} D_{Rényi}(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = D_{\text{Kull}}(\mathbf{p}_{ET}(\boldsymbol{\theta}_0), \mathbf{p}_{ET}(\boldsymbol{\theta})).$$

This measure of divergence was introduced in Rényi (1961) for  $a > 0$  and  $a \neq 1$  and Liese and Vajda (1987) extended it for all  $a \neq 1, 0$ . An interesting divergence measure related to Rényi divergence measure is the Bhattacharya divergence defined as the Rényi divergence for  $a = 1/2$  divided by 4. Other interesting example of divergence measure that is not included in the family of  $\phi$ -divergence measures is the divergence measures introduced by Sharma and Mittal (1997).

In order to unify the previous divergence measures, as well as another divergence measures, Menéndez et al. (1995, 1997) introduced the family of divergences called “ $(h, \phi)$ -divergence measures” in the following way

$$D_\phi^h(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0)) = h(D_\phi(\mathbf{p}_{ET}(\boldsymbol{\theta}), \mathbf{p}_{ET}(\boldsymbol{\theta}_0))),$$

where  $h$  is a differentiable increasing function mapping from  $\left[0, \phi(0) + \lim_{t \rightarrow \infty} \frac{\phi(t)}{t}\right]$  onto  $[0, \infty)$ , with  $h(0) = 0$ ,  $h'(0) > 0$ , and  $\phi \in \Phi$ . In Table 1, these divergence measures are presented, along with the corresponding expressions of  $h$  and  $\phi$ .

Divergence	$h(x)$	$\phi(x)$
Rényi	$\frac{1}{a(a-1)} \log(a(a-1)x + 1), \quad a \neq 0, 1$	$\frac{x^a - a(x-1) - 1}{a(a-1)}, \quad a \neq 0, 1$
Sharma-Mittal	$\frac{1}{b-1} \left\{ [1 + a(a-1)x]^{\frac{b-1}{a-1}} - 1 \right\}, \quad b, a \neq 1$	$\frac{x^a - a(x-1) - 1}{a(a-1)}, \quad a \neq 0, 1$

Table 1: Some specific  $(h, \phi)$ -divergence measures.

Based on the  $(h, \phi)$ -divergence measures we can define two new families of empirical  $(h, \phi)$ -divergence test statistics,

$$S_n^{\phi, h}(\hat{\theta}_{ETEL}, \theta_0) = \frac{2n}{\phi''(1)h'(0)} h\left(D_\phi\left(\left(\mathbf{p}_{ET}(\hat{\theta}_{ETEL}), \mathbf{p}_{ET}(\theta_0)\right)\right)\right) \quad (55)$$

and

$$T_n^{\phi, h}(\hat{\theta}_{ETEL}, \theta_0) = \frac{2n}{\phi''(1)h'(0)} \left( h\left(D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\theta_0))\right) - h\left(D_\phi(\mathbf{u}, \mathbf{p}_{ET}(\hat{\theta}_{ETEL}))\right) \right). \quad (56)$$

The results obtained in this paper for the empirical  $\phi$ -divergence test statistics  $T_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  and  $S_n^\phi(\hat{\theta}_{ETEL}, \theta_0)$  can be obtained for the empirical  $(h, \phi)$ -divergence test statistics defined in (55) and (56).

## 5 Simulation study

The aim of this simulation study is to analyze the performance of the empirical  $\phi$ -divergence test-statistics when the ETEL estimator of an unknown parameter is considered. In this regard, robustness under misspecification and efficiency are studied, based on the design of the simulation study given in Schennach (2007). Let  $X$  be an unknown univariate random variable, with mean  $\theta \in \mathbb{R}$  and variance  $\sigma^2 \in \mathbb{R}^+$  both unknown, but it is supposed to be known that  $\sigma^2 = \theta^2 + 1$ . The corresponding moment based vectorial estimating function is  $\mathbf{g}(X, \theta) = \mathbf{0}_2$ , with  $\mathbf{g}(X, \theta) = (g_1(X, \theta), g_2(X, \theta))^T$ ,

$$g_1(X, \theta) = X - \theta, \quad (57)$$

$$g_2(X, \theta) = X^2 - 2\theta^2 - 1. \quad (58)$$

By modifying (58) to

$$g_2(X, \theta) = X^2 - 2\theta^2 - \delta, \quad \delta \in (-2\theta^2, \infty) - \{1\}, \quad (59)$$

we are considering a misspecified model, with  $\delta$  being a tuning parameter for the model misspecification degree. Since the correctly specified model has a variance equal to  $\theta^2 + \delta$  with  $\delta = 1$ , less variance than the correct one is specified when  $\delta \in (-2\theta^2, 1)$ , while a bigger variance than the correct one is specified when  $\delta \in (1, \infty)$ . The EL estimator of  $\theta$  is given by

$$\hat{\theta}_{EL} = \arg \min_{\theta \in \mathbb{R}} \left( - \sum_{i=1}^n \log p_{i,EL}(\theta) \right),$$

with

$$\begin{aligned} p_{i,EL}(\theta) &= \frac{1}{n} \frac{1}{1 + \sum_{h=1}^2 t_{h,EL}(\theta) g_h(x_i, \theta)}, \quad i = 1, \dots, n, \\ t_{1,EL}(\theta), t_{2,EL}(\theta) &\text{ s.t. } \sum_{i=1}^n \frac{1}{1 + \sum_{h=1}^2 t_{h,EL}(\theta) g_h(x_i, \theta)} g_r(x_i, \theta) = 0, \quad r = 1, 2, \end{aligned} \quad (60)$$

the ET estimator of  $\theta$  by

$$\hat{\theta}_{ET} = \arg \min_{\theta \in \mathbb{R}} \sum_{i=1}^n p_{ET,i}(\theta) \log(p_{ET,i}(\theta)),$$

with

$$p_{i,ET}(\theta) = \frac{\exp \left\{ \sum_{h=1}^2 t_{h,ET}(\theta) g_h(x_i, \theta) \right\}}{\sum_{i=1}^n \exp \left\{ \sum_{h=1}^2 t_{h,ET}(\theta) g_h(x_i, \theta) \right\}}, \quad i = 1, \dots, n, \quad (61)$$

$$t_{1,ET}(\theta), t_{2,ET}(\theta) \quad \text{s.t.} \quad \sum_{i=1}^n \exp \left\{ \sum_{h=1}^2 t_{h,ET}(\theta) g_h(x_i, \theta) \right\} g_r(x_i, \theta) = 0, \quad r = 1, 2,$$

and the ETEL of  $\theta$  estimator by

$$\hat{\theta}_{ETEL} = \arg \min_{\theta \in \mathbb{R}} \left( - \sum_{i=1}^n \log p_{i,ETEL}(\theta) \right),$$

with  $p_{i,ETEL}(\theta) = p_{i,ET}(\theta)$ ,  $i = 1, \dots, n$ . The test-statistics  $T_n^{\phi_\lambda}(\hat{\theta}_\ell, \theta_0)$  and  $S_n^{\phi_\lambda}(\hat{\theta}_\ell, \theta_0)$ , with  $\ell \in \{EL, ET, ETEL\}$ , and

$$\phi_\lambda(x) = \begin{cases} \frac{1}{\lambda(\lambda+1)} (x^{\lambda+1} - x - \lambda(x-1)), & \lambda \in \mathbb{R} - \{0, -1\} \\ \lim_{s \rightarrow 0} \phi_s(x) = x \log x - x + 1, & \lambda = 0 \\ \lim_{s \rightarrow -1} \phi_s(x) = -\log x + x - 1, & \lambda = -1 \end{cases},$$

are the so-called empirical power divergence based test-statistics of Cressie and Read (1984), valid in this new setting for testing

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0, \quad \text{with } \theta_0 = 0. \quad (62)$$

The expressions of the empirical power divergence based test-statistics are

$$T_n^{\phi_\lambda}(\hat{\theta}_\ell, \theta_0) = \begin{cases} \frac{2}{\lambda(1+\lambda)} \left( \sum_{i=1}^n (np_{i,\ell}(\theta_0))^{-\lambda} - \sum_{i=1}^n (np_{i,\ell}(\hat{\theta}_\ell))^{-\lambda} \right), & \lambda \in \mathbb{R} - \{0, -1\} \\ 2 \sum_{i=1}^n \log \left( \frac{p_{i,\ell}(\hat{\theta}_\ell)}{p_{i,\ell}(\theta_0)} \right), & \lambda = 0 \\ 2n \left( \sum_{i=1}^n p_{i,\ell}(\theta_0) \log(np_{i,\ell}(\theta_0)) - \sum_{i=1}^n p_{i,\ell}(\hat{\theta}_\ell) \log(np_{i,\ell}(\hat{\theta}_\ell)) \right), & \lambda = -1 \end{cases},$$

$$S_n^{\phi_\lambda}(\hat{\theta}_\ell, \theta_0) = \begin{cases} \frac{2n}{\lambda(1+\lambda)} \left( \sum_{i=1}^n \frac{p_{i,\ell}^{\lambda+1}(\hat{\theta}_\ell)}{p_{i,\ell}^\lambda(\theta_0)} - 1 \right), & \lambda \in \mathbb{R} - \{0, -1\} \\ 2n \sum_{i=1}^n p_{i,\ell}(\hat{\theta}_\ell) \log \left( \frac{p_{i,\ell}(\hat{\theta}_\ell)}{p_{i,\ell}(\theta_0)} \right), & \lambda = 0 \\ 2n \sum_{i=1}^n p_{i,\ell}(\theta_0) \log \left( \frac{p_{i,\ell}(\theta_0)}{p_{i,\ell}(\hat{\theta}_\ell)} \right), & \lambda = -1 \end{cases},$$

with  $\ell \in \{EL, ET, ETEL\}$ ,  $p_{i,EL}(\theta)$  given by (60) and  $p_{i,ETEL}(\theta) = p_{i,ET}(\theta)$  by (61). It is worth of mentioning that the empirical likelihood ratio test-statistic of Qin and Lawless (1994) matches the case of  $\lambda = 0$  when the EL estimator of  $\theta$  is applied, i.e.  $T_n^{\phi_0}(\hat{\theta}_{EL}, \theta_0) = G_n^2(\hat{\theta}_{EL}, \theta_0)$ .

For the study of the performance of  $T_n^{\phi_\lambda}(\hat{\theta}_\ell, \theta_0)$  and  $S_n^{\phi_\lambda}(\hat{\theta}_\ell, \theta_0)$ , for illustrative purposes, a subset of tuning parameters of the empirical power divergence based test-statistics are considered,  $\lambda \in \{-1, -0.5, 0, \frac{2}{3}\}$ . When the model is correctly specified, the population's distribution is simulated with a standard normal distribution, i.e.  $X \sim \mathcal{N}(\theta, \theta^2 + \delta)$ , with  $\theta = 0$  and  $\delta = 1$  ( $\sigma^2 = 1$ ). When the model is misspecified, two cases are considered, by simulating the population distribution either through  $X \sim \mathcal{N}(\theta, \theta^2 + \delta)$ , with  $\theta = 0$  and  $\delta = 0.7$  ( $\sigma^2 = 0.7 < 1$ ) or  $\theta = 0$  and  $\delta = 1.3$  ( $\sigma^2 = 1.3 > 1$ ). The pseudo true value of the ETEL estimator is  $\theta_{*,ETEL} = \theta_0 = 0$  for  $\delta > \frac{1}{2}$ , and  $t_{*,1,ETEL} = 0$ ,  $t_{*,2,ETEL} = \frac{1-\delta}{2\delta}$ , so even being a misspecified model  $\hat{\theta}_{ETEL}$  is a consistent estimator of the true value of  $\theta$ . Using  $R = 10,000$  replications, the following results are obtained.

In Figure 1 the simulated cumulative distribution functions (CDF) of  $\hat{\theta}_{EL}$ ,  $\hat{\theta}_{ET}$  and  $\hat{\theta}_{ETEL}$  are shown with a sample size of  $n = 1000$ , for the correctly specified model ( $\delta = 1$ ) as well as the two misspecified models ( $\delta \in \{0.7, 1.3\}$ ). Since the sample size is very big, the three types of estimators exhibit almost the same CDF. The gray color line of the figures indicates the theoretical distribution with correct specification, i.e. the reference line to be compared. Under misspecification, as expected according to Schennach (2007), the most robust estimator under misspecification is  $\hat{\theta}_{ET}$  (it is closer to the gray line), the least robust  $\hat{\theta}_{EL}$  (it is further from the gray line), and  $\hat{\theta}_{ETEL}$  tends to be between the two. In addition,  $\hat{\theta}_{ETEL}$  tends to be in between the two in efficiency with respect to the exact size of the asymptotic test for small sample sizes, no as efficient as  $\hat{\theta}_{EL}$  but more efficient than  $\hat{\theta}_{ET}$ . In the same way, we would like to identify a test-statistic  $T_n^{\phi\lambda}(\hat{\theta}_{ETEL}, \theta_0)$  or  $S_n^{\phi\lambda}(\hat{\theta}_{ETEL}, \theta_0)$  with good performance at the same in robustness under misspecification and efficiency.

The simulations showed that in robustness under misspecification  $S_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$  is much worse than  $T_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$ , with  $\ell \in \{EL, ET, ETEL\}$ , for this reason the following figures are focussed only on  $T_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$ . In Figure 2 the simulated CDFs of  $T_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$  are plotted with the three types of estimators and a degree of misspecification equal to  $\delta = 1.3$ , while in Figure 3 are plotted with a degree of misspecification equal to  $\delta = 0.7$ . From them, the test-statistic  $T_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$  with  $\lambda = -1$  seems to be the most robust test-statistic under misspecification. Figure 4 has been plotted to compare the performance of  $T_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$  with  $\lambda = -1$  when different types of estimators are plugged,  $\ell \in \{EL, ET, ETEL\}$ . As expected, the most robust test-statistic is  $T_n^{\phi-1}(\hat{\theta}_{ET}, \theta_0)$ , the worst one  $T_n^{\phi-1}(\hat{\theta}_{EL}, \theta_0)$ , and  $T_n^{\phi-1}(\hat{\theta}_{ETEL}, \theta_0)$  is in between. From Figures 2 and 3, for the misspecified model (either with  $\delta = 1.3$  or  $\delta = 0.7$ ), the exact significance levels can be visually compared with respect to the 0.05 nominal level, comparing the values of the black color curves just at  $\chi_{0.05}^2 = 3.84$  in the abscissa axis, with respect to the gray color curve. In this regard, the exact sizes for  $\delta = 1.3$  are better than for  $\delta = 0.7$ : for ETEL estimators the exact significance levels are 0.048 ( $\lambda = -1$ ), 0.036 ( $\lambda = -0.5$ ), 0.031 ( $\lambda = 0$ ), 0.025 ( $\lambda = \frac{2}{3}$ ) when  $\delta = 1.3$  and 0.176 ( $\lambda = -1$ ), 0.208 ( $\lambda = -0.5$ ), 0.258 ( $\lambda = 0$ ), 0.391 ( $\lambda = \frac{2}{3}$ ) when  $\delta = 0.7$ . The figures of the simulations for  $S_n^{\phi\lambda}(\hat{\theta}_\ell, \theta_0)$ , with  $n = 1000$ , were omitted, but the exact sizes are as follows: the exact significance levels are 0.017 ( $\lambda = -1$ ), 0.017 ( $\lambda = -0.5$ ), 0.017 ( $\lambda = 0$ ), 0.017 ( $\lambda = \frac{2}{3}$ ) when  $\delta = 1.3$  and 0.417 ( $\lambda = -1$ ), 0.417 ( $\lambda = -0.5$ ), 0.418 ( $\lambda = 0$ ), 0.419 ( $\lambda = \frac{2}{3}$ ) when  $\delta = 0.7$ .

Figure 6 and 7 represent, only for  $n = 100$  for illustrative purposes, the asymptotic power based on the power-divergence test statistics  $T_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$  and  $S_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$ ,  $\beta_{T_n^\phi}(\theta^*)$  and  $\beta_{S_n^\phi}(\theta^*)$  when the nominal significance level is  $\alpha = 0.05$ . There are no substantial differences for a generic small or moderate samples size. The test-statistics  $T_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$  and  $S_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$ , with  $\lambda = -1$ , exhibit the exact significance levels closest to the nominal significance level, 0.058 for  $T_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$  and 0.052 for  $S_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$ . In the results obtained in Balakrishnan et al. (2015)  $S_n^{\phi\lambda}(\hat{\theta}_{EL}, \theta_0)$  was found out to be much more efficient than  $T_n^{\phi\lambda}(\hat{\theta}_{EL}, \theta_0)$  with small sample sizes, for being  $S_n^{\phi\lambda}(\hat{\theta}_{EL}, \theta_0)$  closer to the nominal level than  $T_n^{\phi\lambda}(\hat{\theta}_{EL}, \theta_0)$ . Such a difference is less pronounced for  $T_n^{\phi\lambda}(\hat{\theta}_{ETEL}, \theta_0)$  and  $S_n^{\phi\lambda}(\hat{\theta}_{ETEL}, \theta_0)$ . The performance of  $T_n^{\phi\lambda}(\hat{\theta}_{EL}, \theta_0)$  with  $\lambda = -1$  is then relatively good in efficiency with small sample sizes as well as in robustness under misspecification (with small and big sample sizes).

The approximation to the asymptotic power  $\beta_{T_n^\phi}(\theta^*)$ , at  $\theta^* \neq 0$ , of the power-divergence test  $T_n^{\phi\lambda}(\hat{\theta}_{ETEL}, 0)$

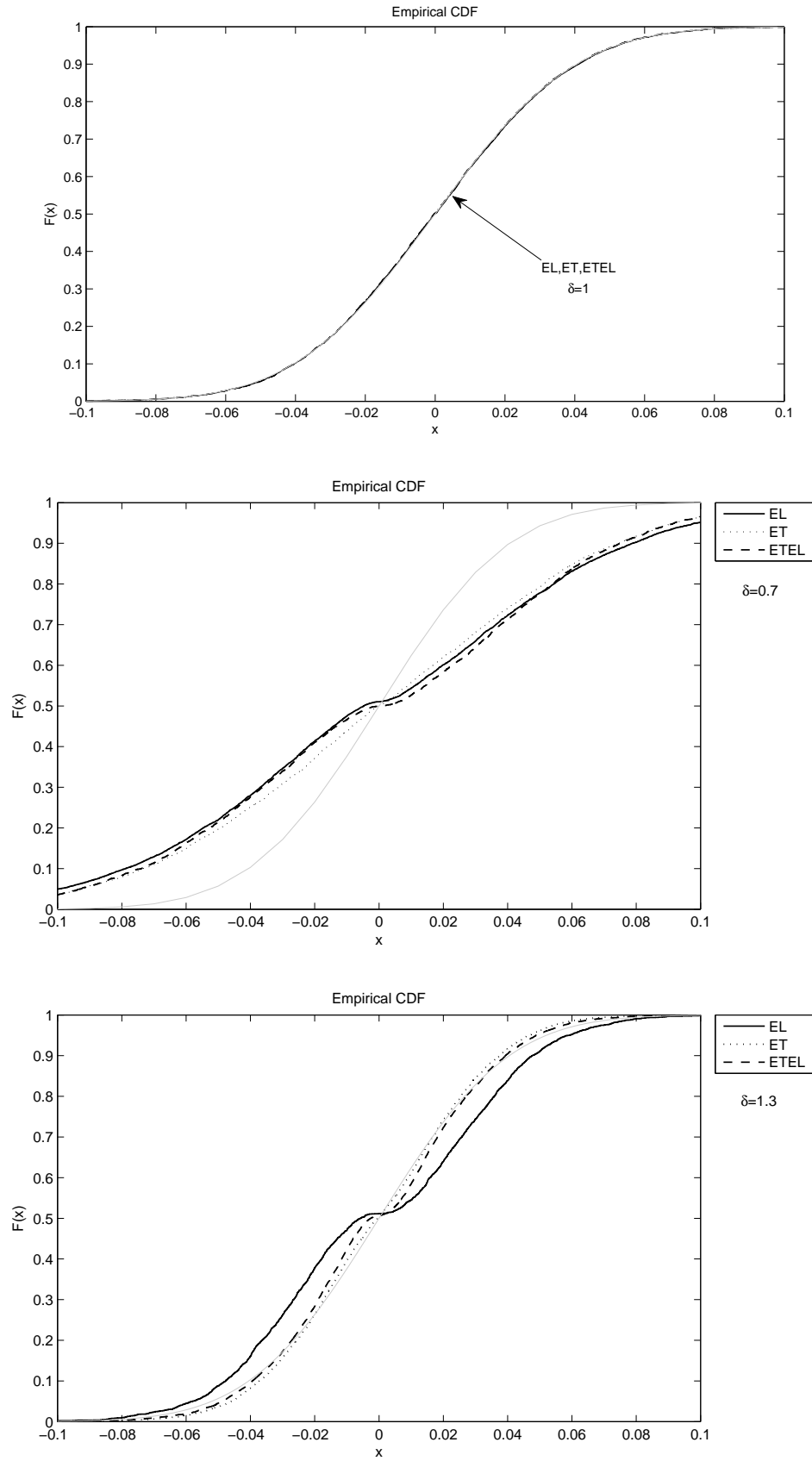


Figure 1: Cumulative distribution function of the three types of estimators, for  $n = 1000$ , when the model is correctly specified (top), and is misspecified ( $\delta = 0.7$ ,  $\delta = 1.3$ ).

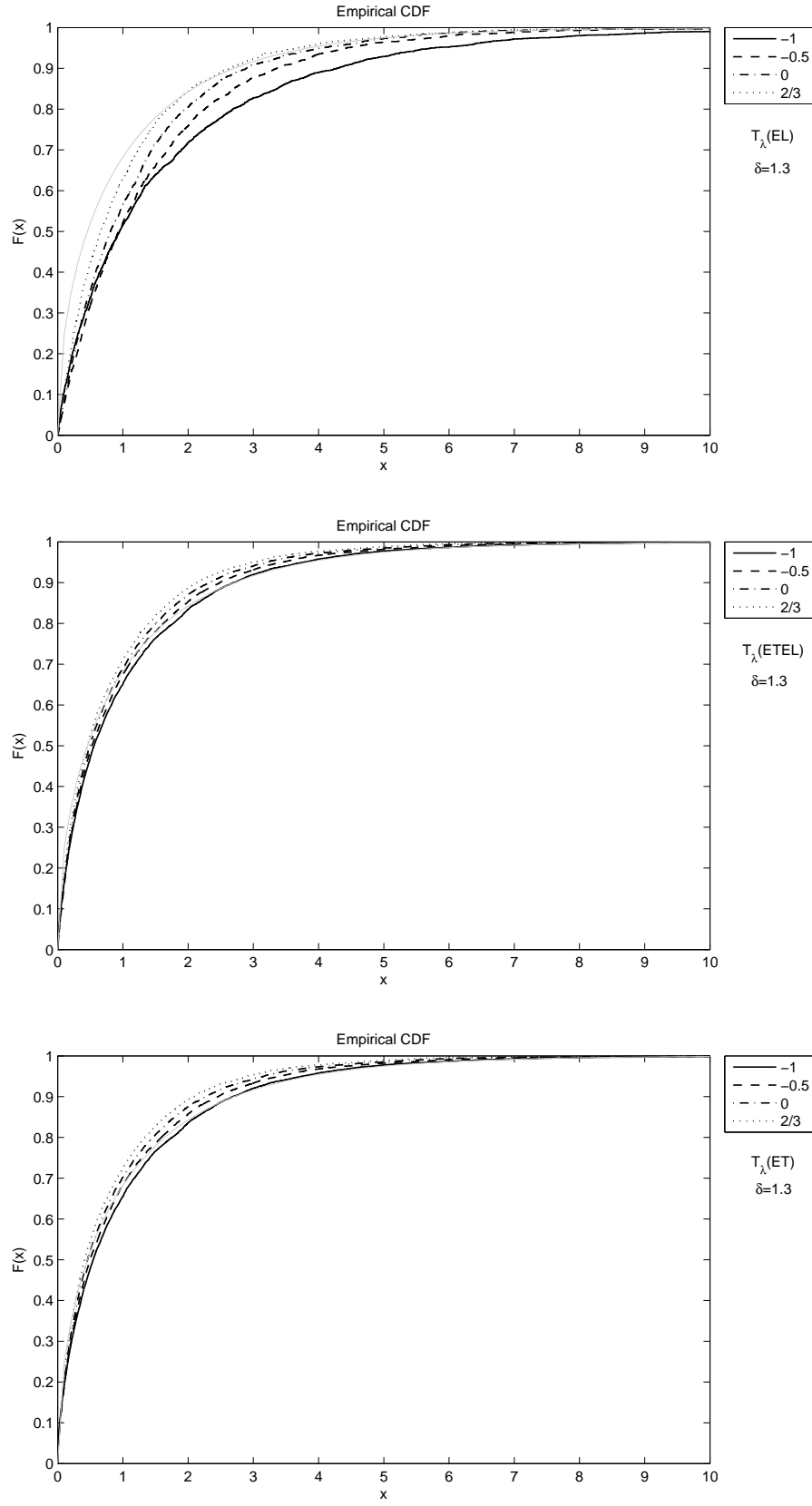


Figure 2: Cumulative distribution function of the empirical power divergence based test-statistics with the three types of estimators, for  $n = 1000$ , when the model is misspecified with  $\delta = 1.3$ .

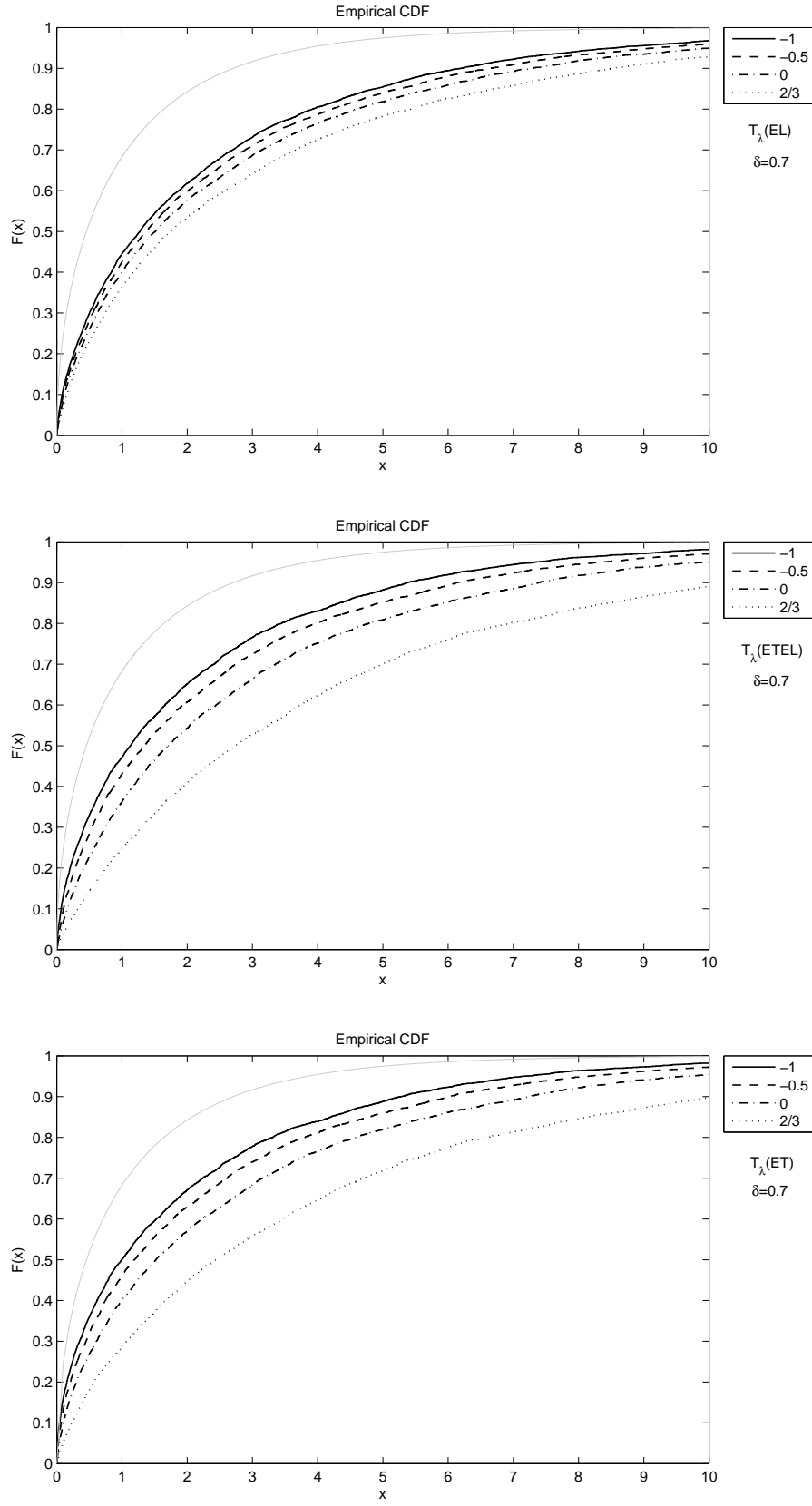


Figure 3: Cumulative distribution function of the empirical power divergence based test-statistics with the three types of estimators, for  $n = 1000$ , when the model is misspecified with  $\delta = 0.7$ .

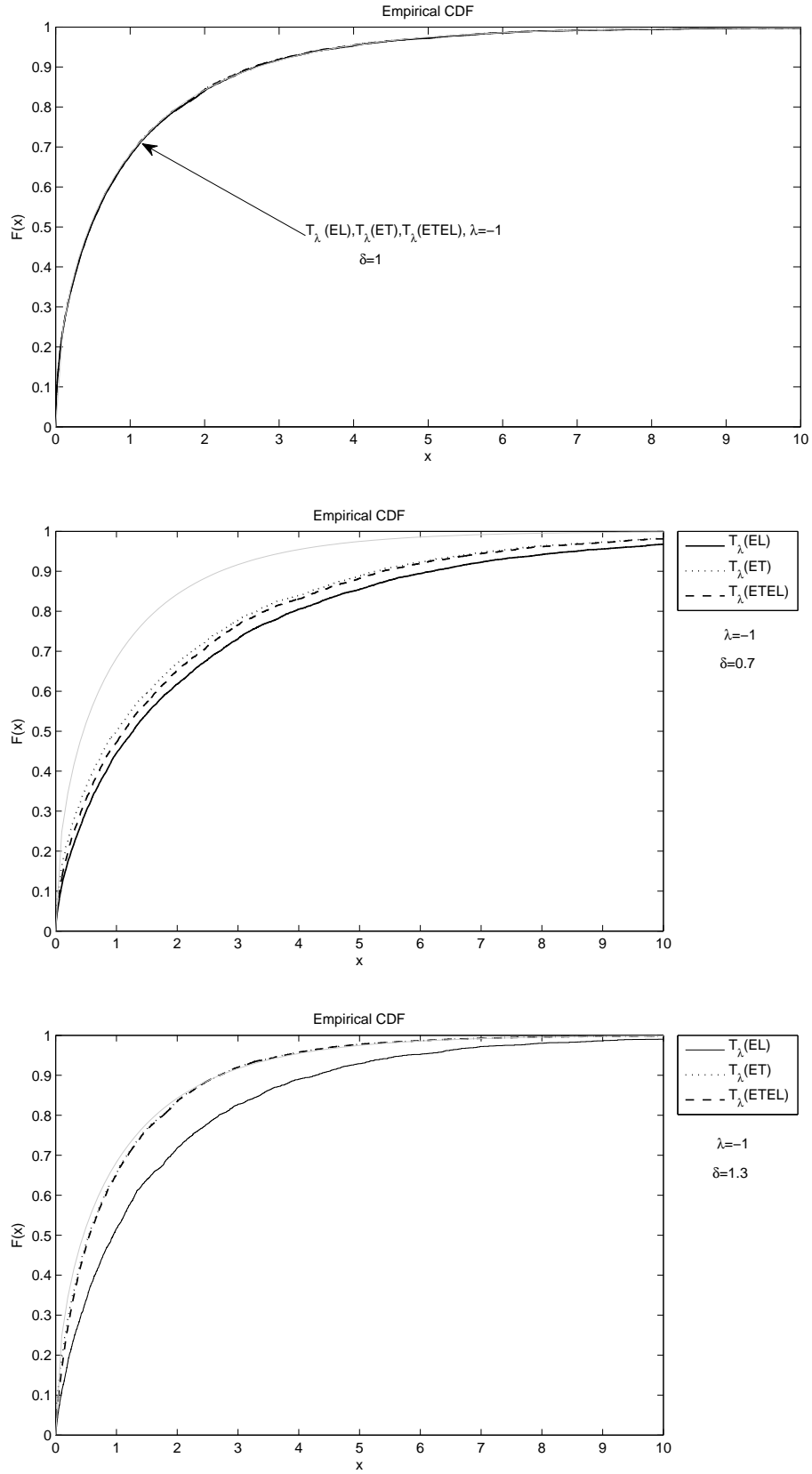


Figure 4: Cumulative distribution function of  $T_n^{\phi-1}$  with the three types of estimators, for  $n = 1000$ , when the model is correctly specified (top), and is misspecified ( $\delta = 0.7$ ,  $\delta = 1.3$ ).



for the correctly specified model, with a significance level  $\alpha$ , is according to Remark 5 and doing some algebraic manipulations,  $\beta_{T_n^{\phi_\lambda}}^*(\theta^*) = 1 - \Phi\left(\nu_{T_n^{\phi_\lambda}}(\theta^*, 0)\right)$ , where

$$\nu_{T_n^{\phi_\lambda}}(\theta^*, 0) = \left(\frac{1}{n} \mathbf{s}_{T_n^{\phi_\lambda}}^T(\theta^*, 0) \mathbf{M}_{T_n}(\theta^*, 0) \mathbf{s}_{T_n^{\phi_\lambda}}(\theta^*, 0)\right)^{-\frac{1}{2}} \left(\frac{\chi_{p,\alpha}^2}{2n} - \mu_{\phi_\lambda}(\theta^*, 0)\right),$$

$$\mu_{T_n^{\phi_\lambda}}(\theta^*, 0) = \begin{cases} \frac{1}{\lambda(\lambda+1)} \left( \frac{\exp\left\{\frac{\lambda(\lambda+1)\theta^{*2}}{2(1-\lambda\theta^{*2})}\right\}}{\sqrt{(1-\lambda\theta^{*2})(\theta^{*2}+1)^\lambda}} - 1 \right), & \lambda \in \mathbb{R} - \{0, -1\} \\ \theta^{*2} - \frac{1}{2} \log(1 + \theta^{*2}) & \lambda = 0 \\ \frac{1}{2} \log(1 + \theta^{*2}) & \lambda = -1 \end{cases},$$

$$\mathbf{s}_{T_n^{\phi_\lambda}}^T(\theta^*, 0) \mathbf{M}_{T_n}(\theta^*, 0) \mathbf{s}_{T_n^{\phi_\lambda}}(\theta^*, 0) = \frac{\theta^{*2} \exp\left\{\theta^{*2} \left(\frac{\lambda(\lambda+1)}{1-\lambda\theta^{*2}} + \frac{1}{2\theta^{*2}+1}\right)\right\}}{\sqrt{(2\theta^{*2}+1)^5 (1-\lambda\theta^{*2})^3 (\theta^{*2}+1)^{\lambda-1}}}$$

$$\times \begin{pmatrix} 1 & (\lambda+2)\theta^* \end{pmatrix} \begin{pmatrix} 2\theta^{*4} + 4\theta^{*2} + 1 & -\frac{\theta}{2\theta^2+1} (\theta^{*4} + 3\theta^{*2} + 1) \\ -\frac{\theta^*}{2\theta^{*2}+1} (\theta^{*4} + 3\theta^{*2} + 1) & \frac{1}{2(2\theta^{*2}+1)^2} (6\theta^{*8} + 16\theta^{*6} + 19\theta^{*4} + 8\theta^{*2} + 1) \end{pmatrix} \begin{pmatrix} 1 \\ (\lambda+2)\theta^* \end{pmatrix}.$$

In particular, Figure 5 shows the approximated and exact asymptotic powers of  $T_n^{\phi_\lambda}(\hat{\theta}_{ETEL}, 0)$  when  $\lambda = -1$ ,  $\beta_{T_n^{\phi_{-1}}}^*(\theta^*)$  and  $\beta_{T_n^{\phi_{-1}}}(\theta^*, 0)$ , for two sample sizes  $n = 100$  and  $n = 200$ . The approximation is quite good for the values not very close to  $\theta_0 = 0$ ,  $\theta^* \notin (-0.11, 0.11)$  when  $n = 100$ , and  $\theta^* \notin (-0.075, 0.075)$  when  $n = 200$ .

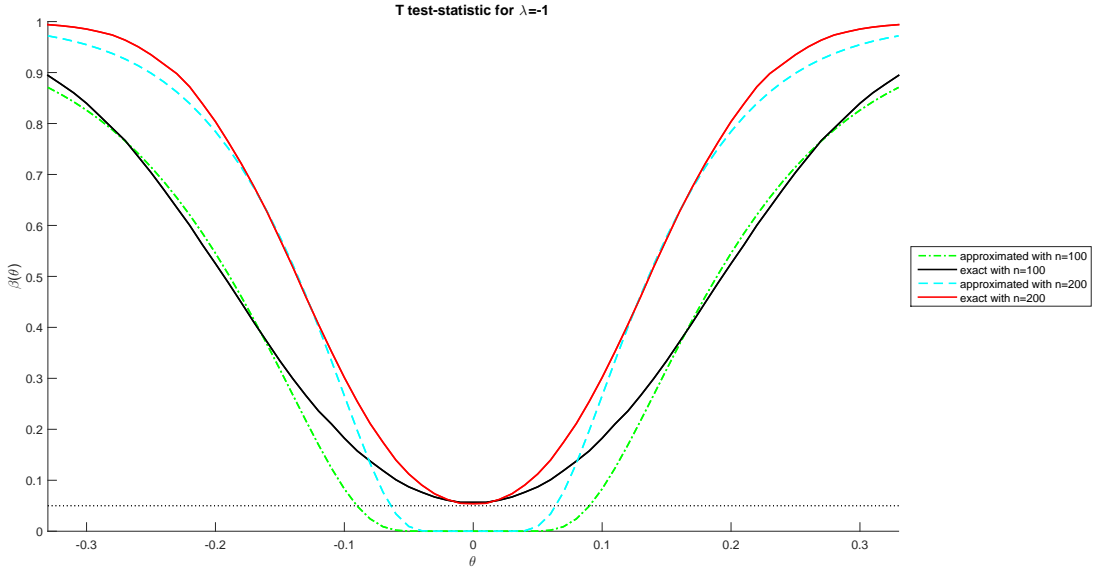


Figure 5:  $\beta_{T_n^{\phi_{-1}}}^*(\theta^*)$  and  $\beta_{T_n^{\phi_{-1}}}(\theta^*, 0)$  when the ETEL estimator is plugged.

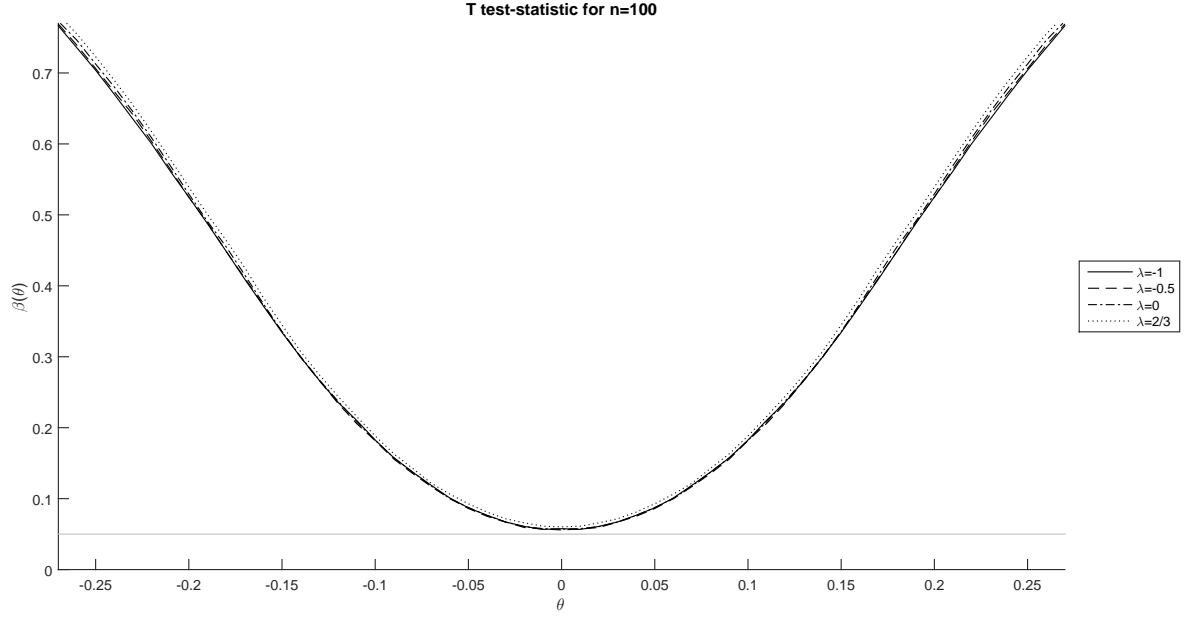


Figure 6: Power function  $\beta_{T_n^{\phi_\lambda}}(\theta^*)$ , for different values of  $\lambda$  when the ETEL estimator is plugged.

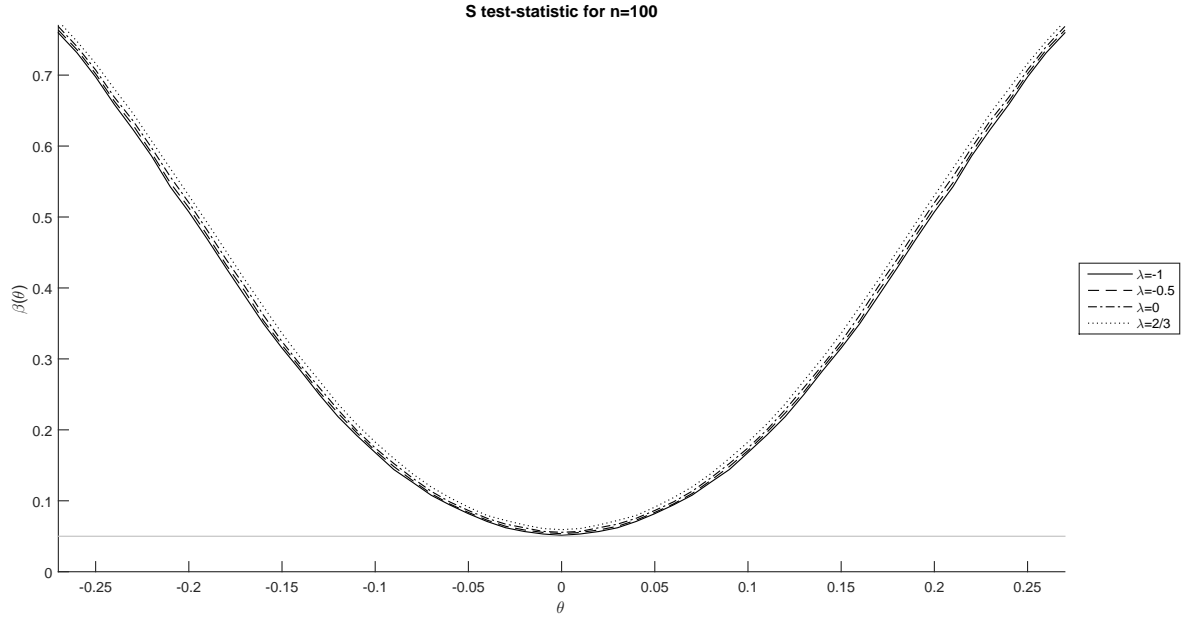


Figure 7: Power function  $\beta_{S_n^{\phi_\lambda}}(\theta^*)$ , for different values of  $\lambda$  when the ETEL estimator is plugged.

## 6 Conclusion

This paper introduces empirical  $\phi$ -divergence test-statistics using exponentially tilted empirical likelihood estimators, as alternative to the empirical likelihood ratio test-statistic. It is shown that these test-statistics follow the same efficiency and robustness patterns of the corresponding estimators, empirical likelihood estimators, exponential tilted estimators and exponentially tilted empirical likelihood estimators. This justifies the practical choice of the exponentially tilted empirical likelihood estimator to be plugged into the empirical  $\phi$ -divergence test-statistics, for being a good compromise between the efficiency of the exact size of the test for small or moderate sample sizes and the robustness under model misspecification. According to the results of the simulation study, the modified empirical likelihood ratio test

$$T_n^{\phi-1}(\hat{\theta}_{ETEL}, \theta_0) = 2n \left( \sum_{i=1}^n p_{i,ET}(\theta_0) \log(np_{i,ET}(\theta_0)) - \sum_{i=1}^n p_{i,ET}(\hat{\theta}_{ETEL}) \log(np_{i,ET}(\hat{\theta}_{ETEL})) \right),$$

exhibits, by far, the best performance.

A possible future research could include a correction of the critical value for  $T_n^{\phi-1}(\hat{\theta}_{ETEL}, \theta_0)$  test-statistic. For instance, in the line of Lee (2014), bootstrap critical values of  $T_n^{\phi-1}(\hat{\theta}_{ETEL}, \theta_0)$  could be studied to be compared with the Wald type test-statistic's bootstrap critical values proposed in the aforementioned paper.

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## Appendix

**Proof of Lemma 10.** Taking into account (12),

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\theta}} p_{ET,i}(\boldsymbol{\theta}) &= \frac{\partial}{\partial \boldsymbol{\theta}} \frac{1}{n} \frac{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\}}{\overline{\exp}_{ET}(\boldsymbol{\theta})} \\
&= \frac{1}{n} \frac{\exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \left[ \frac{\partial}{\partial \boldsymbol{\theta}} (\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})) \overline{\exp}_{ET}(\boldsymbol{\theta}) - \frac{\partial}{\partial \boldsymbol{\theta}} \overline{\exp}_{ET}(\boldsymbol{\theta}) \right]}{\overline{\exp}_{ET}^2(\boldsymbol{\theta})} \\
&= p_{ET,i}(\boldsymbol{\theta}) \left[ \frac{\partial}{\partial \boldsymbol{\theta}} (\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})) - \overline{\exp}_{ET}^{-1}(\boldsymbol{\theta}) \frac{\partial}{\partial \boldsymbol{\theta}} \overline{\exp}_{ET}(\boldsymbol{\theta}) \right],
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\theta}} (\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})) &= \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) + \frac{\partial}{\partial \boldsymbol{\theta}} \mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \\
&= \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) - \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) (\mathbf{t}_{ET}(\boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) + \mathbf{I}_r) \right) \\
&\quad \times \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) \right)^{-1} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}), \\
\frac{\partial}{\partial \boldsymbol{\theta}} \mathbf{t}_{ET}^T(\boldsymbol{\theta}) &= - \left( \sum_{i=1}^n p_{ET,i}(\boldsymbol{\theta}) \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) (\mathbf{t}_{ET}(\boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) + \mathbf{I}_r) \right) \\
&\quad \times \left( \sum_{i=1}^n p_{ET,i}(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) \right)^{-1} \\
&= - \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) (\mathbf{t}_{ET}(\boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) + \mathbf{I}_r) \right) \\
&\quad \times \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) \right)^{-1}
\end{aligned}$$

according to (41) of Schennach (2007), and

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\theta}} \overline{\exp}_{ET}(\boldsymbol{\theta}) &= \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \frac{\partial}{\partial \boldsymbol{\theta}} (\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})) \\
&= \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \right) \mathbf{t}_{ET}(\boldsymbol{\theta}) \\
&\quad - \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) (\mathbf{t}_{ET}(\boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) + \mathbf{I}_r) \right) \\
&\quad \times \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \mathbf{g}^T(\mathbf{X}_i, \boldsymbol{\theta}) \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) \right) \\
&= \overline{\exp_{ET} \mathbf{G}^T}(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) - \widehat{\mathbf{K}}(\boldsymbol{\theta}) \overline{\exp_{ET} \mathbf{g}}(\boldsymbol{\theta}) \\
&= \overline{\exp_{ET} \mathbf{G}^T}(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}), \tag{63}
\end{aligned}$$

with

$$\overline{\exp_{ET} \mathbf{g}}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^n \exp\{\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})\} \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}) = \mathbf{0}_r$$

from (13). Using the previous notation

$$\frac{\partial}{\partial \boldsymbol{\theta}} (\mathbf{t}_{ET}^T(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta})) = \mathbf{G}_{\mathbf{X}_i}^T(\boldsymbol{\theta}) \mathbf{t}_{ET}(\boldsymbol{\theta}) - \widehat{\mathbf{K}}(\boldsymbol{\theta}) \mathbf{g}(\mathbf{X}_i, \boldsymbol{\theta}), \tag{64}$$

and replacing (64) and (63) in the expression of  $\frac{\partial}{\partial \boldsymbol{\theta}} p_{ET,i}(\boldsymbol{\theta})$ , the desired result is obtained. ■

**Proof of Lemma 11.** Taking into account the expression of (25),

$$\frac{\partial}{\partial \boldsymbol{\theta}} D_{\phi}(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta})) = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \boldsymbol{\theta}} (np_{ET,i}(\boldsymbol{\theta})) \psi\left(\frac{1}{np_{ET,i}(\boldsymbol{\theta})}\right).$$

By plugging  $\frac{\partial}{\partial \boldsymbol{\theta}} p_{ET,i}(\boldsymbol{\theta})$  from Theorem 10 into the previous expression, (47) is obtained. Since according to the weak law of large numbers  $\frac{1}{n} \sum_{i=1}^n h(\mathbf{X}_i) \xrightarrow[n \rightarrow \infty]{P} E[h(\mathbf{X})]$  for any integrable function  $h : \mathbb{R}^p \rightarrow \mathbb{R}$ , taking the appropriate functions in the role of  $h$ , the limiting value of  $\frac{\partial}{\partial \boldsymbol{\theta}} D_{\phi}(\mathbf{u}, \mathbf{p}_{ET}(\boldsymbol{\theta}))$  is obtained. ■

