Applying Young diagrams to 2-symmetric fuzzy measures with an application to general fuzzy measures

P. García-Segador
National Statistics Institute
Paseo de la Castellana, 183
28071 Madrid (Spain)
pedro.garcia.segador@ine.es

P. Miranda*
Complutense University of Madrid
Plaza de Ciencias, 3
28040 Madrid (Spain)
pmiranda@mat.ucm.es

Abstract

In this paper we apply Young diagrams, a well-known object appearing in Combinatorics and group representation theory, to study some properties of the polytope of 2-symmetric fuzzy measures with respect to a given partition, a subpolytope of the set of fuzzy measures. The main result in the paper allows to build a simple and fast algorithm for generating points on this polytope in a random fashion. Besides, we also study some other properties of this polytope, as for example its volume. In the last section, we give an application of this result to the problem of identification of general fuzzy measures.

Keywords: Fuzzy measures; 2-symmetric measures; Young tableaux; Ferrers posets; random generation; identification.

1 Introduction

Fuzzy measures over a finite referential set constitute a powerful tool appearing in many different fields, such as Decision Making, Game Theory, Evidence Theory, and many others (see e.g. [20, 18, 19] and the references therein). For this reason, the study of the properties of these measures (or any of their subfamilies) has attracted the attention of many researchers. One of these problems is determining the properties of the set of all fuzzy measures over a referential set of fixed cardinality; in this sense, it is easy to see that this set is a convex bounded polyhedron (i.e. a polytope), and this also happens to many other subfamilies, such as k-additive measures, k-symmetric measures or belief functions, among others [9, 10]. Related to the geometrical structure of fuzzy measures or any subfamily, we have the problem of randomly generating a measure [5].

This is an interesting problem for several reasons. First, and most evident, it is an appealing problem from a mathematical point of view. Next, being able to generate points in a polytope allows to approximate several characteristics, as the center of gravity, the volume, and so on; these problems could be solved numerically by using Monte-Carlo methods if we know how to generate uniformly inside these polytopes.

We state now a more practical reason, related to the problem of identification of fuzzy measures. This problem arises when we have some sample data (perhaps in a non-numerical scale) and we

*Corresponding author: P. Miranda. pmiranda@mat.ucm.es. Tel: (+34) 91 394 44 19. Fax: (+34) 91 394 44 06

look for the fuzzy measure (possibly restricted to a subfamily $\mathcal{F}$) that best fits these data. In [6], a method to deal with the problem of identification of some convex families of fuzzy measures based on genetic algorithms has been proposed; in that paper, the cross-over operator considered is the convex combination of points, taking advantage of the fact that $\mathcal{F}$ can be seen as a polytope. This algorithm is very fast and the simulations carried out suggest that it is stable with respect to the presence of noise. However, the convex combination reduces the search region in each iteration; although a desirable property at first sight, it has the handicap that if the solution is left outside the search region in an iteration, it is not possible to include it in the search region again. To cope with this problem, there are several options; the most natural is to consider the set of vertices as the initial population of the algorithm and use as mutation operator the convex combination of the measure with one of the vertices; unfortunately, for the general case of fuzzy measures and also for many subfamilies, the number of vertices increases too fast [7]. Thus, this solution is most of times unfeasible from a computational point of view. Then, the most practical way of proceeding is to consider as initial population a set of fuzzy measures selected randomly with respect to the uniform distribution on the set of all fuzzy measures in $\mathcal{F}$. In other words, we want to select a set of fuzzy measures in $\mathcal{F}$ such that no region takes advantage of a larger probability than other region with the same volume.

As stated before, for general fuzzy measures and many subfamilies, the problem of random generation consists in generating points randomly in a polytope. However, this is a complex problem that has not been satisfactorily solved for general polytopes; indeed, there are many different techniques to deal with this problem, as the grid method [13], the sweep-plane method [28] and triangulation methods [13]; in the case of fuzzy measures, it can be seen that it is a special case of polytope, namely an order polytope [36], whose subjacent poset is the Boolean poset. The same happens for some subfamilies of fuzzy measures, as k-symmetric measures [9] (but it does not hold for other important families, as k-additive measures). In this sense, triangulations methods are specially appealing for order polytopes because, as it is proved in [30] pag. 304, the problem of generating a point in an order polytope can be turned into generating a linear extension of the subjacent poset in a random way.

On the other hand, Brightwell and Winkler have shown in [3] that, for general posets, this problem, as well as the problem of counting the number of linear extensions, are $\sharp$ P-complete problems, and consequently, it is not possible to derive a satisfactory procedure for solving these problems for general posets. Then, this problem has been treated by many researchers, aiming to derive either techniques for a particular class of posets, or to propose algorithms that are close to randomness and whose complexity is reduced (see e.g. [23, 34, 25] for generating a random linear extension of a poset and [26, 40, 24] for counting the number of linear extensions). Indeed, this problem is a hot topic in Computer Sciences nowadays.

In this paper, we deal with the polytope of 2-symmetric measures. Briefly speaking, k-symmetric measures arise when some elements of the referential set X present the same behavior and thus can be grouped in a subset of indifferent elements (see Section 2 below). This is a situation that appears in many practical situations and, thus defined, k-symmetric measures allow a reduction in the number of coefficients (and thus, the complexity) needed to define the corresponding measure. Therefore, k-symmetric measures define a gradation between symmetric measures (where we just care about the number of elements in the subset) and general fuzzy measures. Moreover, from a combinatorial point of view, the set of k-symmetric measures with respect to a given partition of X is an order polytope, so it suffices to consider the subjacent poset and solve the problem of random generating a linear extension.

We show in the paper that this poset is a special case of Ferrers posets. These posets are closely related to Young diagrams, a tool appearing in Combinatorics to represent partitions of natural numbers. We show that Young diagrams are a suitable way to represent the posets corresponding to 2-symmetric measures. And from the properties of Young diagrams, we derive a procedure to generate randomly linear extensions of Ferrers posets. Besides, we show that Young diagrams can be used to recover some properties of 2-symmetric measures in an elegant and easy way.

The rest of the paper goes as follows: we introduce the basic concepts about k-symmetric measures and order polytopes in next section. In Section 3 we introduce Young diagrams and Ferrers posets, together with the Hook theorem, that constitutes the basis of our algorithm for random generation. Section 4 shows the relation between 2-symmetric measures and these combinatorial objects. In Section 5, we develop the algorithm for random generation of 2-symmetric measures. Section 6 presents an application of these results, related to the problem of identification of general fuzzy measures. We finish with conclusions and open problems.

2 Basic concepts on k-symmetric measures and order polytopes

2.1 The set $\mathcal{FM}(A_1, A_2)$

Let $X = \{1, \ldots, n\}$ be a finite referential set of n elements and let us denote by $A, B, \ldots$ the subsets of X. The set of subsets of X is denoted by $\mathcal{P}(X)$. A **fuzzy measure** [39] over X is a set function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying $\mu(\emptyset) = 0, \mu(X) = 1$, (boundary conditions) and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$ (monotonicity). Depending on the context, elements of X have different meanings (criteria in Multicriteria Decision Making, players in Game Theory, and so on). We will denote by $\mathcal{FM}(X)$ the set of all fuzzy measures over X.

Fuzzy measures have proved to be an interesting tool in many different fields, as they are able to model many different situations; for example, in Multicriteria Decision Making fuzzy measures are able to model interactions among criteria, as well as situations of veto or favor [16]. On the other hand, this flexibility has to be paid via an increase in the complexity; in this sense, for a referential set of n elements, $2^n - 2$ values are needed to completely define the fuzzy measure. In order to reduce this complexity, several attempts have been proposed. For example, we could just reduce the number of coalitions allowed to happen; this situation is perfectly justified in many cases in Game Theory (see e.g. [17]). Another way to reduce the complexity is to consider some subfamilies of fuzzy measures, aiming to combine flexibility with a reduction in the number of coefficients; following this line, many subfamilies have been proposed, as for example k-additive measures [15, 16], k-symmetric measures [31], k-intolerant measures [29], and many others.

Let us introduce k-symmetric measures. For this concept, the notion of subset of indifference is the crux of the issue.

Definition 1. *[31] Let $\mu : \mathcal{P}(X) \to [0, 1]$ be a fuzzy measure. Given a subset $A \subseteq X$, we say that A is a* **subset of indifference** *for μ if for any $B_1, B_2 \subset A$, such that $|B_1| = |B_2|$ and for any $C \subseteq X \setminus A$, then*

$$\mu(B_1 \cup C) = \mu(B_2 \cup C).$$

From the point of view of Game Theory, this definition translates the idea that we do not care about which players inside A are included in a coalition, and we just need to know how many of them take part in it. Remark that if A is a subset of indifference, so is any $B \subseteq A$. Moreover, for a given fuzzy measure μ, X can be partitioned in $\{A_1, ..., A_k\}$ subsets of indifference; there are several partitions of X in subsets of indifference, but it can be proved that there is a partition that is the coarsest one [31], being any other refinement of this partition another partition in subsets of indifference for μ.

Definition 2. *[31] Let $\mu \in \mathcal{FM}(X)$. We say that μ is a $k-$***symmetric measure** *if and only if the (unique) coarsest partition of the referential set in subsets of indifference has k non-empty subsets.*

In particular, if we consider the partition $\{X\}$ we recover the set of symmetric fuzzy measures, i.e. measures μ satisfying $\mu(A) = \mu(B)$ whenever $|A| = |B|$. These measures and their corresponding Choquet integral are very important in Fuzzy Logic, where they are related to OWA operators [41].

If $\{A_1, A_2, \cdots, A_k\}$ is a partition of X, the set of all fuzzy measures μ such that $\{A_1, A_2, \cdots, A_k\}$ is a partition of indifference for μ is denoted by $\mathcal{FM}(A_1, A_2, \cdots, A_k)$. Note that $\mu \in \mathcal{FM}(A_1, A_2, \cdots, A_k)$ does not imply that $\{A_1, A_2, \cdots, A_k\}$ is the coarsest partition for μ; indeed, all symmetric measures belong to $\mathcal{FM}(A_1, A_2, \cdots, A_k)$, no matter the partition.

As all elements in a subset of indifference have the same behavior, when dealing with a fuzzy measure in $\mathcal{FM}(A_1, ..., A_k)$, it suffices to know the number of elements of each A_i that belong to a given subset C of the universal set X. Therefore, the following result holds:

Lemma 1. *[31] If $\{A_1, ..., A_k\}$ is a partition of X, then any $C \subseteq X$ can be identified with a k-dimensional vector $(c_1, ..., c_k)$ with $c_i := |C \cap A_i|$.*

Then, $c_i \in \{0, ..., |A_i|\}$ and in order to build a $k-$symmetric measure we just need $(|A_1| + 1) \times (|A_2| + 1) \times \cdots \times (|A_k| + 1) - 2$ coefficients, a number far away from $2^n - 2$ needed for a general fuzzy measure.

2.2 Order polytopes

Let us now turn to the problem of random generation of fuzzy measures. For this problem, we have to note that the set $\mathcal{FM}(X)$ is a polytope, and so are many subfamilies of $\mathcal{FM}(X)$; in particular, it is an easy exercise [7] to show that $\mathcal{FM}(A_1, A_2, \cdots, A_k)$ is a polytope. As explained in the introduction, there are several methods to generate random points in a polytope; one of these methods is the *triangulation method*; let us explain this method with more detail.

Consider $n+1$ affine independent points in $\mathbf{R}^m, m \geq n$, i.e. $n+1$ points of $\mathbf{R}^m$ in general position. The convex hull of these points is called a **simplex**. This notion is a generalization of the notion of triangle for the m-dimensional space. The random generation in simplices is very simple and fast [35], as any point in the simplex can be written as a unique convex combination of the vertices. Thus, it just suffices to generate such convex combination, and this can be done, e.g., by ordering the points and sampling uniformly n times in $[0, 1]$, assigning each value to the corresponding point and finally normalizing.

The triangulation method is based on the decomposition of the polytope into simplices; once the decomposition is obtained, we assign to each simplex a probability proportional to its volume; next, these probabilities are used for selecting one of the simplices; finally, a random m-uple in the simplex is generated.

However, in general it is not easy to split a polytope into simplices. Moreover, even if we are able to decompose the polytope in a suitable way, we have to deal with the problem of determining the volume of each simplex in order to properly select one of them. This is the *Achilles heel* of the triangulation method and the reason for which it is not very popular for general polytopes. In our case, we take advantage of the fact that $\mathcal{FM}(X)$ and $\mathcal{FM}(A_1, A_2, \cdots, A_k)$ belong to a very special class of polytopes called **order polytopes**.

The notion of order polytope is deeply related to finite partially ordered sets (posets for short). For a general introduction on the theory of posets see [11, 32]). A (finite) poset, denoted by $(P, \preceq)$, is a finite set P with p elements endowed with a partial order $\preceq$ (a reflexive, antisymmetric and transitive relation); we will often denote the poset $(P, \preceq)$ by P for short. Posets can be represented through *Hasse diagrams*. The Hasse diagram is a graph where for $a, b \in P$, $a \prec b$ if and only if there is a sequence of connected lines upwards from a to b (see Fig. 1).

Given a poset $(P, \preceq)$, it is possible to associate to P a polytope $O(P)$ over $\mathbb{R}^p$, called the **order polytope** of P (cf. [36]), formed by the p-uples f of real numbers indexed by the elements of P satisfying

- $0 \leq f(a) \leq 1$ for every a in P, and
- $f(a) \leq f(b)$ whenever $a \preceq b$ in P.

Thus, the polytope $O(P)$ consists of (the p-uples of images of) the order-preserving functions from P to $[0, 1]$. $O(P)$ is a 0/1-polytope [36], i.e., its extreme points are all in $\{0, 1\}^p$. In fact, it is easy to see that the extreme points of $O(P)$ are exactly (the characteristic functions of) the filters of P (i.e., subsets F of P satisfying that if $x \in F$ and $x \preceq y$, then $y \in F$). Applied to distributive lattices, the notion of order polytope has been also defined in [27] with the name of **geometric realization**.

It can be easily seen that the polytope $\mathcal{FM}(X)$ is the order polytope of the poset $(P, \preceq)$ where $P = \mathcal{P}(X) \backslash \{X, \emptyset\}$ and $\preceq$ is the inclusion between subsets [8], i.e., the Boolean lattice. Similarly, the set $\mathcal{FM}(A_1, ..., A_k)$ is the order polytope associated to the poset $(P(A_1, ..., A_k), \preceq)$, where

$$P(A_1, ..., A_k) := \{(i_1, ..., i_k) : i_j \in \{0, ..., |A_j|\}, i, j \in \mathbb{Z}\} \backslash \{(0, ..., 0), (|A_1|, ..., |A_k|)\}, \qquad (1)$$

and $\preceq$ is given by $(c_1, ..., c_k) \preceq (b_1, ..., b_k) \Leftrightarrow c_i \leq b_i, i = 1, ..., k$ (see [8]). The Hasse diagrams of the Boolean lattice of $|X| = 4$ and $P(\{1, 2\}, \{3, 4\})$ are given in Figure 1. Notice the difference in the number of elements between these two posets.

Thus, the problem of random generation of points in $\mathcal{FM}(X)$ and $\mathcal{FM}(A_1, ..., A_k)$ reduces to obtain a random procedure for generating points in an order polytope. Now, when dealing with order polytopes, the following result can be applied (see [30], pag. 304):

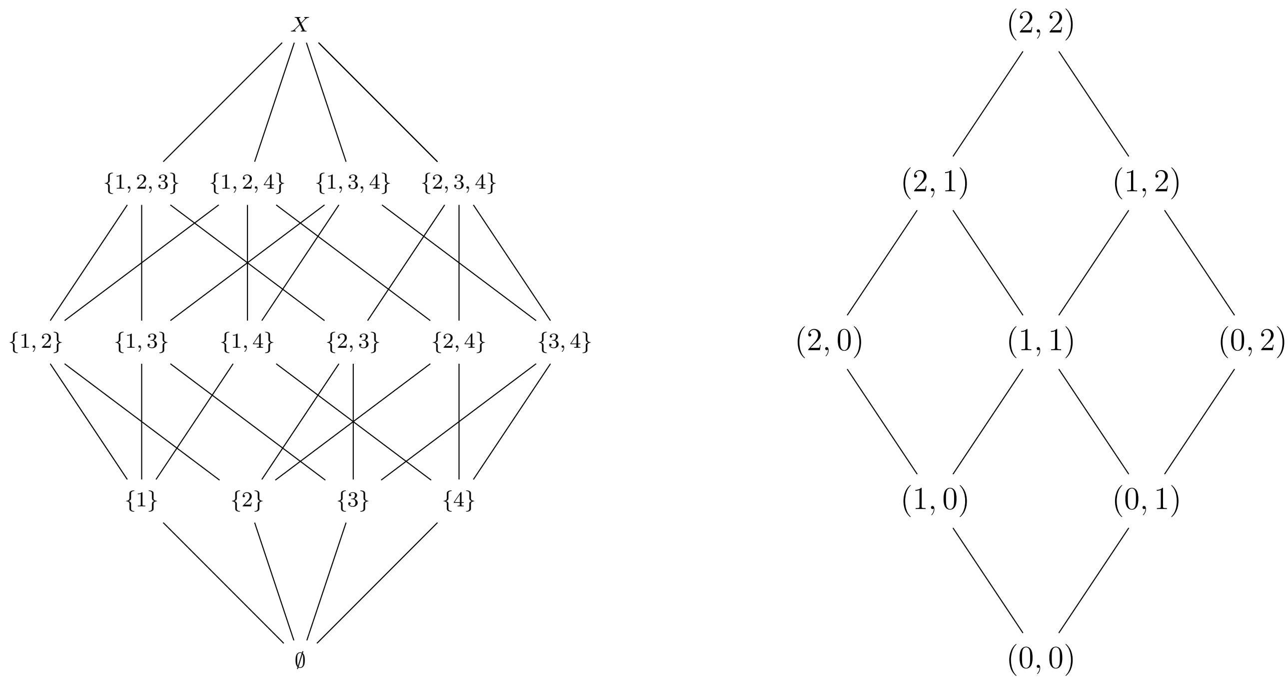
Theorem 1. *Let $(P, \preceq)$ be a poset of p elements.*

- *If $\preceq$ is a total order on P, then the corresponding order polytope is a simplex of volume $\frac{1}{p!}$.*
- *For any partial ordering $\preceq$ on P, the simplices of the order polytope of $(P, \leq)$, where $\leq$ is a linear extension[1] of $\preceq$, cover the order polytope of $(P, \preceq)$ and have disjoint interiors. Consequently, $vol(O(P, \preceq)) = \frac{1}{p!}e(P)$, where $e(P)$ denotes the number of linear extensions of poset $(P, \preceq)$.*

These results are also outlined in [36]. Consequently, it suffices to generate randomly a linear extension of $\preceq$ and then generate a point in the corresponding simplex. However, the problem of generating a random linear extension of a general poset is a $\sharp$ P-complete problem [3]. The case of the Boolean poset is especially complex, and in fact it can be proved that any poset can be seen as a subposet of the Boolean lattice [37]. Therefore, obtaining procedures with low complexity valid for a class of order polytopes is relevant.

[1]A linear extension of a poset $(P, \preceq)$ is a total order $\leq$ on P that is compatible with $\preceq$, in the sense that $x \preceq y$ implies $x \leq y$.

Figure 1: Hasse diagrams of the Boolean poset for $|X| = 4$ (left) and the poset $P(\{1,2\},\{3,4\})$ (right).



3 Young diagrams and Ferrers posets

In this section, we are going to introduce the basic properties of Young diagrams. Young diagrams are not only a useful combinatorial structure but also play an important role in other branches of mathematics such as representation theory or Schubert calculus [22].

Consider a natural number n. A **partition** of n (do not confuse this concept with the partition of a set) is a sequence of natural numbers $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 1$ and $\sum_{i=1}^{k} \lambda_i = n$. Each partition represents a way to obtain n as a sum of natural numbers. Partitions of n are represented through Young diagrams (see e.g., [33, 1] for the basic properties of this object).

Definition 3. *Let $\boldsymbol{\lambda}$ be a partition of a natural number $n \in \mathbb{N}$. Then, we define the* **Young diagram** *(or* **Ferrers diagram***) of shape $\boldsymbol{\lambda}$, $\mathbf{X}(\boldsymbol{\lambda})$, as an array of cells, $x_{i,j}$, arranged in left-justified rows, $i = 1, \cdots, k$ and $j = 1, \ldots, \lambda_i$.*

Example 1. *For instance, if $\boldsymbol{\lambda} = (5, 4, 4, 1)$ we obtain the Young diagram for $n = 14$ shown in Figure 2 left.*

Definition 4. *Let $\boldsymbol{\lambda}$ be a partition of n. A* **Young tableau** *(plural* **Young tableaux***) of shape $\boldsymbol{\lambda}$ is an assignment of the integers $1, \ldots, n$ to the cells in the diagram of shape $\boldsymbol{\lambda}$. A Young tableau is said to be* **standard** *if all rows and columns form increasing sequences.*

Example 2. *A (standard) Young tableau for the Young diagram of shape $\boldsymbol{\lambda} = (5, 4, 4, 1)$ is presented in Figure 3.*

Let us now introduce Hook-length formula. For this, some previous concepts are needed.

Definition 5. *Let $\mathbf{X}(\boldsymbol{\lambda})$ be a Young diagram of shape $\boldsymbol{\lambda}$. For a cell (i, j) in the diagram, the* **Hook** *of cell (i, j), denoted $H_{\boldsymbol{\lambda}}(i, j)$, consists of the cells that are either below (i, j) in column j or to the right of (i, j) in row i, along with (i, j) itself. That is:*

Figure 2: Young diagram associated to $\boldsymbol{\lambda} = (5, 4, 4, 1)$ and its corresponding Ferrers poset.

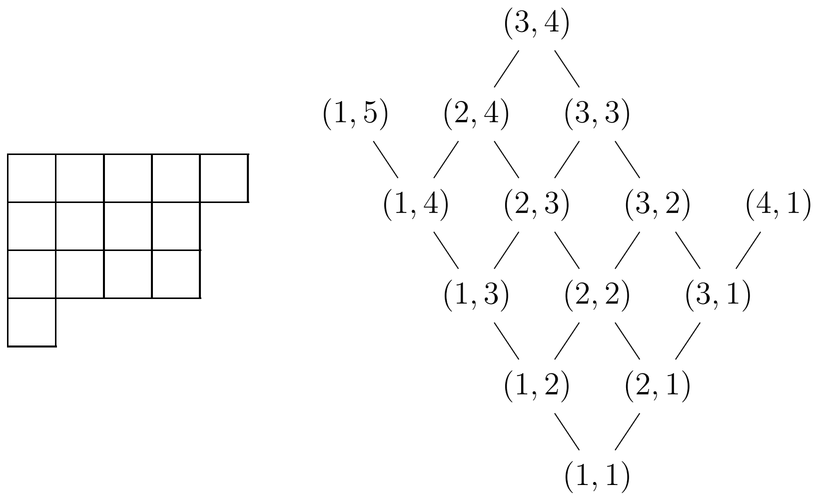


Figure 3: (Standard) Young tableau of shape (5,4,4,1).

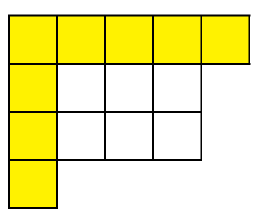
1	4	6	7	10
2	8	11	12	
3	9	13	14	
5				

$$H_{\boldsymbol{\lambda}}(i, j) := \{(a, b) \in \mathbf{X}(\boldsymbol{\lambda}) | (a = i, b \geq j) \vee (a \geq i, b = j)\}.$$

The **Hook length** $h_{\boldsymbol{\lambda}}(i, j)$ *is the number of cells in the hook* $H_{\boldsymbol{\lambda}}(i, j)$*.*

Example 3. *For the example with* $\boldsymbol{\lambda} = (5, 4, 4, 1)$*, the set* $H(1, 1)$ *is drawn in Figure 4.*

Figure 4: Hook $H_{\boldsymbol{\lambda}}(1, 1)$ with $\boldsymbol{\lambda} = (5, 4, 4, 1)$.



In this case, $h_{\boldsymbol{\lambda}}(1, 1) = 8$*.*

The number of standard Young tableaux of shape $\boldsymbol{\lambda}$ can be written in terms of the Hook lengths of the different cells. This is the famous Hook-length formula.

Theorem 2. *[33, 1] Let* $\boldsymbol{\lambda}$ *be a partition of* n *and* $d_{\boldsymbol{\lambda}}$ *be the number of standard Young tableaux of shape* $\boldsymbol{\lambda}$*. Then,*

$$d_{\boldsymbol{\lambda}} = \frac{n!}{\prod_{(i,j) \in \mathbf{X}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(i, j)}.$$

Finally, let us introduce Ferrers posets. For a given Young diagram, we can build a poset $(P, \preceq)$ in the following way. Elements of P are the cells (written as the pair of coordinates) and given two cells $(i, j), (k, l)$, we define

$$(i, j) \preceq (k, l) \Leftrightarrow i \leq k, j \leq l.$$

This poset is known as the **Ferrers poset** associated to shape $\boldsymbol{\lambda}$, denoted $\mathbf{P}(\boldsymbol{\lambda})$. We also call **normalized Ferrers poset**, $\mathbf{P}^*(\boldsymbol{\lambda})$ to the poset resulting from removing maximum and minimum from $\mathbf{P}(\boldsymbol{\lambda})$.

For example, the Ferrers poset associated to shape $(5, 4, 4, 1)$ can be seen in Figure 2 right.

Remark 1.

- *Note that the same Ferrers poset arises if we swap rows and columns and build the corresponding Young diagram, i.e., the Ferrers poset is the same for shapes* $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$ *and* $\boldsymbol{\eta} = (\eta_1, ..., \eta_l)$ *where* $\eta_i := \sharp\{\lambda_j : \lambda_j \geq i\}$.

- *Remark also that* (i, j) *covers* $(i-1, j)$ *and* $(i, j-1)$. *In terms of the Young diagram, this means that a cell covers just the cell to the left and the cell to the top. Following the terminology of graphs, we will say that two cells are* **adjacent** *if one of them covers the other in the corresponding Ferrers poset, i.e., when they are consecutive cells in a column or a row.*

  We define a **path** *in a Young diagram as a sequence of adjacent cells. Finally, we will say that a subset* F *of cells in the Young diagram is* **connected** *if there is a path inside* F *connecting any pair of cells in* F. *This will play an important role in the following section.*

Now, for a fixed shape, observe that any standard Young tableau can be related to a possible linear extension of the corresponding Ferrers poset, where the labels give the position of each element in the linear extension (label one corresponds to the first element, label two to the second one and so on); reciprocally, given a linear extension of the Ferrers poset, we can build a standard Young tableau just assigning to each cell the position of the cell in the linear extension. Therefore, we have as many standard Young tableaux as linear extensions of the Ferrers poset and applying Theorem 2, the following holds for the number of linear extensions.

Corollary 1. *Let* $\boldsymbol{\lambda}$ *be a partition of* n*, then:*

$$e(\mathbf{P}(\boldsymbol{\lambda})) = \frac{n!}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}.$$

4 $\mathcal{FM}(A_1, A_2)$ as an order polytope

We start with a fundamental property relating $\mathcal{FM}(A_1, A_2)$ to the normalized Ferrers poset $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} := (|A_1| + 1, |A_1| + 1, \overset{|A_2|+1}{\ldots}, |A_1| + 1)$.

Lemma 2. *Let* $\{A_1, A_2\}$ *be a partition of the referential set* X *and let us denote* $|A_1| = a_1$ *and* $|A_2| = a_2$. *The set* $\mathcal{FM}(A_1, A_2)$ *can be seen as an order polytope associated to the normalized Ferrers poset* $\mathbf{P}^*(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} := (a_1 + 1, a_1 + 1, \overset{a_2+1}{\ldots}, a_1 + 1)$.

Proof. As explained before (Eq. (1)), the set $\mathcal{FM}(A_1, A_2)$ is an order polytope whose subjacent poset is $P(A_1, A_2)$. Let us consider $P^+(A_1, A_2)$, where we have added $\emptyset$ and X, given by $(0, 0)$ and

(a_1, a_2), resp. to $P(A_1, A_2)$; then, $P^+(A_1, A_2)$ can be identified to a $(a_1+1) \times (a_2+1)$ table, where entry (i, j) corresponds to element $(i-1, j-1)$. But this table is the Young diagram of shape $\boldsymbol{\lambda} = (a_1+1, a_1+1, \overset{a_2+1}{\cdots}, a_1+1)$. Thus, $P^+(A_1, A_2)$ can be associated to $\mathbf{P}(\boldsymbol{\lambda})$ and consequently, $P(A_1, A_2)$ can be associated to $\mathbf{P}^*(\boldsymbol{\lambda})$. □

Swapping positions of A_1 and A_2, we conclude that $\mathcal{FM}(A_1, A_2)$ can be associated to the Ferrers poset $\mathbf{P}(\boldsymbol{\eta})$ with $\boldsymbol{\eta} = (a_2+1, a_2+1, \overset{a_1+1}{\cdots}, a_2+1)$. Compare this with Remark 1.

An interesting consequence which follows from the last theorem gives us the exact volume of the polytope $\mathcal{FM}(A_1, A_2)$.

Theorem 3. *Let $\{A_1, A_2\}$ be the coarsest partition of X in subsets of indifference and let $|A_1| = a_1$ and $|A_2| = a_2$. Then,*

$$Vol\,(\mathcal{FM}(A_1, A_2)) = [(a_1+1)(a_2+1)]\,[(a_1+1)(a_2+1)-1] \prod_{k=0}^{a_1} \frac{k!}{(a_2+1+k)!}$$

Proof. By Theorem 1, we know that for an order polytope,

$$vol(O(P, \preceq)) = \frac{1}{|P|!} e(P),$$

where $e(P)$ denotes the number of linear extensions of poset $(P, \preceq)$. Thus, it suffices to find the number of linear extensions of the subjacent poset. For $\mathcal{FM}(A_1, A_2)$, we have to sort out the number of linear extensions of $P(A_1, A_2)$. Note that the number of linear extensions of this poset is the same as the polytope $P^+(A_1, A_2)$ defined in Lemma 2 or equivalently, to poset $\mathbf{P}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} := (a_1+1, a_1+1, \overset{a_2+1}{\cdots}, a_1+1)$. Now, by Corollary 1, for a shape $\boldsymbol{\lambda}$,

$$e(\mathbf{P}(\boldsymbol{\lambda})) = \frac{[(a_1+1)(a_2+1)]!}{\prod_{(i,j) \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(i, j)}.$$

Let us then find the values of $h_{\boldsymbol{\lambda}}(i, j)$ in this case. Note that for row i, the values of $h_{\boldsymbol{\lambda}}(i, j)$, where $j = 1, ..., |A_2|+1$ are $|A_1|+1-i+|A_2|+1, ..., |A_1|+1-i+1$ resp. (see Figure 5),

Figure 5: Hook lengths for a rectangular Young tableau.

7	6	5	4	3
6	5	4	3	2
5	4	3	2	1

Thus,

$$\prod_{(i,j) \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(i, j) = \prod_{i=1}^{a_1+1} \frac{(|A_1|+1+|A_2|+1-i)!}{(|A_1|+1-i)!} = \prod_{k=0}^{a_1} \frac{(|A_2|+1+k)!}{k!}.$$

Consequently,

$$
\begin{aligned}
Vol\,(\mathcal{FM}(A_1, A_2)) &= \frac{1}{[(a_1+1)\times(a_2+1)-2]!} e(P(A_1, A_2)) \\
&= \frac{1}{[(a_1+1)\times(a_2+1)-2]!} e(P^+(A_1, A_2)) \\
&= \frac{1}{[(a_1+1)\times(a_2+1)-2]!} [(a_1+1)(a_2+1)]! \prod_{k=0}^{a_1} \frac{k!}{(a_2+1+k)!} \\
&= [(a_1+1)(a_2+1)]\,[(a_1+1)(a_2+1)-1] \prod_{k=0}^{a_1} \frac{k!}{(a_2+1+k)!}.
\end{aligned}
$$

□

Some values of this volume for different values of a_1 and a_2 can be seen in the next table.

$a_1 \backslash a_2$	1	2	3	4	5
1	1	$\frac{6\cdot5}{3!4!}$	$\frac{8\cdot7}{4!5!}$	$\frac{10\cdot9}{5!6!}$	$\frac{12\cdot11}{6!7!}$
2		$\frac{9\cdot8\cdot2!}{3!4!5!}$	$\frac{12\cdot11\cdot2!}{4!5!6!}$	$\frac{15\cdot14\cdot2!}{5!6!7!}$	$\frac{18\cdot17\cdot2!}{6!7!8!}$
3			$\frac{16\cdot15\cdot2!3!}{4!5!6!7!}$	$\frac{20\cdot19\cdot2!3!}{5!6!7!8!}$	$\frac{24\cdot23\cdot2!3!}{6!7!8!9!}$
4				$\frac{25\cdot24\cdot2!3!4!}{5!6!7!8!9!}$	$\frac{30\cdot29\cdot2!3!4!}{6!7!8!9!10!}$
5					$\frac{36\cdot35\cdot2!3!4!5!}{6!7!8!9!10!11!}$

It must be remarked at this point that obtaining the volume of an order polytope is a complex problem depending on the number of linear extensions (and counting linear extensions is another complex problem), so that usually bounds are considered. In this case, some bounds for the volume of $\mathcal{FM}(A_1, A_2)$ were given in [8].

Young tableaux also gives an interesting combinatorial approach to the adjacency in $\mathcal{FM}(A_1, A_2)$.

Definition 6. *Given a $n \times m$ grid, a* **staircase walk** *is a path from $(0,0)$ to (n,m) which uses just up and right steps.*

An example of a staircase walk in a 4×4 grid can be seen in Figure 6. Staircase walks play an important role in Combinatorics, since they are intimately linked to Catalan numbers [38].

Let us now show the relationship between vertices and staircase walks, and apply it to characterize adjacency in $\mathcal{FM}(A_1, A_2)$.

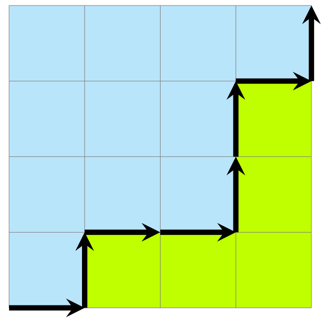
Proposition 1. *Let $\{A_1, A_2\}$ be the coarsest partition of X in subsets of indifference and let $|A_1| = a_1$ and $|A_2| = a_2$. Then,*

i) There is a bijection between the set of vertices of $\mathcal{FM}(A_1, A_2)$ and the staircase walks on a $(a_1+1)\times(a_2+1)$ grid which do not cross $(0, a_2+1)$ nor $(a_1+1, 0)$.

ii) $\mathcal{FM}(A_1, A_2)$ has $\binom{a_1+a_2+2}{a_1+1} - 2$ vertices.

iii) Let F_1 and F_2 be two vertices of $\mathcal{FM}(A_1, A_2)$. F_1 and F_2 are adjacent if and only if the set of cells between the two associated staircase paths is connected, as defined in Remark 1.

Figure 6: Staircase walk in a 4×4 grid.



Proof.

i) By Lemma 2, we know that $\mathcal{FM}(A_1, A_2)$ is an order polytope associated to the normalized Ferrers poset $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} := (a_1+1, a_1+1, \overset{a_2+1}{\ldots}, a_1+1)$. Since $\mathcal{FM}(A_1, A_2)$ is an order polytope, its vertices are the characteristic functions of the filters of $\mathbf{P}^*(\boldsymbol{\lambda})$. Let us then characterize filters in this poset. For this, note that $(i, j) \leq (k, l)$ in $\mathbf{P}^*(\boldsymbol{\lambda})$ means $i \leq k, j \leq l$. Consequently, a filter of $\mathbf{P}(\boldsymbol{\lambda})$ translates in the Young diagram as a set of cells satisfying that if $(i, j) \in F$ then $H_{\boldsymbol{\lambda}}(i, j) \subseteq F$. Therefore, a filter defines a group of cells whose border is given by a staircase path from $(a_1+1, 1)$ to $(1, a_2+1)$, or equivalently, a path in a $(a_1+1) \times (a_2+1)$ grid from $(0, 0)$ to (a_1, a_2). For example, the cells appearing to the right of the staircase in Figure 6 define a filter of $\mathbf{P}^*(\boldsymbol{\lambda})$. Finally, we have to take into account that the underlying poset is $\mathbf{P}^*(\boldsymbol{\lambda})$ instead of $\mathbf{P}(\boldsymbol{\lambda})$, whence we have to remove two possible paths, namely the path crossing $(0, a_2+1)$ and the path crossing $(a_1+1, 0)$.

ii) It suffices to remark that the number of staircase walks in a $(a_1+1) \times (a_2+1)$ grid is given by

$$\binom{(a_1+1)+(a_2+1)}{a_1+1} = \binom{a_1+a_2+2}{a_1+1}.$$

By *i)*, we just should remove 2 staircase walks, whence the result.

iii) To prove this, we will use the fact that in an order polytope two vertices are adjacent to each other if and only if their associated filters satisfy either $F_1 \subset F_2$ or $F_2 \subset F_1$ and the difference $F_1 \setminus F_2$ or $F_2 \setminus F_1$ is connected [8]. Consider two vertices in these conditions and let us assume $F_2 \subset F_1$; suppose $F_1 \setminus F_2$ is connected as poset. Note that by *i)* the elements in $F_1 \setminus F_2$ correspond to the cells between the two staircase paths associated to both filters. Now, given two cells $(i_1, j_1), (i_2, j_2)$ in $F_1 \setminus F_2$, there exists a path

$$(i_1, j_1) = a_1 - a_2 - \ldots - a_r = (i_2, j_2)$$

in $F_1 \setminus F_2$ such that either a_i covers a_{i+1} or the other way round. On the other hand, (i, j) covers (k, l) if and only if either $i = k+1$ or $j = l+1$. But this means that cells $(i+1, j+1)$ and $(k+1, l+1)$ are adjacent in $\mathbf{P}(\boldsymbol{\lambda})$. Thus, connection of $F_1 \setminus F_2$ is equivalent to connection of the corresponding set of cells $F_1 \setminus F_2$ in the Young diagram, in the sense of Remark 1.

□

5 A procedure for random generation in $\mathcal{FM}(A_1, A_2)$

In this section, we tackle the problem of generating measures in $\mathcal{FM}(A_1, A_2)$ in a random way. As explained in Section 2, this is equivalent to generate a linear extension in a random way for the underlying poset $P(A_1, A_2)$. Now, as explained in Section 4, generating a linear extension of $P(A_1, A_2)$ is equivalent to generate a linear extension of $\mathbf{P}^*(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (|A_1|+1, |A_1|+1, \overset{|A_2|+1}{\dots}, |A_1|+1)$; and finally, as stated in Section 3, this is equivalent to generate a standard Young tableau for a rectangular Young diagram of shape $\boldsymbol{\lambda}$. The main difficulty here is the following: once an element i is selected as the last one in the linear extension of poset $\mathbf{P}^*$, this element is removed and we have to select another one from the poset $\mathbf{P}^* \backslash \{i\}$ to be the element before i in the linear extension; and this has to be done in a way such that all linear extensions have the same probability.

Definition 7. *Given a Young diagram of shape $\boldsymbol{\lambda}$, we say that a position (i, j) is* **maximal** *if there is no $(k, l) \neq (i, j)$ in the diagram satisfying $i \leq k, j \leq l$.*

Example 4. *For the Young diagram in Figure 2, it can be seen that we have three maximal cells, namely (1,5), (3,4) and (4,1).*

Note that at each step, we aim to select a maximal element of the diagram. For a $(|A_1|+1) \times (|A_2|+1)$ rectangular Young diagram, the only maximal element in the linear extension is the one in position $(|A_1|+1, |A_2|+1)$. Next step is to select the previous element in the linear extension and two candidates (maximal elements) arise: $(|A_1|, |A_2|+1)$ and $(|A_1|+1, |A_2|)$; the question now is: which probability should be assigned to select $(|A_1|, |A_2|+1)$? This translates to step i: once several elements have been selected, we have some other positions that are candidates to be chosen as the previous element in the linear extension; which are the corresponding probabilities for each of them?

At this point, it should be noted that when an element is selected to be the last one, we obtain a new Young diagram (not necessarily a rectangular one), as next lemma shows:

Lemma 3. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of n. If a maximal cell is removed from the $i-th$ row in $\mathbf{X}(\boldsymbol{\lambda})$, then we obtain another Young diagram $\mathbf{X}(\boldsymbol{\lambda}')$ with $\lambda'_i = \lambda_i - 1, \lambda'_j = \lambda_j, j \neq i$.*

Proof. Since $\mathbf{X}(\boldsymbol{\lambda})$ is a Young diagram and the removed element is a maximal element, it follows that below and on the right side of the the cell associated to this maximal element there is no cell. Then, $\lambda_i > \lambda_{i+1}$, whence $\lambda'_i = \lambda_i - 1 \geq \lambda_{i+1} = \lambda'_{i+1}$. We conclude that $\boldsymbol{\lambda}'$ is a partition of $n-1$ and $\mathbf{X}(\boldsymbol{\lambda}')$ is its corresponding Young diagram. □

Consequently, the problem reduces to obtain a procedure to select a maximal element in a general Young diagram so that we obtain a random standard Young tableau. The main result of the section solves this question.

Theorem 4. *Let $\mathbf{P}(\boldsymbol{\lambda})$ be a Ferrers poset with n elements and $x \in \mathbf{P}(\boldsymbol{\lambda})$ be a maximal element. Then, the probability of x being the last element in a linear extension of $\mathbf{P}(\boldsymbol{\lambda})$ is given by:*

$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{1}{n} \prod_{z \in H^{-1}_{\boldsymbol{\lambda}}(x)} \frac{h_{\boldsymbol{\lambda}}(z)}{h_{\boldsymbol{\lambda}}(z) - 1},$$

where $H^{-1}_{\boldsymbol{\lambda}}(x) = \{z \in \mathbf{P}(\boldsymbol{\lambda}) \mid x \in H_{\boldsymbol{\lambda}}(z)$ and $z \neq x\}$.

Proof. We start noting that

$$e\left(\mathbf{P}(\boldsymbol{\lambda})\right) = \sum_{x \text{ maximal of } \mathbf{P}(\boldsymbol{\lambda})} e\left(\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}\right)$$

and

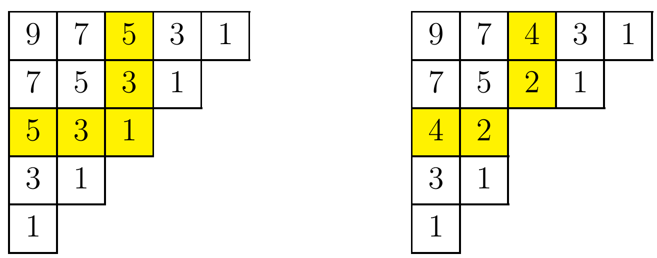
$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{e\left(\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}\right)}{e\left(\mathbf{P}(\boldsymbol{\lambda})\right)}.$$

From Lemma 3, $\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}$ is a Ferrers poset with partition $\boldsymbol{\lambda}'$, with $\lambda'_i = \lambda_i - 1, \lambda'_k = \lambda_k, k \neq i$ if $x = (i, j)$. Then, both $e\left(\mathbf{P}(\boldsymbol{\lambda}) \setminus \{x\}\right)$ and $e\left(\mathbf{P}(\boldsymbol{\lambda})\right)$ can be computed from Corollary 1, whence

$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{\dfrac{(n-1)!}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda}')} h_{\boldsymbol{\lambda}'}(z)}}{\dfrac{n!}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}} = \frac{1}{n} \frac{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda}')} h_{\boldsymbol{\lambda}'}(z)}.$$

Observe that if we remove a maximal element from a Young diagram, then the only elements with different hook lengths are the elements in the same row or in the same column as the removed maximal element (see Figure 7). Therefore,

Figure 7: Hook lengths before (left) and after (right) removing a maximal element.



$$P\left(x \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{1}{n} \frac{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda})} h_{\boldsymbol{\lambda}}(z)}{\prod_{z \in \mathbf{P}(\boldsymbol{\lambda}')} h_{\boldsymbol{\lambda}'}(z)} = \frac{1}{n} \frac{\prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} h_{\boldsymbol{\lambda}}(z)}{\prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} h_{\boldsymbol{\lambda}'}(z)} = \frac{1}{n} \prod_{z \in H_{\boldsymbol{\lambda}}^{-1}(x)} \frac{h_{\boldsymbol{\lambda}}(z)}{h_{\boldsymbol{\lambda}'}(z)},$$

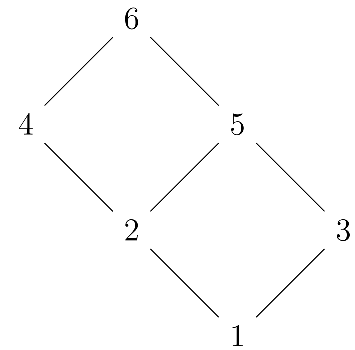
where in the last step we have used that $h_{\boldsymbol{\lambda}}(x) = 1$ because it is a maximal element.

Finally, since we have removed x we obtain $h_{\boldsymbol{\lambda}'}(z) = h_{\boldsymbol{\lambda}}(z) - 1$ for every element z in the same row or column as x. Therefore, we get the desired formula. □

Theorem 4 provides the probability of selecting a maximal element as the previous element in a linear extension at each step. Therefore, we can state the following procedure for deriving a random standard Young tableau and thus, a random 2-symmetric measure.

Figure 8: Standard Young tableau associated to $\boldsymbol{\lambda} = (3, 3)$ (left) and its Ferrers poset (right).

1	2	3
4	5	6



SAMPLING ALGORITHM FOR 2-SYMMETRIC MEASURES $\mathcal{F}M(A_1, A_2)$

1. Consider a Young diagram of shape $\boldsymbol{\lambda} = (|A_1|+1, |A_1|+1, \overset{|A_2|+1}{\cdots}, |A_1|+1)$.
2. **Sampling algorithm for standard Young tableaux.**
   (a) Select a maximal cell (i, j) of the Young diagram with probability
   $$P\left((i,j) \mid \mathbf{P}(\boldsymbol{\lambda})\right) = \frac{1}{n} \prod_{(k,l) \in H_{\boldsymbol{\lambda}}^{-1}(i,j)} \frac{h_{\boldsymbol{\lambda}}(k,l)}{h_{\boldsymbol{\lambda}}(k,l) - 1}.$$
   (b) Remove this cell and repeat the previous step for the new Young diagram.
3. Given the standard Young tableau obtained in the previous step, build the corresponding linear extension of the Ferrers poset, thus obtaining a linear extension ϵ of the poset $(P, \preceq)$.
4. Remove the first $(0, 0)$ and last element $(|A_1|+1, |A_2|+1)$ of the linear extension. Thus, we obtain a linear extension ϵ^* of the poset $\mathbf{P}^*(\boldsymbol{\lambda})$.
5. **Sampling the values of the 2-symmetric measure.**
   (a) Generate a $[(a_1+1)(a_2+1) - 2]$ vector $\vec{u}$ of random variables $U(0, 1)$.
   (b) Sort $\vec{u}$, to get a $[(a_1+1)(a_2+1) - 2]$ vector $\vec{v}$ with the values generated in the previous step in increasing order.
   (c) Assign the value $v[k]$ to $\mu(i, j)$ if the element associated to the cell $(i+1, j+1)$ is placed at position k in ϵ^*. Let us denote this by $\epsilon^*[(i+1, j+1)] = k$.

Example 5. *Let $X = \{1, 2, 3\}$, $A_1 = \{1, 2\}$ and $A_2 = \{3\}$. Therefore $n = 3, a_1 = 2$ and $a_2 = 1$. Then by Lemma 2, this polytope is associated to the Young diagram of shape $\boldsymbol{\lambda} = (3, 3)$. The initial Young diagram and its Ferrers poset (with its labeling) are given in Figure 8.*

- *At the beginning we have just one maximal cell, namely $(2, 3)$ (element 6 in the Ferrers poset), so we remove it and $\epsilon((2, 3)) = 6$. That is, the element associated to subset $(2, 3)$ is at position 6 in the linear extension ϵ. Update $\boldsymbol{\lambda}' = (3, 2)$.*

- *Now* $(2,2)$ *and* $(1,3)$ *are maximal cells, corresponding to elements 5 and 3 in the Ferrers poset. We compute its probabilities through Theorem 4.*

$$P\left((2,2) \mid \mathbf{P}(\boldsymbol{\lambda}')\right) = \frac{1}{5}\cdot\frac{3}{2}\cdot\frac{2}{1} = \frac{3}{5}, \quad P\left((1,3) \mid \mathbf{P}(\boldsymbol{\lambda}')\right) = \frac{1}{5}\cdot\frac{3}{2}\cdot\frac{4}{3} = \frac{2}{5}.$$

  Suppose $(2,2)$ *is selected,* $\epsilon(2,2) = 5$*. Update* $\boldsymbol{\lambda}'$ *to* $\boldsymbol{\lambda}'' = (3,1)$*.*

- *Now* $(2,1)$ *and* $(1,3)$ *are maximal cells, corresponding to elements 3 and 4 in the Ferrers poset. We compute its probabilities through Theorem 4.*

$$P\left((2,1) \mid \mathbf{P}(\boldsymbol{\lambda}'')\right) = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}, \quad P\left((1,3) \mid \mathbf{P}(\boldsymbol{\lambda}'')\right) = \frac{1}{4}\cdot\frac{2}{1}\cdot\frac{4}{3} = \frac{2}{3}.$$

  Suppose $(2,1)$ *is selected,* $\epsilon(2,1) = 4$*. Update* $\boldsymbol{\lambda}''$ *to* $\boldsymbol{\lambda}''' = (3,0) \equiv 3$*.*

- *The remaining poset is a chain, so we have just one maximal in each step and* $\epsilon(1,3) = 3$*,* $\epsilon(1,2) = 2$ *and* $\epsilon(1,1) = 1$*. Then we get the standard tableau shown in Figure 8 (left). Therefore,* $\epsilon = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$ *or with the poset labeling* $\epsilon = \{1,2,4,3,5,6\}$*.*

Now we obtain ϵ^* *from* ϵ *by removing the maximum and minimum,* $\epsilon^* = \{(1,2),(1,3),(2,1),(2,2)\} = \{2,4,3,5\}$*.*

Finally, we have to generate 4 *variables* $U(0,1)$ *in order to sample inside the simplex* ϵ^**. Using any mathematical software (for example R), we obtain say*

$$(0.77, 0.65, 0.73, 0.09).$$

Now, $v = (0.09, 0.65, 0.73, 0.77)$*, and* μ *is given by:*

$$\mu(0,0) = 0.00,\ \mu(0,1) = 0.09\ \mu(0,2) = 0.65,\ \mu(1,0) = 0.73,\ \mu(1,1) = 0.77\ \mu(1,2) = 1.00.$$

Next, let us study the computational complexity.

Proposition 2. *The computational complexity of the previous algorithm is* $O(n^2)$*, where* $n = (a_1 + 1)(a_2 + 1)$*.*

Proof. We will count the number of operations that the algorithm performs to obtain a random 2-symmetric measure.

- **Initializing**: Build the initial hook matrix. The complexity is n, since we just have to save $n = (a_1+1)\cdot(a_2+1)$ numbers in memory, corresponding to the hook length of the n cells.

- **Updating position vector**: When an element is removed, we should identify the new maximal cells. The number of maximal elements is at most $min\{a_1+1, a_2+1\}$ and we repeat this search n times, one for each iteration, we need say $(a_1+1)\cdot n \leq n^2$ computations at most.

- **Updating hook matrix**: When removing an element, we have to subtract 1 to the elements above and to the left of it. The number of such cells is limited by $(a_1+1)+(a_2+1) \leq n$. And taking it for the n iterations, we obtain the upper bound n^2.

- **Computing probability vectors for maximals**: For each combination of maximals we should compute the probability of each maximal.

  First, let us assume w.l.g. that $a_1 \leq a_2$, and that this holds for the Young diagram arising at each iteration. Note that the number of maximals is bounded by 1 in the first iteration, by 2 for iterations 2,3, by 3 for iterations 4, 5, 6 and so on. Next, after iteration $1+2+3+4+...+a_1$ the number of maximals is bounded by a_1+1 (the number of lines of the initial Young diagram) and this remains for $(a_1+1)(a_2-a_1)+1$ iterations, i.e., until the last $1+2+3+4+...+a_1$ iterations, where the number of maximals is again bounded by $a_1, a_1-1, ...$ Now, for maximal (i,j), the number of products to compute the probability of this maximum is $2*(i+j-2)+1$. Therefore, we have to compute

$$\sum_{t=1}^{n} \sum_{\{(i,j) \text{ } maximal \text{ } in \text{ } iteration \text{ } t\}} [2*(i+j-2)+1] .$$

  We are going to split this sum into the two parts mentioned above.

  For the first $1+2+3+4+...+a_1$ iterations, we know that there are a_1 kinds of iterations depending on the bound for the number of maximals. Moreover, for the kind of iteration p there are p of these iterations (as many as cells in the diagonal), and in each of these iteration there are at most p maximals; for each maximal we should do $2 \cdot (i+j-2)+1 \leq 2(a_1+a_2)$ computations at most. This also applies for the last $1+2+3+4+...+a_1$ iterations, so we have to double the bound.

$$2\sum_{p=1}^{a_1} 2(a_1+a_2)p^2 = 4(a_1+a_2)\frac{a_1(a_1+1)(2a_1+1)}{6} \leq \frac{16}{6}\cdot(a_1+1)^3(a_2+1) \leq 3\cdot(a_1+1)^2(a_2+1)^2 = 3n^2.$$

  For the second part, we work with the $(a_1+1)(a_2-a_1)+1$ iterations related to the cells in the middle of the Young tableau. In each of these iterations, there are at most (a_1+1) maximals and for each maximal we should do $2 \cdot (i+j-2)+1 \leq 2(a_1+a_2)$ computations at most. Therefore we have:

$$2(a_1+a_2)\cdot(a_1+1)^2\cdot(a_2-a_1+1) \leq 4\cdot(a_1+1)^2(a_2+1)^2 = 4n^2,$$

  whence the quadratic complexity is obtained.

- **Sampling**: We sample n uniform random variables. Obviously, it takes a complexity of n.

By adding these steps we observe that the important part is the computation of the probability vector which reaches a quadratic complexity. □

6 Application: the problem of identification

In this last section, let us present an application of the previous results. First, we need to introduce Choquet integral.

Definition 8. *The* **Choquet integral** *of a function*

$$f : X \to \mathbb{R}^+$$

with respect to a fuzzy measure μ *on* X *is defined by*

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n}(f((i)) - f((i-1)))\mu(B_i),$$

where $\{(1), \ldots, (n)\}$ *is a permutation of the set* $\{1, \ldots, n\}$ *satisfying*

$$0 = f((0)) \leq f((1)) \leq ... \leq f((n)),$$

and

$$B_i = \{(i), ..., (n)\}.$$

Consider a problem in the framework of Multicriteria Decision Making; then, we have a set of $|X| = n$ criteria and we have to choose between a set of objects; each object is given an overall value y from the partial evaluations on each criterion, denoted $x_1, ..., x_n$; it could be the case that y, $(x_1, ..., x_n)$ or both of them are ordinal values, not necessarily numerical; we assume that the problem can be modeled via a Choquet model, i.e., we assume that y is the Choquet integral [4] of a function f defined by $f(i) = x_i, i = 1, ..., n$, with respect to an unknown fuzzy measure μ (possibly restricted to a subfamily of fuzzy measures). The goal is to identify the fuzzy measure μ modeling this situation and for this, we have a sample of m objects for which we know both the partial scores $(x_1^i, ..., x_n^i), i = 1, ...m$, and the overall score $y_i, i = 1, ..., m$ (possibly affected by some random noise). Then, we look for a fuzzy measure (not necessarily unique) that best fits these data. If the quadratic error is considered, this amounts to looking for a fuzzy measure μ_0 minimizing

$$F(\mu) := \sum_{i=1}^{m} \left(\mathcal{C}_\mu(x_1^i, ..., x_n^i) - y_i\right)^2,$$

where $\mathcal{C}_\mu$ denotes Choquet integral. Several techniques have been proposed to solve this problem [21, 42, 2]. One of them is based on genetic algorithms [14]. Genetic algorithms are general optimization methods based on the theory of natural evolution; starting from an initial population, at each iteration some individuals are selected with probability proportional to their fitness (measured according to the function that we want to optimize) and new individuals are generated from them using a cross-over operator. These new individuals replace the old ones (their parents) and the process continues until an optimum is found or the maximum number of generations is reached; besides, some other individuals may mutate via a mutation operator. Finally, the best individual in the last population is returned as a possible solution to the problem.

In [6], a procedure based on genetic algorithms has been proposed. In this procedure, the cross-over operator is given by the convex combination between parents and the mutation operator is given by a convex combination between the selected measure and another one. The algorithm seems to be fast and work properly. However, it has the drawback that the search region reduces in each iteration; and this forbids to go back if the solution of the problem is left outside in an iteration. To avoid this, the natural option is to take advantage of the polyhedral structure of $\mathcal{FM}(X)$ and consider the set of vertices of $\mathcal{FM}(X)$ as the initial population; and use a vertex selected at random for the mutation operator. However, the number of vertices of $\mathcal{FM}(X)$ is too large to be used in practice [6]. Therefore, we have to look for another solution and a natural one is to consider a random sample of $\mathcal{FM}(X)$; in this case, the problem relies on the fact that no random procedure for generating points in $\mathcal{FM}(X)$ has been developed. In this section, we are going to develop a procedure based on the previous results on 2-symmetric measures.

To apply the previous results on 2-symmetric measures to this problem, the following result is the cornerstone. It tells that the whole set of fuzzy measures $\mathcal{FM}(X)$ can be recovered from 2-symmetric measures, just allowing different partitions. First, let us define

$$\mathcal{FMS}_2 := \bigcup_{A\subseteq X} \mathcal{FM}(A, A^c),$$

the set of all fuzzy measures being at most 2-symmetric for a partition.

Proposition 3. *Let X be a referential set of n elements and consider $\mu \in \mathcal{FM}(X)$. Then:*

There exists $k, k' \in \mathbb{N}$ such that μ can be written as

$$\mu = \mu_1 \vee \mu_2 \vee \cdots \vee \mu_k$$

and also

$$\mu = \mu_1^* \wedge \mu_2^* \wedge \cdots \wedge \mu_{k'}^*$$

where $\mu_i \in \mathcal{FMS}_2$, $\forall i \in 1, \cdots, k$ and $\mu_i^* \in \mathcal{FMS}_2$, $\forall i \in 1, \cdots, k'$.

Proof.

$$\mu = \bigvee_{\emptyset\subset A\subset X} \mu_A,$$

where μ_A is defined by

$$\mu_A(C) := \begin{cases} \mu(A) & \text{if} \quad C \supseteq \text{A}, \\ 0 & \text{otherwise} \end{cases}$$

As $\mu_A \in \mathcal{FM}(A, A^c)$, the result holds.

For the second statement, note that

$$\mu = \bigwedge_{\emptyset\subset A\subset X} \mu_A^*,$$

where μ_A^* is given by

$$\mu_A^*(C) := \begin{cases} \mu(A) & \text{if} \quad A \supseteq \text{C}, \\ 1 & \text{otherwise} \end{cases}$$

As $\mu_A^* \in \mathcal{FM}(A, A^c)$, the result holds.

□

Based on the previous proposition, we could modify the procedure in [6], and consider as cross-over operator

$$\mu_1 \oplus_\lambda \mu_2 := \lambda\left(\mu_1 \vee \mu_2\right) + (1-\lambda)\left(\mu_1 \wedge \mu_2\right), \ \lambda \in [0,1], \tag{2}$$

instead of the convex combination. This cross-over operator shares similar properties to the convex combination, in the sense that it is not necessary to check at each iteration whether the children are inside the set $\mathcal{FM}(X)$. In other words, we can achieve any element of $\mathcal{FM}(X)$ from a proper random initial population of elements in $\mathcal{FMS}_2$.

To study the behavior of this algorithm, we have conducted a simulation study. We have studied two different situations.

- **Case 1: Identification of 2-symmetric measures for a fixed partition.** We try to identify a measure $\mu \in \mathcal{FM}(A, A^c)$ chosen at random. The data to identify the measure are the 3^n points consisting in all possible n-vectors $(x_1, ..., x_n)$, where $x_i \in \{0, 0.5, 1\}$ (this is the same data as considered in [6]). For any possible vector, the overall value y is the corresponding Choquet integral with respect to μ. The initial population consists in 30 measures in $\mathcal{FM}(A, A^c)$ selected at random via the algorithm presented in the previous section. At each step, the population splits into three groups. The first group consists in the measures with the highest scores (lowest mean Choquet errors $F(\mu)$) computed with the first quartile. The second group includes the elements with the lowest scores, using the third quartile. Finally, the third group are the rest of the elements. The cross-over operator is the convex combination. In each step, we apply the cross-over operator to groups 2 and 3. The mutation operator is as follows: the elements of group 2 are replaced by new ones generated at random from $\mathcal{FM}(A, A^c)$; In addition, groups 1 and 3 are mutated by adding gaussian noise on some coordinates chosen at random. As the algorithm progresses, the three types of population change and better scored elements arrive to group 1. This method stops when the mean Choquet error is below a threshold ($\epsilon = 10^{-8}$) or a maximum number of iterations is achieved.

- **Case 2: Identification of general fuzzy measures.** In this case, we try to identify a measure $\mu \in \mathcal{FM}(X)$ chosen at random. For this generation we can use some high complexity algorithm such as an acceptance-rejection method. As in the previous case, the data are the 3^n points consisting in all possible n-vectors $(x_1, ..., x_n)$, where $x_i \in \{0, 0.5, 1\}$ and the overall value y is the corresponding Choquet integral with respect to μ. The initial population is a set of $n = 30$ 2-symmetric fuzzy measures in $\mathcal{FMS}_2$ selected at random; for this, we have first to choose a partition $\{A, A^c\}$ at random, and this can be done using the volume of each $\mathcal{FM}(A, A^c)$ (that has been obtained in Corollary 1) to know the probability of each partition; next, once A is fixed, we generate a measure in $\mathcal{F}M(A, A^c)$ via the algorithm proposed in the previous section. The population again splits into three groups as defined above. The rest of the algorithm is also similar as above but selecting the cross-over operator given by (2). At each step, we apply the cross-over operator for a value of λ selected at random in [0,1] to groups 2 and 3. The mutation operator is also similar as above. The elements of group 2 are replaced by the new ones generated at random from $\mathcal{FMS}_2$. In addition, groups 1 and 3 are mutated by adding gaussian noise on some coordinates chosen at random. This method stops when the mean Choquet error is below 10^{-8} or a maximum number of iterations is achieved.

We have repeated both models 50 times for each size n. The mean identification errors for the different referential sizes are given in Table 1. From these values, it can be seen that the algorithm seems to perform very well in both cases.

Obviously, when we identify a general fuzzy measure by using an initial random population of 2-symmetric measures we get a higher identification error. However, this error is acceptable even for big values of n. Moreover, this algorithm get lower errors than other methods in the literature (see [6]). This way we have managed to achieve a way of avoiding the computational problems related to the huge number of vertices in $\mathcal{FM}(X)$ and identifying fuzzy measures by using the combinatorial properties of $\mathcal{FMS}_2$.

Identification Algorithm Performance		
Referencial Size	2-Symmetric	Fuzzy measures
$n = 3$	1.388e-05	2.653e-05
$n = 4$	3.918e-05	4.128e-05
$n = 5$	6.365e-05	2.413e-04
$n = 6$	7.831e-05	4.480e-04
$n = 7$	8.134e-05	7.279e-04
$n = 8$	1.915e-04	9.011e-04
$n = 9$	2.353e-04	9.545e-04
$n = 10$	2.561e-04	1.025e-03

Table 1: Identification errors.

7 Conclusions and open problems

In this paper we have developed an algorithm to generate 2-symmetric measures with respect to a fixed partition in a random way. The novelty in the paper is the use of combinatorial tools, in this case Young diagrams, to sort out the problem. The procedure proposed in the paper is appealing from a visual point of view, easy to implement and has a reduced computational cost. Moreover, we have obtained several results related to the geometrical structure of $\mathcal{FM}(A, A^c)$, as the volume or the adjacency structure. All these results have been applied in the last section to the problem of identification of fuzzy measures. Two types of algorithms have been proposed. In the first case we identify 2-symmetric random measures by using genetic algorithms with the convex combination as cross-over operator. In the second case, we use Proposition 3 to construct a new cross-over operator which let us identify general fuzzy measures with a low mean Choquet error. Although these results do not provide us with a random sampler in $\mathcal{FM}(X)$ because a fuzzy measure can be put as a maximum or minimum of 2-symmetric measures in different ways, this could be a startpoint for a fast pseudo-random procedure.

Of course, next step should be extending this procedure for general k-symmetric measures. However, many technical problems arise, showing the difficulties that random generation prompts. The first one is to extend Hook theorem for three-dimensional Young diagrams; this is a very complex problem that has not been solved yet; indeed, generalizations of Hook theorem for (2-dimensional) Young diagrams where some cells inside the diagram are not considered has not been solved in general (see shifted tableaux [1]).

Another interesting problem arises from Proposition 3 and consits in determining the minimum number $k(n)$ of 2-symmetric measures needed to write a fuzzy measure $\mu \in \mathcal{FM}(X)$ as maximum or minimum of 2-symmetric measures. In this sense, it can be shown that $k(n) = k'(n), \forall n \in \mathbb{N}$ and the first three values of $k(n)$ are $k(1) = k(2) = 1$ and $k(3) = 2$. However, the problem of computing the exact value of $k(n)$ for a general natural number seems to be a deep and complex combinatorial problem.

Finally, we have characterized when two vertices in $\mathcal{FM}(A_1, A_2)$ are adjacent. Thus, given a vertex $\mu \in \mathcal{FM}(A_1, A_2)$, how can we count the number of adjacent vertices to μ?

Acknowledgements

This paper has been supported by the Spanish Grant MTM-2015-67057.

References

[1] J. Bandlow. An elementary proof of the hook formula. *The Electronic Journal of Combinatorics*, (15), 2008. Research paper 45.

[2] G. Beliakov, R. Mesiar, and L. Valášková. Fitting generated aggregation operators to empirical data. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12(2):219–236, 2004.

[3] G. Brightwell and P. Winkler. Counting linear extensions. *Order*, 8(3):225–242, 1991.

[4] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, (5):131–295, 1953.

[5] E. F. Combarro, I. Díaz, and P. Miranda. On random generation of fuzzy measures. *Fuzzy Sets and Systems*, 228:64–77, 2013.

[6] E. F. Combarro and P. Miranda. Identification of fuzzy measures from sample data with genetic algorithms. *Computers and Operations Research*, 33(10):3046–3066, 2006.

[7] E. F. Combarro and P. Miranda. On the polytope of non-additive measures. *Fuzzy Sets and Systems*, 159(16):2145–2162, 2008.

[8] E. F. Combarro and P. Miranda. Adjacency on the order polytope with applications to the theory of fuzzy measures. *Fuzzy Sets and Systems*, 180:384–398, 2010.

[9] E. F. Combarro and P. Miranda. Characterizing isometries on the order polytope with an application to the theory of fuzzy measures. *Information Sciences*, 180:384–398, 2010.

[10] E. F. Combarro and P. Miranda. On the structure of the k-additive fuzzy measures. *Fuzzy Sets and Systems*, 161:2314–2327, 2010.

[11] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, 2002.

[12] D. Denneberg. *Non-additive measures and integral*. Kluwer Academic, Dordrecht (The Netherlands), 1994.

[13] L. Devroye. *Non-uniform random variate generation*. Springer-Verlag, New York, 1986.

[14] D. E. Goldberg. *Genetic algorithms in search, optimization and machine learning*. Addison-Wesley, 1989.

[15] M. Grabisch. k-order additive discrete fuzzy measures. In *Proceedings of 6th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU)*, pages 1345–1350, Granada (Spain), 1996.

[16] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, (92):167–189, 1997.

[17] M. Grabisch. Ensuring the boundedness of the core of games with restricted cooperation. *Annals of Operations Research*, 191:137–154, 2011.

[18] M. Grabisch. *Set functions, games and capacities in Decision Making*, volume 46 of *Theory and Decision Library*. Springer, 2016.

[19] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Cambridge University Press, 2009.

[20] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy Measures and Integrals- Theory and Applications*. Number 40 in Studies in Fuzziness and Soft Computing. Physica–Verlag, Heidelberg (Germany), 2000.

[21] M. Grabisch and J.-M. Nicolas. Classification by fuzzy integral-performance and tests. *Fuzzy Sets and Systems, Special Issue on Pattern Recognition*, (65):255–271, 1994.

[22] P. Griffiths and J. Harris, editors. *Principles of Algebraic Geometry*. Wiley Classics Library. Wiley, 1994.

[23] M. Huber. Fast perfect sampling from linear extensions. *Discrete Mathematics*, 306:420–428, 2006.

[24] A.D. Kalvin and Y.L. Varol. On the generation of all topological sortings. *J. Algorithms*, (4):150–162, 1983.

[25] A. Karzanov and L. Khachiyan. On the conductance of order Markov chains. *Order*, 8(1):7–15, 1995.

[26] D.E. Knuth and J. Szwarcfiter. A structured program to generate all tpological sorting arrangements. *Inform. Process. Lett.*, (2):153–157, 1974.

[27] G. Koshevoy. Distributive lattices and product of capacities. *Journal of Mathematical Analysis and Applications*, 219:427–441, 1998.

[28] J. Leydold and W. Hörmann. A Sweep-Plane Algorithm for Generating Random Tuples in Simple Polytopes. *J. Math. Comp.*, (67):1617–1635, 1998.

[29] J.-L. Marichal. Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral. *European Journal of Operational Research*, 155(3):771–791, 2004.

[30] Jiri Matousek. *Lectures on Discrete Geometry*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.

[31] P. Miranda, M. Grabisch, and P. Gil. p-symmetric fuzzy measures. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10 (Suppl.):105–123, 2002.

[32] J. Neggers and H. S. Kim. *Basic posets*. World Scienctific, 1998.

[33] I. Pak, J.C. Novelli, and A. Stoyanovskii. A direct bijective proof of the hook-length formula. *Discrete Mathematics and Theoretical Computer Science*, (1):53–67, 1997.

[34] G. Pruesse and F. Ruskey. Generating linear extensions fast. *SIAM Journal on Computing*, (23):373–386, 1994.

[35] R.Y. Rubinstein and D.P. Kroese. *simulation and the Monte Carlo method*. Wiley-Interscience, 2007.

[36] R. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

[37] R. Stanley. *Enumerative Combinatorics.* Cambridge University Press, Cambridge (UK), 2012.

[38] R. Stanley. *Catalan numbers.* Cambridge University Press, Cambridge (UK), 2015.

[39] M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, 1974.

[40] Y.L. Varol and D. Rotem. An algorithm to generate all topological sorting arrangements. *Comput. J.*, (24):83–84, 1981.

[41] R. R. Yager. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man and Cybernetics*, (18):183–190, 1988.

[42] J. Wang Z. Wang, K. Xu and G. Klir. Using genetic algorithms to determine nonnegative monotone set functions for information fusion in environments with random perturbation. *International Journal of Intelligent Systems*, 14:949–962, 1999.