Bottom-Up: A new algorithm to generate random linear extensions of a poset

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Abstract

In this paper we present a new method for deriving a random linear extension of a poset. This new strategy combines Probability with Combinatorics and obtains a procedure where each minimal element of a sequence of subposets is selected via a probability distribution. The method consists in obtaining a weight vector on the elements of P, so that an element is selected with a probability proportional to its weight. From some properties on the graph of adjacent linear extensions, it is shown that the probability distribution can be obtained solving a linear system; the number of equations involved in this system relies on the number of what we have called positioned antichains, that allows a reduced number of equations; finally, we give some examples of the applicability of the algorithm. This procedure cannot be applied to any poset, but it is exact when it can be used. Moreover, the method is quick and easy to implement. Besides, it allows a simple way to derive the number of linear extensions of a given poset.

Keywords: Poset; linear extension; random generation.

1 Introduction

One of the most interesting problems when dealing with posets is to generate a linear extension in a random fashion (see for example [8]). Brightwell and Winkler have shown in [2] that, for general posets, this problem, as well as the problem of counting the number of linear extensions of a poset, are ♯ P-complete problems, and consequently, it is not possible to derive a satisfactory procedure for solving these problems for general posets.

On the other hand, this problem has drawn the attention of many researchers and many algorithms have been proposed to cope with it (see e.g. [6, 25, 11, 20] for generating a random linear extension of a poset and [18, 16, 12] for counting the number of linear extensions).

In this sense, an interesting research line is to provide an algorithm that allows to solve the problem for general posets and such that the computational cost for each step is bounded; these algorithms are called loopless-free and several algorithms fulfilling this condition have been proposed in the literature (see e.g. [13]).

Another interesting line to deal with the problem is based on Markov chains and follows the work of Karzanov and Khachiyan [12] (see also [14] for an introduction on mixing Markov chains). The idea is to generate a sequence of linear extensions and consider the n-th term of the sequence; it has been proved [10] that it is possible to obtain a bound n such that for the n-th element in the sequence, all linear extensions of the poset have the same probability of appearence, no matter the initial linear extension considered; thus, a way to generate a linear extension is to follow the sequence until the n-th term. Some works in this line can be seen e.g. in [1, 3].

Interestingly enough, the problem of generating a linear extension of a poset in a random fashion appears in other domains. For example, consider the set of non-additive measures [24, 4, 7]; non-additive measures are a fundamental tool appearing in many different fields, as for example Multicriteria Decision Making or Cooperative Game Theory, among many others (see [9]); the set of all non-additive measures on a finite referential is a bounded polytope; now, consider the problem of generating a random non-additive measure; there are several methods to generate points in a polytope, as the grid method [8], the sweep-plane method [15] and triangulation methods [8]; it can be easily proved that the set of non-additive measures is an order polytope (see [22] for the concept and properties of these polytopes) whose subjacent poset is the Boolean poset; then, triangulations methods are specially appealing for order polytopes because, as it is proved in [17] pag. 304, the problem of generating a point in an order polytope can be turned into generating a linear extension of the subjacent poset in a random way.

The main difficulty when trying to generate a random linear extension relies in the combinatorial nature of the problem. Usually, the quantities appearing in the problem are very large and grow very fast when the cardinality of the poset grows, and then are considered untractable (see unsolvable) for generating a linear extension in a random way.

In this paper we present a new procedure for generating linear extensions, that we have called Bottom-Up method. Briefly speaking, we look for a vector of weights w^* on the elements of P, so that an element has a probability to be selected as the next element of the linear extension given by the quotient between its weight and the sum of all elements that can be selected for this position. The problem we solve in the paper is how to obtain a weight vector so that any linear extension has the same probability of being obtained. We will see that this can be done solving a linear system of equations; moreover, this system does not depend on the number of linear extensions, but on the number of what we have called positioned antichains, whose number is usually very small compared with the number of linear extensions. As it will become apparent below, the algorithm is simple and fast; besides, it allows to compute the number of linear extensions of the poset very easily. Moreover, once w^* is obtained, the computational cost of deriving a random extension is very reduced and thus, this procedure is specially appealing when several linear extensions are needed. On the other hand, this method cannot be applied to any poset and indeed, we show that the existence of a suitable weight w^* depends on whether a linear system has infinite solutions. We show some examples of families of posets where it can be applied, illustrating the procedure; we also provide some sufficient conditions on the polytope to admit such a weight vector and study how this weight vector can be obtained.

The rest of the paper is presented as follows: In order to fix the notation and to be self-contained, in next section we give the basic definitions and facts that will be needed to explain the procedure. The Bottom-Up method is developed in Section 3. Section 4 shows some examples of families of polytopes that allow the existence of a weight vector. We finish with the conclusions and open problems. In order to avoid missing the thread of the paper, the technical proof of Theorem 3 is given in an appendix.

2 Basic concepts and tools on posets

For a general introduction on the theory of posets see [5, 19]. Let us consider a finite set P endowed with a partial order $\preceq$ (a reflexive, antisymmetric and transitive relation). The pair $(P, \preceq)$ is a **partial order set** or **poset** for short; with some abuse of notation, we will usually omit $\preceq$ and write P instead of $(P, \preceq)$ when referring to posets. Elements of P are denoted x, y and so on, and also $a_1, a_2, ...$; if $|P| = n$, we will also use the notation $P = \{1, ..., n\}$. Subsets of P are denoted by capital letters $A, B, ...$; a poset can be represented through *Hasse diagrams*. An element x such that $x \not\prec y, \forall y \in P$ is called a **maximal element**; similarly, if x is such that $y \not\prec x, \forall y \in P, x$ is called a **minimal element**; we will denote by $\mathcal{M}(P)$ the set of minimal elements of poset P and $m(P) = |\mathcal{M}(P)|$. We say that y **covers** x, denoted $x \lessdot y$, if $x \preceq y$ and there is no $z \in P \setminus \{x, y\}$ satisfying $x \preceq z \preceq y$. For an element x, we define its **level** recursively as follows: maximal elements are in level 0, denoted L_0; then, maximal elements of $P \setminus L_0$ are in level L_1; in general, L_i is the set of maximal elements of $P \setminus (L_0 \cup ... \cup L_{i-1})$. For a poset P, we can define the **dual poset** $P^\partial = (P, \preceq_\partial)$ such that $x \preceq_\partial y \Leftrightarrow y \preceq x$.

A **chain** is a poset such that $\preceq$ is a total order; we will denote the chain of n elements by $\mathbf{n}$; similarly, an **antichain** is a poset where $\preceq$ is given by $x \preceq y \Leftrightarrow x = y$; we will denote the antichain of n elements by $\bar{n}$.

Given an element x, we denote by $\downarrow x$ the subposet of P whose elements are $\{y : y \preceq x\}$ and by $\downarrow \hat{x} :=\downarrow x \setminus \{x\}$; similarly, we denote by $\uparrow x$ the subposet of P whose elements are $\{y : x \preceq y\}$ and by $\uparrow \hat{x} :=\uparrow x \setminus \{x\}$. These notions can be extended for a general subset A, thus obtaining $\downarrow A, \uparrow \hat{A}, \downarrow A$ and $\uparrow \hat{A}$. Finally, we will denote by $\updownarrow A$ the set of elements related to any element of A and by $\updownarrow \hat{A} =\updownarrow A \setminus A$. An **ideal** or **downset** I of P is a subset of P such that if $x \in I$ then $\downarrow x \subseteq I$; we will denote the set of all ideals of P by $\mathcal{I}(P)$ and $i(P) = |\mathcal{I}(P)|$. Symmetrically, a subset F of P is a **filter** or **upset** if for any $x \in F$ and any $y \in P$ such that $x \preceq y$, it follows that $y \in F$; we denote by $\mathcal{F}(P)$ the set of filters of P.

Two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ are **isomorphic** if there is a bijection $f : P \to Q$ such that $x \preceq_P y \Leftrightarrow f(x) \preceq_Q f(y)$. If two posets are isomorphic, then their corresponding Hasse diagrams are the same up to differences in the names of the elements.

Two elements $x, y \in P$ are said to be **interchangeable** if there is an automorphism $f : P \to P$ such that $f(x) = y$ and $f(y) = x$.

The **direct sum** of two posets $(P, \preceq_P), (Q, \preceq_Q)$, denoted $P \oplus Q$ is a poset $(P \cup Q, \preceq_{P \oplus Q})$ where $x \preceq_{P \oplus Q} y$ whenever $x, y \in P$ and $x \preceq_P y$, or $x, y \in Q$ and $x \preceq_Q y$, or $x \in P, y \in Q$. A poset is **irreducible** by direct sum if it cannot be written as a direct sum of two posets. Similarly, the **disjoint union** of two posets $(P, \preceq_P), (Q, \preceq_Q)$, denoted $P \uplus Q$ is a poset $(P \cup Q, \preceq_{P \uplus Q})$ where $x \preceq_{P \uplus Q} y$ whenever $x, y \in P$ and $x \preceq_P y$, or $x, y \in Q$ and $x \preceq_Q y$. A poset which cannot be written as disjoint union of two posets is called **connected**. Obviously, the Hasse diagram of a connected poset is also a connected graph.

A **linear extension** of $(P, \preceq)$ is a sorting of the elements of P that is compatible with $\preceq$, i.e. $x \preceq y$ implies that x is before y in the sorting. Linear extensions will be denoted ϵ_1, ϵ_2 and so on and the i-th element of ϵ is denoted $\epsilon(i)$. We will denote by $\mathcal{L}(P)$ the set of all linear extensions of poset $(P, \preceq)$ and by $e(P) = |\mathcal{L}(P)|$. The aim of the paper is to generate a linear extension in a random fashion. Two linear extensions are said to be related by a **transposition** if they are identical except for the swapping of two elements; if these elements are consecutive, the linear extensions are related by an **adjacent transposition**.

For $\mathcal{L}(P)$ two graphs can be assembled: the first one is the **transposition graph**, denoted $(\mathcal{L}(P), \tau)$; this graph has all the elements in $\mathcal{L}(P)$ as vertices and edges exist between linear extensions that are related by a transposition. The second one is the **adjacent transposition graph**, denoted $(\mathcal{L}(P), \tau^*)$, where edges exist between linear extensions that are related by an adjacent transposition. It is well-known [21] that both graphs are connected.

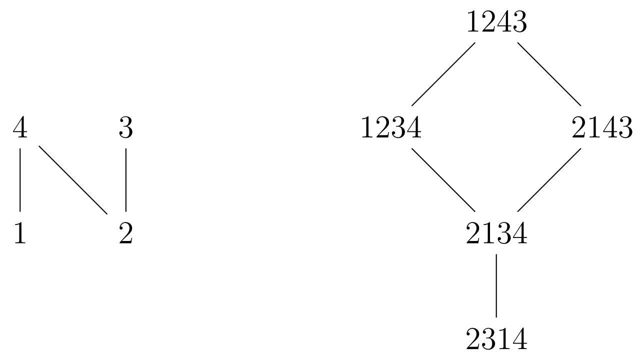
When working with finite posets, it is sometimes convenient to denote elements as natural numbers. A **labeling** is a bijective mapping $L : \{1, 2, \ldots, |P|\} \to P$ (see [19]). There are $n!$ ways to define a labeling. A labeling is **natural** if $x \preceq y$ implies $L^{-1}(x) \leq L^{-1}(y)$ with the natural numbers order. It is well-known that every finite poset admits a natural labeling.

Example 1. *Consider the poset N, given by four elements $1, 2, 3, 4$ and whose corresponding Hasse diagram is given in Figure 1 left. The linear extensions of this poset N are*

$$(1, 2, 3, 4), (1, 2, 4, 3), (2, 1, 3, 4), (2, 1, 3, 4), (2, 1, 4, 3), (2, 3, 1, 4)$$

and the corresponding transposition graph and adjacent transposition graph (they are the same for this poset) are given in Figure 1 right. Note that we have used a natural labeling for N.

Figure 1: N poset and its (adjacent) transposition graph.



3 Bottom-Up method

Let us consider a poset $(P, \preceq)$ and let us treat the problem of building a linear extension. The first element of the linear extension is a minimal element of P, say x_1; next, an element in $\mathcal{M}(P \backslash \{x_1\})$ is selected, say x_2; then, a minimal element of $P \backslash \{x_1, x_2\}$, and so on. In order to generate a random linear extension, the problem we have to face is the way each minimal element is selected. In this section, we develop a procedure that assigns to each element of the poset a weight value, so that the probability of selecting this element in a step is proportional to the quotient between its weight and the sum of the weights of all minimal elements of the corresponding subposet. The basic steps of our algorithm are:

BOTTOM-UP ALGORIHM

1. INITIALIZATION: For any $x \in P$, assign a weight value $w_x > 0$.
2. MAIN STEP: Starting with $P' = P$:
   (a) Select a minimal element x of P' with probability the quotient between w_x and the sum of the weights of $\mathcal{M}(P')$.
   (b) Remove this element and repeat the previous step for the new poset $P' \backslash \{x\}$.

Lemma 1. *Let $w = (w_1, ..., w_n)$ be a weight vector such that $w_i > 0 \, \forall i$, and suppose that we derive a linear extension via the previous algorithm. Then, we obtain a probability distribution on $\mathcal{L}(P)$.*

Proof: By construction, we know that $P(\epsilon) > 0$ for any $\epsilon \in \mathcal{L}(P)$. Thus, it suffices to show $\sum_{\epsilon \in \mathcal{L}(P)} P(\epsilon) = 1$. We will prove it by induction on $|P|$.

For $|P| = 2$ we have a chain or an antichain, and the result trivially holds in both cases.

Assume the result holds for $|P| \leq n$ and consider the case $|P| = n+1$. Let $m_1, ..., m_r$ be the minimal elements of P. Then, one of them is the first element in the linear extension and thus,

$$\sum_{\epsilon \in \mathcal{L}(P)} P(\epsilon) = \sum_{i=1}^{r} P(\epsilon(1) = m_i) \sum_{\epsilon' \in \mathcal{L}(P \backslash \{m_i\})} P(\epsilon') = \sum_{i=1}^{r} \frac{w_{m_i}}{\sum_{i=1}^{r} w_{m_r}} \sum_{\epsilon' \in \mathcal{L}(P \backslash \{m_i\})} P(\epsilon').$$

Now, applying the induction hypothesis, $\sum_{\epsilon' \in \mathcal{L}(P \backslash \{m_i\})} P(\epsilon') = 1 \, \forall i = 1, ..., r$, and the result holds. □

With some abuse of notation, we will denote $w_A = \sum_{x \in A} w_x$. Note that for a given linear extension ϵ and a weight function w, the probability of appearance of ϵ is given by

$$P(\epsilon) = \frac{w_{\epsilon(1)}}{w_{\mathcal{M}(\mathcal{P})}} \times \frac{w_{\epsilon(2)}}{w_{\mathcal{M}(P \backslash \{\epsilon(1)\})}} \times \cdots = \prod_{i=1}^{n} \frac{w_{\epsilon(i)}}{w_{\mathcal{M}(P \backslash \{\epsilon(1), ..., \epsilon(i-1)\})}}. \quad (1)$$

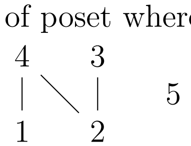
We look for a weight vector satisfying that all linear extensions share the same probability, that we will denote by w^*. The critical point in this procedure is the way we assign weights to elements in P so that w^* serves this purpose. Two questions arise:

1. Is it possible to derive such a weight for any poset?
2. If a poset admits a weight vector in these conditions, how can it be sorted out?

Answer to the first question is negative, as next example shows:

Example 2. *Consider the poset given in Figure 2*

Figure 2: Example of poset where w^* does not exist.



Consider the following pairs of linear extensions: $\epsilon_1 = (1,2,4,3,5), \epsilon_2 = (1,2,4,5,3)$, $\epsilon_3 = (5,2,1,3,4), \epsilon_4 = (5,2,3,1,4)$ *and* $\epsilon_5 = (1,5,2,3,4), \epsilon_6 = (5,1,2,3,4)$. *Then, if* w^* *exists,* $P(\epsilon_1) = P(\epsilon_2)$ *leads to* $w^*_3 = w^*_5$*; next,* $P(\epsilon_3) = P(\epsilon_4)$ *leads to* $w^*_1 = w^*_3 + w^*_4$*; finally,* $P(\epsilon_5) = P(\epsilon_6)$ *leads to* $w^*_1 = w^*_5$. *As a conclusion, no* w^* *satisfying* $w^*_i > 0, \forall i \in P$ *exists.*

If a poset P admits a weight vector $w^* > 0$, we say that P is **BU-feasible**.

Assume P is BU-feasible; we treat the problem of obtaining w^* below. Let us start with an example.

Example 3. *Consider poset* N *and let* $w = (w_1, w_2, w_3, w_4)$ *be a (possible) vector of weights. Then, for each linear extension of poset* N*, we obtain the probabilities:*

$$\epsilon_1 = (1,2,3,4) \Rightarrow p(\epsilon_1) = \frac{w_1}{w_1+w_2} \times \frac{w_2}{w_2} \times \frac{w_3}{w_3+w_4} \times \frac{w_4}{w_4}$$

$$\epsilon_2 = (1,2,4,3) \Rightarrow p(\epsilon_2) = \frac{w_1}{w_1+w_2} \times \frac{w_2}{w_2} \times \frac{w_4}{w_3+w_4} \times \frac{w_3}{w_3}$$

$$\epsilon_3 = (2,1,3,4) \Rightarrow p(\epsilon_3) = \frac{w_2}{w_1+w_2} \times \frac{w_1}{w_1+w_3} \times \frac{w_3}{w_3+w_4} \times \frac{w_4}{w_4}$$

$$\epsilon_4 = (2,1,4,3) \Rightarrow p(\epsilon_4) = \frac{w_2}{w_1+w_2} \times \frac{w_1}{w_1+w_3} \times \frac{w_4}{w_3+w_4} \times \frac{w_3}{w_3}$$

$$\epsilon_5 = (2,3,1,4) \Rightarrow p(\epsilon_5) = \frac{w_2}{w_1+w_2} \times \frac{w_3}{w_1+w_3} \times \frac{w_1}{w_1} \times \frac{w_4}{w_4}$$

In order to sampling uniformly, we should get a vector $w = (w_1, w_2, w_3, w_4)$ *such that* $p(\epsilon_1) = p(\epsilon_2) = p(\epsilon_3) = p(\epsilon_4) = p(\epsilon_5)$. *It is easy to see that* $w^* = (2,3,1,1)$ *satisfies these conditions.*

Note that we have two problems to face: first, as it can be seen in the previous example, we have to solve a non-linear system of equations; this is the case for general posets. Second, the system involves $|\mathcal{L}(P)| - 1$ equations and relies on the knowledge of the whole set of linear extensions; therefore, the problem is untractable at this stage. In next results we will see that it is possible to derive an equivalent linear system involving a reduced number of equations and such that it does not depend on the knowledge of $\mathcal{L}(P)$.

We start transforming the system into an equivalent one just involving linear equations.

Lemma 2. *Let us consider two adjacent linear extensions* ϵ_1, ϵ_2 *in* $(\mathcal{L}(P), \tau^*)$*; then, the equation* $p(\epsilon_1) = p(\epsilon_2)$ *is linear.*

Proof: Let ϵ_1 and ϵ_2 be two adjacent linear extensions in $(\mathcal{L}(P), \tau^*)$. Then, they can be written as $\epsilon_1 = (a_1, a_2, \ldots, a_k, x, y, b_1, b_2, \ldots, b_s)$ and $\epsilon_2 = (a_1, a_2, \ldots, a_k, y, x, b_1, b_2, \ldots, b_s)$. Now, denoting $P_i = P \backslash \{\epsilon_1(1), ..., \epsilon_1(i-1)\}$ and $P'_i = P \setminus \{\epsilon_2(1), ..., \epsilon_2(i-1)\}$, then

$$P(\epsilon_1) = \frac{w_{a_1}}{w_{\mathcal{M}(P_1)}} \times \cdots \times \frac{w_{a_k}}{w_{\mathcal{M}(P_k)}} \times \frac{w_x}{w_{\mathcal{M}(P_{k+1})}} \times \frac{w_y}{w_{\mathcal{M}(P_{k+2})}} \times \frac{w_{b_1}}{w_{\mathcal{M}(P_{k+3})}} \times \cdots \times \frac{w_{b_s}}{w_{P_n}}.$$

$$P(\epsilon_1) = \frac{w_{a_1}}{w_{\mathcal{M}(P'_1)}} \times \cdots \times \frac{w_{a_k}}{w_{\mathcal{M}(P'_k)}} \times \frac{w_y}{w_{\mathcal{M}(P'_{k+1})}} \times \frac{w_x}{w_{\mathcal{M}(P'_{k+2})}} \times \frac{w_{b_1}}{w_{\mathcal{M}(P'_{k+3})}} \times \cdots \times \frac{w_{b_s}}{w'_{P_n}}.$$

Notice that $w_{\mathcal{M}(P_i)} = w_{\mathcal{M}(P'_i)}$, if $i \neq k+2$. Consequently,

$$P(\epsilon_1) = P(\epsilon_2) \Leftrightarrow w_{\mathcal{M}(P'_{k+2})} = w_{\mathcal{M}(P_{k+2})},$$

whence we have obtained a linear equation. □

From this result we can transform the system of equations into a linear system.

Theorem 1. *Let P be a finite poset. The system of $|\mathcal{L}(P)|-1$ non-linear equations $p(\epsilon_1) = p(\epsilon_2) = \ldots = p(\epsilon_{e(P)})$ can be transformed into a linear system of the same number of equations.*

Proof: Consider a linear extension ϵ_1; as $(\mathcal{L}(P), \tau^*)$ is connected, there exists ϵ_2 being adjacent to ϵ_1. Applying the previous lemma, we conclude that $p(\epsilon_1) = p(\epsilon_2)$ is a linear equation. Now, there exists $\epsilon_3 \in \mathcal{L}(P)\backslash\{\epsilon_1, \epsilon_2\}$ adjacent to one of them, say ϵ_2; and again, $p(\epsilon_2) = p(\epsilon_3)$ is linear. Acting like this, we obtain a system of $e(P) - 1$ linear equations. □

Example 4. *In next table we show the way to obtain the weight vector for poset N. In the first column, we consider the set of adjacent linear extensions. Although there are five pairs of adjacent linear extensions (see Fig. 1), note that we just need four of them in order to define the system involving all linear extensions. Moreover, remark that there are just three different equations, as some of them coincide.*

Linear extensions	*Equation*	*Incomparable pair*	V
$p(\epsilon_1) = p(\epsilon_2)$	$w_3 = w_4$	$\{3, 4\}$	$\emptyset$
$p(\epsilon_1) = p(\epsilon_3)$	$w_2 = w_1 + w_3$	$\{1, 2\}$	$\emptyset$
$p(\epsilon_2) = p(\epsilon_4)$	$w_2 = w_1 + w_3$	$\{1, 2\}$	$\emptyset$
$p(\epsilon_3) = p(\epsilon_4)$	$w_3 = w_4$	$\{3, 4\}$	$\emptyset$
$p(\epsilon_3) = p(\epsilon_5)$	$w_1 = w_3 + w_4$	$\{1, 3\}$	$\emptyset$

Therefore, we obtain the following linear system:

$$\begin{cases} w_3 = w_4 \\ w_2 = w_1 + w_3 \\ w_1 = w_3 + w_4 \end{cases}$$

whose solution is $w^* = (2\lambda, 3\lambda, \lambda, \lambda), \lambda > 0$.

Now, a problem may be considered: how can the adjacent linear extensions be chosen? Of course, the natural answer is to consider a Hamiltonian path in the graph $(\mathcal{L}(P), \tau^*)$; however, such a path does not exist in general and it is indeed a problem that has attracted the attention of many researchers (see [21] and references therein). We will show below that the problem can be solved in a more suitable way without the need of considering linear extensions and connections between them, so we can avoid this problem.

Let us now deal with the problem of reducing the number of equations of the linear system. To shed light on what follows, let us have a look to the system obtained in Example 4. As we have seen in this example, it could be the case that some equations coming from different pairs of adjacent linear extensions in $(\mathcal{L}(P), \tau^*)$ coincide. This is the case for $p(\epsilon_1) = p(\epsilon_2)$ and $p(\epsilon_3) = p(\epsilon_4)$, or $p(\epsilon_1) = p(\epsilon_3)$ and $p(\epsilon_2) = p(\epsilon_4)$. Let us take a deeper look at these equations; as they come from adjacent linear extensions, they differ in two consecutive uncomparable elements that have swapped positions, the position of the other elements remaining unaltered; thus, as we have shown in Lemma 2, the equation depends on these pairs of elements; moreover, as the equations relies on the minimal elements of the corresponding subposets when these elements are selected, we have to take into account the elements that have been selected before (or equivalently, that will be selected after); the elements in $\downarrow \hat{x} \cup \downarrow \hat{y}$ are of course selected before x, y, and elements in $\uparrow \hat{x} \cup \uparrow \hat{y}$ are selected after; then, it just suffices to know which are the elements outside these subsets that have been selected before x, y; this is set V in the fourth column and it is an ideal of $P \backslash \updownarrow \{x, y\}$. This leads us to the following definition.

Definition 1. *Let P be a finite poset. We say that $(a = \{x, y\}, V)$ is a* **positioned antichain** *if $a = \{x, y\}$ is an antichain in P and V is an ideal of $P \setminus \updownarrow \{x, y\}$. Notice that V can be the empty set.*

Let us denote by $\mathcal{PA}(P)$ the set of all positioned antichains for poset P and by $pa(P)$ its cardinality. Remark that for any pair of adjacent linear extensions in $(\mathcal{L}(P), \tau^*)$, a positioned antichain is associated to the pair, but it could be the case that several pairs of adjacent linear extensions share the same positioned antichain, as we have seen in the previous example. Therefore, the linear system based on adjacent linear extensions can be transformed into another one based on positioned antichains. This is formally shown below.

Definition 2. *Let P be a finite poset and (a, V) a positioned antichain of P with $a = \{x, y\}$. Define the* **set of linear extensions generated** *by (a, V) as*

$$\mathcal{G}(a, V) := \{\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \mathcal{L}(P) : \epsilon_1 \in \mathcal{L}(\downarrow \widehat{x} \cup \downarrow \widehat{y} \cup V), \epsilon_2 \in \mathcal{L}(a), \epsilon_3 \in \mathcal{L}(\uparrow \widehat{x} \cup \uparrow \widehat{y} \cup V_a^c)\}$$

where $V_a^c := P \setminus (\updownarrow \{x, y\} \cup V)$. Notice that $\epsilon_2 = (x, y)$ or $\epsilon_2 = (y, x)$.

Lemma 3. *Let us consider a pair of adjacent linear extensions (ϵ_1, ϵ_2) in $(\mathcal{L}(P), \tau^*)$. Then, there is a unique positioned antichain (a, V) such that $\epsilon_1, \epsilon_2 \in \mathcal{G}(a, V)$. We call it the* **positioned antichain associated** *to the pair (ϵ_1, ϵ_2).*

Proof: Let $\epsilon_1 = (a_1, a_2, \ldots, a_k, x, y, b_1, b_2, \ldots, b_s)$ and $\epsilon_2 = (a_1, a_2, \ldots, a_k, y, x, b_1, b_2, \ldots, b_s)$. Now we necessarily have to choose $a = (x, y)$ and $V = (P \setminus \updownarrow \{x, y\}) \cap \{a_1, a_2, \ldots, a_k\}$. Since ϵ_1 and ϵ_2 are linear extensions, then V is an ideal of $P \setminus \updownarrow \{x, y\}$. Finally $\epsilon_1, \epsilon_2 \in \mathcal{G}(a, V)$. □

Note that for any positioned antichain $(a = \{x, y\}, V)$, it is always possible to obtain a pair of linear extensions (ϵ_1, ϵ_2) such that ϵ_1 and ϵ_2 are adjacent in $(\mathcal{L}(P), \tau^*)$ and whose associated positioned antichain is (a, V). To see this, it just suffices to consider $\epsilon_1 = (\epsilon_1^1, x, y, \epsilon_1^3) \in \mathcal{G}(a, V)$ and $\epsilon_2 = (\epsilon_1^1, y, x, \epsilon_1^3)$.

Theorem 2. *Let us consider two pairs of adjacent linear extensions (ϵ_1, ϵ_2) and (ϵ_3, ϵ_4) in $(\mathcal{L}(P), \tau^*)$ and suppose they share the same positioned antichain. Then, the linear equation for $p(\epsilon_1) = p(\epsilon_2)$ and $p(\epsilon_3) = p(\epsilon_4)$ is the same. Consequently, it just suffices to consider the linear equations corresponding to different positioned antichains.*

Proof: Let us denote by (a, V) with $a = \{x, y\}$ the common positioned antichain associated to both pairs. From the proof of Lemma 2, we know that the linear equations for $p(\epsilon_1) = p(\epsilon_2)$ and $p(\epsilon_3) = p(\epsilon_4)$ only depend on the terms corresponding to the (consecutive) positions for x, y in the linear extension; on the other hand, these terms depend only on the elements appearing before in the linear extension. As these elements are in both cases $\downarrow \widehat{x} \cup \downarrow \widehat{y} \cup V$, the result holds. □

Definition 3. *We define the* **linear equation associated** *to the positioned antichain $(a = \{x, y\}, V)$ the equation given by*

$$w(\mathcal{M}(P \setminus (\downarrow \widehat{a} \cup V \cup \{x\}))) = w(\mathcal{M}(P \setminus (\downarrow \widehat{a} \cup V \cup \{y\}))). \tag{2}$$

Note that this equation arises for $p(\epsilon_1) = p(\epsilon_2)$ where (ϵ_1, ϵ_2) is a pair of two adjacent linear extensions whose associated positioned antichain is (a, V).

Now, the following question is of relevance: Does the system based on positioned antichains involve a reduced number of equations? For this, we have to compare $pa(P)$ and $e(P)$; the following holds:

Theorem 3. *Let P be a finite poset. Then, $pa(P) \leq e(P)$.*

Proof: See appendix. □

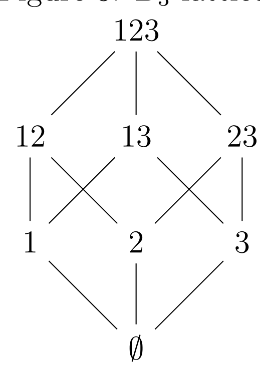
In general, $pa(P) < e(P)$ and $pa(P)$ is very small compared to $e(P)$. Note however that it could be the case that the number of linear extensions and the number of positioned antichains could be the same.

Example 5. *Consider the antichain of three elements, $\bar{3}$. In this case, there are six linear extensions. On the other hand, each pair of elements are uncomparable, and V can be either the element outside the antichain or the empty set; then, there are six positioned antichains.*

Finally, it is important to note that the number of equations could be further reduced, as redundances may appear. Next example illustrates this situation and shows how positioned antichains can reduce the complexity of the problem.

Example 6. *Let us consider the case of the Boolean poset of order three B_3 (Figure 3), that has 48 linear extensions.*

Figure 3: B_3 lattice.



Note that the $\emptyset$ and the total set have fixed positions and any weight is valid for these elements. If we remove these two elements we obtain an irreducible poset. We have the following positioned antichains and equations:

Positioned Antichain	*Equation*	*positioned Antichain*	*Equation*
$(12,13)$ $V=\emptyset$	$w_{13}=w_{12}$	$(12,13)$ $V=23$	$w_{13}=w_{12}$
$(12,23)$ $V=\emptyset$	$w_{12}=w_{23}$	$(12,23)$ $V=13$	$w_{12}=w_{23}$
$(13,23)$ $V=\emptyset$	$w_{13}=w_{23}$	$(13,23)$ $V=12$	$w_{13}=w_{23}$
$(1,23)$ $V=\emptyset$	$w_1=w_{12}+w_{13}+w_{23}$	$(2,13)$ $V=\emptyset$	$w_2=w_{12}+w_{13}+w_{23}$
$(3,12)$ $V=\emptyset$	$w_3=w_{12}+w_{13}+w_{23}$	$(1,2)$ $V=\emptyset$	$w_1=w_2$
$(1,2)$ $V=3$	$w_1+w_{23}=w_2+w_{13}$	$(1,3)$ $V=\emptyset$	$w_1=w_3$
$(1,3)$ $V=2$	$w_1+w_{23}=w_3+w_{12}$	$(2,3)$ $V=\emptyset$	$w_2=w_3$
$(2,3)$ $V=1$	$w_2+w_{13}=w_3+w_{12}$		

Note that we have reduced the number of equations from 47 to 15. In fact, the number of equations can be further reduced to:

$$\begin{cases} w_{12}=w_{13}=w_{23} \\ w_1=w_{12}+w_{13}+w_{23} \\ w_2=w_{12}+w_{13}+w_{23} \\ w_3=w_{12}+w_{13}+w_{23} \end{cases}$$

Then a solution is $w^* = (w_1^*, w_2^*, w_3^*, w_{12}^*, w_{13}^*, w_{23}^*) = (3,3,3,1,1,1)$ *or, with* $w_\emptyset^* = 1$ *and* $w_{123}^* = 1$, $w^* = (1,3,3,3,1,1,1,1)$.

Remark 1. *Let P be BU-feasible. Then, $\forall \epsilon \in \mathcal{L}(P)$, it is possible to obtain $P(\epsilon)$ applying Eq. (1). On the other hand, $P(\epsilon) = \frac{1}{|\mathcal{L}(P)|}$, whence $|\mathcal{L}(P)| = \dfrac{1}{P(\epsilon)}$.*

For example, considering B_3, it can be checked that:

$P(\epsilon = (\emptyset, 1, 2, 3, 12, 13, 23, 123)) = 1 \cdot \dfrac{3}{9} \cdot \dfrac{3}{6} \cdot \dfrac{3}{4} \cdot \dfrac{1}{3} \cdot \dfrac{1}{2} \cdot 1 \cdot 1 = \dfrac{1}{48} \Rightarrow |\mathcal{L}(B_3)| = 48.$

Thus, our procedure also provides an easy way to obtain the number of linear extensions. Note that, as already pointed out in the introduction, the problem of counting all linear extensions of a poset is a hard problem.

Let us now show some properties regarding BU-feasibility.

Proposition 1. *Let $P_1, ..., P_n$ be a group of finite posets and consider $P := P_1 \oplus P_2 \oplus ... \oplus P_n$. Then, there exists a weight vector w_P^* if and only if there exists $w_{P_i}^*, i = 1, ..., n$. In other words, P is BU-feasible if and only if P_i is BU-feasible $\forall i = 1, ..., n$. Moreover, $w_P^* = (w_{P_1}^*, ..., w_{P_n}^*)$.*

Proof: As $P = P_1 \oplus P_2 \oplus ... \oplus P_n$, a positioned antichain in P_i is a positioned antichain in P.

By definition of $\oplus$, given $x \in P_i$ and $y \in P_j$ with $i < j$, $x \preceq_P y$; consequently, any antichain of order two in P consists in two elements in the same P_i that are not related. Consider now a positioned antichain $(a = \{x, y\}, V)$ in P. Then, x, y belong to the same P_i; on the other hand, V is an ideal of $P \setminus \updownarrow \{x, y\}$; as $x, y \succeq z, \forall z \in P_j, j < i$ and $x, y \preceq z, \forall z \in P_k, k > i$, we conclude that $V \subset P_i$ and thus, $(\{x, y\}, V)$ is a positioned antichain in P_i. As a result,

$$\mathcal{PA}(P) = \bigcup_{i=1}^{n} \mathcal{PA}(P_i).$$

Finally, for a positioned antichain $(a = \{x, y\}, V)$ in P_i, notice that the minimal elements when introducing x and y in the linear extension belong to P_i; therefore, we obtain a linear equation that just involves elements in P_i, and it is the same equation arising when dealing with P_i instead of P.

Therefore, the equations for P can be divided in n groups of equations, each of them just involving elements of a P_i. Thus, it is possible to obtain a weight vector w_P^* if and only if it is possible to obtain a group of n weight vectors $w_{P_1}^*, ..., w_{P_n}^*$ and in this case $w_P^* = (w_{P_1}^*, ..., w_{P_n}^*)$. ☐

Now, we prove an important property about BU-feasible posets.

Theorem 4. *Let P be a BU-feasible finite poset with weight vector w_P^*. If $F \in \mathcal{P}$, then F is BU-feasible and $w_F^* := w_{P|F}^*$ is a possible weight vector for subposet F.*

Proof: It suffices to prove that every linear equation associated to the linear system generated for F is in the linear system generated for P. Consider a positioned antichain $(a_F = \{a_1, a_2\}, V_F)$ in F; the associated equation for this positioned antichain is given by (see Eq. (2))

$$w(\mathcal{M}(F \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_1\}))) = w(\mathcal{M}(F \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_2\}))).$$

Consider (a, V), where $a = a_F$ and $V = V_F \cup F_a^c$ where $F_a^c := F^c \setminus \updownarrow a$. First, note that V is an ideal of $P \setminus \updownarrow a$; as F^c is an ideal of P, then $F^c \setminus \updownarrow a = F_a^c$ is an ideal of $P \setminus \updownarrow a$; take $x \in V$ and $z \preceq x, z \in P \setminus \updownarrow a$; then, if $x \in F_a^c$ so is z; suppose on the other hand that $x \in V_F$; if $z \in F$, then $z \in V_F$ because V_F is an ideal of F; otherwise, $z \in F^c$ and $z \in P \setminus \updownarrow a$, whence $z \in F_a^c$.

The associated equation is

$$w(\mathcal{M}(P \setminus (\downarrow \widehat{a} \cup V \cup \{a_1\}))) = w(\mathcal{M}(P \setminus (\downarrow \widehat{a} \cup V \cup \{a_2\}))) \Leftrightarrow$$

$$w(\mathcal{M}(P \setminus (\downarrow \widehat{a_F} \cup (V_F \cup F^c_a) \cup \{a_1\}))) = w(\mathcal{M}(P \setminus (\downarrow \widehat{a_F} \cup (V_F \cup F^c_a) \cup \{a_2\}))) \Leftrightarrow$$

$$w(\mathcal{M}(P \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_1\} \cup (F^c \setminus \uparrow a_F)))) = w(\mathcal{M}(P \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_2\} \cup (F^c \setminus \uparrow a_F)))) \Leftrightarrow$$

$$w(\mathcal{M}(P \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_1\} \cup F^c))) = w(\mathcal{M}(P \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_2\} \cup F^c))) \Leftrightarrow$$

$$w(\mathcal{M}(F \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_1\}))) = w(\mathcal{M}(F \setminus (\downarrow \widehat{a_F} \cup V_F \cup \{a_2\}))).$$

Observe that we have used above that $F^c \setminus \uparrow a_F = F^c$ since $\uparrow a_F \subseteq F$. Finally, as the system for P has a positive solution, then so has the one for F, and a possible solution is the restriction of w^*_P to F. $\square$

Note however that this does not mean that BU-feasibility for a subposet F implies BU-feasibility for P.

Corollary 1. *Let $P_1, P_2, \cdots, P_n$ be a collection of finite connected posets. If $P = P_1 \uplus P_2 \uplus \cdots \uplus P_n$ is BU-feasible, then each P_i is BU-feasible and $w^*_{P_i} = w^*_{P|P_i}$ is a possible weight vector.*

Proof: As P_i is a filter of P, it just suffices to apply Theorem 4. $\square$

Lemma 4. *Let P be a finite poset and let $\mathbf{M} = L_0 = \{m_1, m_2, \ldots, m_s\}$ be the set of maximal elements of P. Then, if P is BU-feasible, then $w^*_{m_i} = w^*_{m_j}, \forall m_i, m_j \in \mathbf{M}$.*

Proof: For every pair of maximal elements m_i, m_j, consider the positioned antichain $(\{m_i, m_j\}, V = P \setminus (\downarrow m_i \cup \downarrow m_j))$. Then, $P \setminus (\downarrow \widehat{m_i} \cup \downarrow \widehat{m_j} \cup V) = \{m_i, m_j\}$, whence $\mathcal{M}(P \setminus (\downarrow \widehat{m_i} \cup \downarrow \widehat{m_j} \cup V \cup m_j) = \{m_i\}$ and $\mathcal{M}\,(P \setminus (\downarrow \widehat{m_i} \cup \downarrow \widehat{m_j} \cup V \cup m_i)) = \{m_j\}$ and thus by Eq. (2), the corresponding equation is $w_{m_i} = w_{m_j}$. $\square$

Next, we are going to study a useful tool to compute w^*.

Proposition 2. *Let P be a BU-feasible finite poset and $x, y \in P$. If x and y are interchangeable, then $w^*_x = w^*_y$.*

Proof: Note that if x, y are interchangeable, then they are in the same level. Then, we will prove the result by induction on the level of x, y.

If $x, y \in L_0$, then they are maximal and $w^*_x = w^*_y$ by Lemma 4.

Suppose the result holds for $L_0, L_1, ..., L_k$ and take $x, y \in L_{k+1}$ being interchangeable. Notice that x and y are not related to each other, so we can take the positioned antichain of order 2 given by $(\{x, y\}, V)$ with $V = P \setminus \updownarrow \{x, y\}$. Applying Eq. (2), we derive the equation

$$w^*_x + w^*_{z_1} + w^*_{z_2} + \ldots + w^*_{z_s} = w^*_y + w^*_{z^*_1} + w^*_{z^*_2} + \ldots + w^*_{z^*_s},$$

where $z_1, ..., z_s = \mathcal{M}(P \setminus \{V \cup \downarrow \hat{x} \cup \downarrow \hat{y} \cup y\})$ and $z^*_1, ..., z^*_s = \mathcal{M}(P \setminus \{V \cup \downarrow \hat{x} \cup \downarrow \hat{y} \cup x\})$. As $\uparrow x \cong \uparrow y$, there are as many z_i as z^*_i. Besides, we can suppose that $\{z_i, ..., z_s\} \cap \{z^*_1, ..., z^*_s\} = \emptyset$. Moreover, $\uparrow x \cap L_k \cong \uparrow y \cap L_k$, whence we can assume that the elements z_i and z^*_i are interchangeable (reordering the indices if necessary). Applying the induction hypothesis $w^*_{z^*_i} = w^*_{z_i}$, $\forall i \in \{1, 2, \ldots, s\}$, whence $w^*_x = w^*_y$. $\square$

Remark 2. *It should be noted that we cannot assume that two elements x, y in the poset are interchangeable if $w_x^* = w_y^*$. Consider for example poset N. In this case, $w_4^* = w_3^*$ by Lemma 4. However, 4 and 3 are not interchangeable.*

Let us now deal with the problem of BU-feasibility of a poset P. From Proposition 1, we just need to focus on the case of posets P that are irreducible by direct sum $\bigoplus$.

Theorem 5. *Let P be a finite poset irreducible by direct sum $\oplus$. Then, P is BU-feasible if and only if the system has infinitely many solutions.*

Proof: $\Rightarrow$) Note that the system based on positioned antichains has the null vector as trivial solution, but this vector is not a valid weight vector as weights should be positive. Thus, BU-feasibility implies that the linear system has infinitely many solutions.

$\Leftarrow$) Let us prove that if the system has infinite many solutions, then it is possible to obtain a weight vector; i.e. we shall prove that a vector with positive coordinates can be always obtained.

Consider an element in L_{k+1}. As P is irreducible, there are at least two elements in L_k and there exists at least $x \in L_{k+1}$ such that it is not covered by any element in L_k. Denote by y an element in L_k such that $x \not\preceq y$ and let us denote by $m_1, ..., m_r$ the set of elements in L_k covering x; consider the positioned antichain $(\{x, y\}, V)$ where $V = P \setminus \updownarrow \{x, y\}$. Then, from Def. 3, the corresponding equation is given by

$$w_x = w_{m_1} + w_{m_2} + \ldots + w_{m_r} + w_y. \tag{3}$$

Consider now an element $z \in L_{k+1}$ such that it is covered by any element in L_k. Since there exists $x \in L_{k+1}$ that is not covered by all elements in L_k, let us consider the positioned antichain $(\{z, x\}, V)$ where $V = P \setminus \updownarrow \{x, z\}$. Again, this leads to the equation

$$w_z = w_x + \sum_{y \in L_k, x \not\preceq y} w_{m_k}. \tag{4}$$

First, let us show that if the system has infinite many solutions, then there exists a solution such that $w_{m_i} \neq 0$ for a maximal element. For if $w_{m_i} = 0$ for all maximal elements, we can apply Eqs. (3) and (4) to conclude that $w_x = 0, \forall x \in L_1$; but then, we can repeat the procedure to conclude $w_x = 0, \forall x \in L_2$; and so on. Then, $w_x = 0, \forall x \in P$, whence the system has just an only solution.

Assume then that we have a solution for the system satisfying $w_{m_i} \neq 0$, for a maximal element m_i. We can fix this value to be $w_{m_i} = 1$. Now, consider the positioned antichain given by $(\{m_i, m_j\}, V = P \setminus (\downarrow m_i \cup \downarrow m_j))$, that leads to $w_{m_i} = w_{m_j}$. Consequently, all maximal elements have the same positive weight.

Now, let us show that by induction that if $x \preceq y$, then $w_x > w_y$.

Let us consider an element in L_1. If $L_1 = \emptyset$, then P is an antichain and the results holds. In other case, as P is irreducible, there are at least two maximal elements and there exists at least $x \in L_1$ such that it is not covered by any element in L_0. Let us prove that $w_x > 1$. Denote by y an element in L_0 such that $x \not\preceq y$ and let us denote by $m_1, ..., m_r$ the set of elements in L_0 covering x; consider the positioned antichain $(\{x, y\}, V)$ where $V = P \setminus \updownarrow \{x, y\}$. Then, from Eq. (3), the corresponding equation is given by $w_x = w_{m_1} + w_{m_2} + \ldots + w_{m_r} + w_y = \sum_{m_i \in L_0, x \preceq m_i} w_{m_i} + 1 \geq 2$.

Consider now an element $z \in L_1$ such that it is covered by any element in L_0. Since there exists $x \in L_1$ that is not covered by all elements in L_0, let us consider the positioned antichain $(\{z, x\}, V)$ where $V = P \setminus \updownarrow \{x, z\}$. Eq. (4) leads to the equation $w_z = w_x + \sum_{y \in L_0, x \not\preceq y} w_{m_k} = |L_0| + 1 \geq 3$.

Then, we have proved that every element in the second level has a weight greater than 1.

Now suppose that the result holds for elements in levels $0, ..., k$ and let us prove it for elements in L_{k+1}. The reasoning is pretty similar. Since P is irreducible by direct sum, there are at least two elements in L_k and there exists at least $x \in L_{k+1}$ such that it is not covered by any element in L_k; let us choose an element y in L_k such that $x \not\preceq y$ and let us prove that $w_x > w_y$. Let us denote by $m_1, ..., m_r$ the set of elements in L_k covering x; consider the positioned antichain $(\{x, y\}, V)$ where $V = P \setminus \updownarrow \{x, y\}$. Then, the corresponding equation is given by $w_x = w_{m_1} + w_{m_2} + \ldots + w_{m_r} + w_y > w_y$.

Consider now an element $z \in L_{k+1}$ such that it is covered by any element in L_k. Choose $x \in L_{k+1}$ that is not covered by all elements in L_k and let us consider the positioned antichain $(\{z, x\}, V)$ where $V = P \setminus \updownarrow \{x, z\}$. Again, we obtain the equation $w_z = w_x + \sum_{y \in L_0, x \not\preceq y} w_{m_k} > w_x$. Then, for the elements in L_k covering x, we obtain $w_z > w_x > w_{m_i}$; for the elements $y \in L_k$ not covering x, we have by the previous case, $w_z > w_x > w_y$. □

As a direct consequence of this result, the following holds.

Corollary 2. *Let P be a finite poset irreducible by direct sum $\oplus$. If $|\mathcal{PA}(P)| < |P|$, then there is a positive solution w^*.*

Joining Theorem 5, Proposition 1 and Corollary 2, the following results hold.

Corollary 3. *Let $P = P_1 \oplus P_2 \oplus \ldots \oplus P_n$. Then, there exists w_P^* if and only if the linear systems associated to $P_i, i = 1, ..., n$ have infinitely many solutions.*

Corollary 4. *Let $P = P_1 \oplus P_2 \oplus \ldots \oplus P_n$. If $|\mathcal{PA}(P_i)| < |P_i|, i = 1, ..., n$, then there exists a weight vector w_P^*.*

Note that this condition is sufficient but not necessary (see for example the boolean lattice B_3 developed in Example 6).

Thus, the final version of the Bottom-Up algorithm is as follows:

BOTTOM-UP ALGORIHM

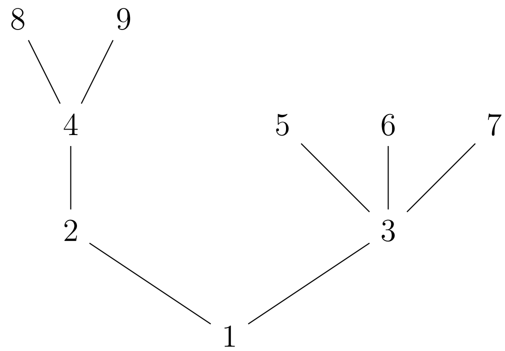
1. ASSIGNING WEIGHTS
   - (a) Compute all possible positioned antichains.
   - (b) For any positioned antichain, build the corresponding linear equation.
   - (c) Solve the system of linear equations and choose a weight vector (if possible).
2. MAIN STEP: Starting with $P' = P$:
   - (a) Select a minimal element x of P' with probability the quotient between w_x and the sum of the weights of $\mathcal{M}(P')$.
   - (b) Remove this element and repeat the previous step for the new poset $P' \setminus \{x\}$.

4 Examples

In this section, we apply the Bottom-Up method to some families of posets in order to illustrate the performance of this algorithm.

Example 7. *In this example we are going to show that every rooted tree is BU-feasible. A rooted tree [23] is a poset satisfying that its Hasse diagram (considered as an undirected graph) is connected and has no cycles, and it has just one minimal element (called root). An example of rooted tree can be seen in Figure 4.*

Figure 4: Rooted Tree.



Consider $(a = \{x, y\}, V)$ *a positioned antichain for a rooted tree. Note that, as* P *has no cycles, for any* $z \in \mathcal{M}(P \backslash (\downarrow \hat{a} \cup V)), z \neq x, y$, *it follows* $z \in \mathcal{M}(P \backslash (\downarrow \hat{a} \cup V \cup \{x\})) \cap \mathcal{M}(P \backslash (\downarrow \hat{a} \cup V \cup \{y\}))$, *whence we conclude that for two positioned antichains* (a, V) *and* (a, V'), *they share the same associated equation. In other words, ideal* V *is not relevant.*

Now, for $x \in P$, *let us consider* $\lambda_x := |\{y \in T | y \succeq x\}|$. *Now, as* P *has no cycles, it is easy to see by induction on the level that* $\lambda_x = 1 + \sum_{x \lessdot h} \lambda_h$. *Then, for* $a = \{x, y\}$, *the associated equation is*

$$w_x + \sum_{y \lessdot z} w_z = w_y + \sum_{x \lessdot h} w_h.$$

Taking $w_x = \lambda_x$, *we have*

$$w_x + \sum_{y \lessdot z} w_z = \lambda_x + \sum_{y \lessdot z} \lambda_z = 1 + \sum_{x \lessdot h} \lambda_h + \sum_{y \lessdot z} \lambda_z = \sum_{x \lessdot h} \lambda_h + 1 + \sum_{y \lessdot z} \lambda_z = \sum_{x \lessdot h} \lambda_h + \lambda_y = w_y + \sum_{x \lessdot h} w_h.$$

Consequently, any rooted tree is BU-feasible and $w_x^* = \lambda_x$ *is a possible solution. Observe that the last argument remains valid for disjoint union of rooted trees by Corollary 1.*

Finally, let us compute $e(T_n)$ *for any rooted tree with* n *elements. Again, we have to keep in mind that* $\lambda_x = 1 + \sum_{x \lessdot h} \lambda_h$. *Then, it can be easily seen by induction that* $\sum_{x \in \mathcal{M}(P)} \lambda_x = |P|$ *for any* P *being a disjoint union of rooted trees. Therefore, for any linear extension* ϵ *of a rooted tree,*

$$P(\epsilon) = \frac{\lambda_{root}}{n} \cdot \frac{\lambda_{x_1}}{n-1} \cdot \frac{\lambda_{x_2}}{n-2} \cdots,$$

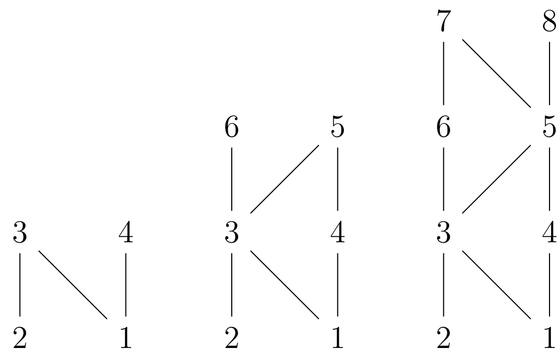
whence

$$e(T_n) = \frac{n!}{\prod_{x \in T_n} \lambda_x}.$$

This formula was already shown by Stanley [23].

Example 8. *Let us consider the following family of posets* P_n, *where* P_n *has* $n+1$ *levels, starting on level 0 for minimals. In each level, we have an antichain of two elements, and they are related* *to the previous and the next level in the following way: one of the elements is related to any element of these levels, while the other one is related only to one element in each level; to fix notation, we assume that level i consists in $\{2i+1, 2i+2\}$ and odd numbers are related to all the elements of the previous and next level, while even numbers are related to odd numbers in the previous and next level. Figure 5 shows the Hasse diagrams of P_1, P_2 and P_3.*

Figure 5: P_1, P_2 and P_3 .



Thus labeled, the possible positioned antichains for P_n are: $(\{2i-1, 2i\}, V = \emptyset)$ and $(\{2i, 2i+2\}, V = \emptyset)$. Consequently, there are $2n+1$ positioned antichains and $2n+2$ elements. By Corollary 2, we conclude that P_n is BU-feasible. Let us then compute a weight vector w^*.

Applying Lemma 4, we can assign $w_{2n+1} = 1, w_{2n+2} = 1$ to the two top elements. Now, considering the positioned antichain $(\{2n, 2n+2\}, V = \emptyset)$, we obtain $w_{2n} = w_{2n+1} + w_{2n+2} = 2$. For $(\{2n-1, 2n\}, V = \emptyset)$, we obtain $w_{2n-1} = w_{2n} + w_{2n+1} = 3$. And we can continue this process until we reach the first level. For example, for P_3 we obtain the vector $w^* = (17, 12, 7, 5, 3, 2, 1, 1)$.

In order to derive a general formula, let us consider the reversed order, so that element i is assigned to w^*_{2n+2-i}; for example, for P_3 we would obtain the vector $w^* = (1, 1, 2, 3, 5, 7, 12, 17)$. Let us denote by o_n the n-th odd number of this sequence and e_n the n-th even number; then, it is easy to show by induction that

$$\begin{cases} o_n = o_{n-1} + e_{n-1} \\ e_n = o_n + o_{n-1} \\ o_1 = e_1 = 1 \end{cases}$$

Merging the second equation into the first one we have $o_n = 2o_{n-1} + o_{n-2}$ with $o_0 = 0$ and $o_1 = 1$. Let us solve this recursive equation through generating functions. Let $F(x) := \sum_{k=0}^{\infty} x^k o_k$. Observe that:

$$F(x) = o_0 + o_1 x + \sum_{k=2}^{\infty} x^k o_k = x + 2\sum_{k=2}^{\infty} x^k o_{k-1} + \sum_{k=2}^{\infty} x^k o_{k-2} = x + 2xF(x) + x^2 F(x).$$

Thus, $F(x) = \dfrac{x}{1-2x-x^2}$.

On the other hand, $\dfrac{x}{1-2x-x^2}$ can be written as $F(x) = \dfrac{A}{1-\alpha x} + \dfrac{B}{1-\beta x} = \dfrac{A+B-(\beta A + \alpha B)x}{(1-\alpha x)(1-\beta x)}$.

As $(1-\alpha x)(1-\beta x) = 1-2x-x^2$, we obtain that α and β satisfy

$$\begin{cases} \alpha\beta = -1 \\ \alpha + \beta = 2 \end{cases}$$

whence $\alpha = \frac{2+\sqrt{4+4}}{2} = 1+\sqrt{2}$ *and* $\beta = \frac{2-\sqrt{4+4}}{2} = 1-\sqrt{2}$. *On the other hand,*

$$\left.\begin{array}{r} A+B=0 \\ \beta A+\alpha B=-1 \end{array}\right\} \Rightarrow A = \frac{1}{2\sqrt{2}},\ B = -\frac{1}{2\sqrt{2}}.$$

Now,

$$F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A\sum_{k=0}^{\infty}\alpha^k x^k + B\sum_{k=0}^{\infty}\beta^k x^k,$$

whence

$$o_n = A\alpha^n + B\beta^n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$$

and

$$\begin{aligned} e_n = o_n + o_{n-1} &= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n + (1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right] \\ &= \frac{1}{2\sqrt{2}}\left[(2+\sqrt{2})(1+\sqrt{2})^{n-1} - (2-\sqrt{2})(1-\sqrt{2})^{n-1}\right] \end{aligned}$$

Finally, let us obtain the number of linear extensions for P_n. *Note that:*

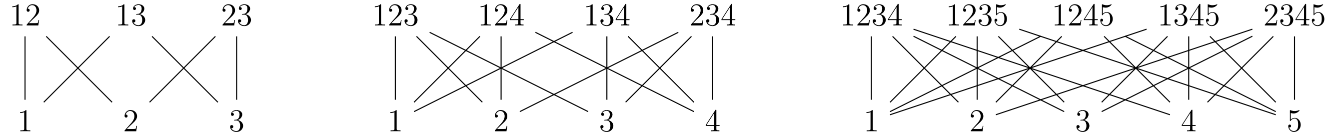
$$\begin{aligned} P(\epsilon = (1,2,3,4,,...,2n+2)) &= \frac{o_{n+1}}{o_{n+1}+e_{n+1}}\cdot 1\cdot\frac{o_n}{o_n+e_n}\cdot 1\cdot\frac{o_{n-1}}{o_{n-1}+e_{n-1}}\cdot\ldots\frac{o_2}{o_2+e_2}\cdot 1\cdot\frac{1}{o_1+e_1}\cdot 1 \\ &= \frac{o_{n+1}}{o_{n+2}}\cdot 1\cdot\frac{o_n}{o_{n+1}}\cdot 1\cdot\frac{o_{n-1}}{o_n}\cdot\ldots\frac{o_2}{o_3}\cdot 1\cdot\frac{1}{o_2}\cdot 1 \\ &= \frac{1}{o_{n+2}} \end{aligned}$$

whence $e(P_n) = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+2} - (1-\sqrt{2})^{n+2}\right]$. *The first values of* $e(P_n)$ *are given in next table.*

n	*2*	*3*	*4*	*5*
$e(P_n)$	*12*	*29*	*70*	*169*

Example 9. *Let* $n \in \mathbb{N}, N = \{1,...,n\}$ *and consider the poset* P_n *consisting on all the subsets of* N *that are either singleton or whose complementary is a singleton, and consider the order relation given by* $x \prec y \Leftrightarrow x \subset y$. *Figure 6 shows* P_3, P_4 *and* P_5.

Figure 6: P_3, P_4 and P_5 .



Let us first find out the possible positioned antichains. We have three different cases:

- *Case 1: Derived from $\{i, j\}$. We have $\binom{n}{2}$ possibilities. For each of them, we have the $n-2$ singletons that are incomparable to both of them, so that we have 2^{n-2} possible choices of V.*

- *Case 2: Derived from $\{N\backslash\{i\}, N\backslash\{j\}\}$. As in the previous case, there are $\binom{n}{2}$ possibilities. For each of them, we have the $n-2$ subsets of cardinality $n-1$ that are incomparable to both of them, so that we have 2^{n-2} possible choices of V.*

- *Case 3: Derived from $\{i, N\backslash\{i\}\}$. In this case, there are n possibilities and any other element in the poset compares to one of them, so that $V = \emptyset$.*

Then, we have $n(n-1)2^{n-2} + n$ positioned antichains, a number much larger than $|P_n| = 2n$ and consequently, Corollary 2 cannot be applied. However, we will show that in this case it is possible to find a weight vector w^*.

Remark that by Lemma 4, we know that if w^* exists, then all maximal elements have the same weight, say 1; thus $w^*_{N\backslash\{i\}} = 1, \forall i \in N$. Now, take $i \in N$ and consider the positioned antichain $(\{i, N\backslash\{i\}\}, \emptyset)$; then, it follows that $w^*_i = n$. It suffices to show that no contradiction arises for any other condition derived from other positioned antichain.

- *For $(\{i, j\}, V)$ we derive $w^*_i = w^*_j$.*

- *For $(\{N\backslash\{i\}, N\backslash\{j\}\}, V)$ we derive $w^*_{N\backslash\{i\}} = w^*_{N\backslash\{j\}}$.*

Thus, our solution fits all the equations. For example, for P_3, P_4 and P_5 the corresponding weight vectors are $(3,3,3,1,1,1)$, $(4,4,4,4,1,1,1,1)$ and $(5,5,5,5,5,1,1,1,1,1)$, respectively, and for P_n we obtain $(n, ..., n, 1, ..., 1)$.

Finally, let us use this vector to compute $e(P_n)$. If we choose an extension ϵ whose first n elements are the singletons, we obtain:

$$p(\epsilon) = \frac{n}{n^2} \cdot \frac{n}{(n-1)n} \cdot \frac{n}{(n-2)n} \cdots \frac{n}{2n} \cdot \frac{n}{n+1} \cdot \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{n}{n!(n+1)!}.$$

Therefore,

$$e(P_n) = (n+1)!(n-1)!.$$

5 Conclusions and open problems

In this paper we have introduced a new procedure for generating a random linear extension of a poset. The idea behind the algorithm is to derive a weight vector on the elements of P so that it can be used for generating next element in the linear extension. Once this weight vector is obtained, it suffices to select a minimal element with probability given by the quotient between its weight and the sum of the weights of the whole set of minimal elements. The method is exact and simple to use; besides, it allows to compute the number of linear extensions of the polytope. The main aspect of the algorithm is the way of deriving the vector of weights; in this sense, we show that whenever this vector exists, it can be obtained solving a linear system of equations; in a first step we show that this system has $e(P)-1$ equations, so it could be a big (see untractable) linear system and the knowledge of all elements of $\mathcal{L}(P)$ is needed; next, we show that in general many of these equations are the same and thus, they can be removed, and we develop a way to reduce the number of equations involved, based on what we have called positioned antichains.

There are some open problems that we aim to treat in the future. First, we have the problem of obtaining necessary and sufficient conditions on the poset to apply the procedure; following this line, we have already obtained Th. 5, but it could be interesting to derive other results based on the structure of P. Another interesting problem would be to compare this procedure with other procedures appearing in the literature, in the sense of their computational costs; in this sense, it should be noted that BU-algorithm is very fast once w^* is obtained; then, it seems to be an appealing solution if many random linear extensions are needed. Finally, we have seen that an equivalent system can be stated if we consider positioned antichains instead of adjacent linear extensions, leading in general to a reduction in the number of equations; on the other hand, we have already seen that the number of equations needed can be less than the number of the positioned antichains; a deeper study on how the number of equations can be reduced seems interesting.

BU-algorithm is based on minimal elements. Similar procedures could be developed if attention is fixed on maximal elements or other positions in the linear extension. This could increase the range of applicability of this philosophy; for example, starting from last element would be a suitable choice for rooted trees when the root is the maximal element.

Acknowledgements

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Appendix: Proof of Theorem 3

In an attempt to clarify the proof, we have considered some previous results before the main part of the proof. In what follows, we will denote P_x the poset $P \setminus \{x\}$.

Lemma 5. *Let P be a finite poset. The following holds:*

i) For every finite posets P and Q, $e(P \oplus Q) = e(P)e(Q)$ and $e(P \uplus Q) = \binom{|P|+|Q|}{|P|} e(P)e(Q)$.

ii) For every $x \in \mathcal{M}(P)$, $i(P_x) \leq i(P) \leq 2i(P_x)$.

iii) $pa(P) = \sum_{(x,y)\in\mathcal{A}_2(P)} i(P \setminus \updownarrow \{x, y\})$, where $\mathcal{A}_2(P)$ denotes the set of antichains of two elements of P.

iv) $pa(P_1 \oplus P_2) = pa(P_1) + pa(P_2)$.

v) $pa(P_1 \uplus P_2) = i(P_2)pa(P_1) + i(P_1)pa(P_2) + \sum_{(x,y)\in P_1\times P_2} i((P_1 \setminus \updownarrow \{x\}) \uplus (P_2 \setminus \updownarrow \{y\}))$.

vi) Let P be a poset with $m(P) = 2$. Suppose that P has a minimal element x_1 less than every non-minimal element in P; then, $pa(P) = pa(P_{x_1}) + 1$. Therefore, if P^* is a poset with an only minimum, then $pa(P^* \uplus \mathbf{1}) = pa(P^*_{x_1} \uplus \mathbf{1}) + 1$.

vii) Let P be a poset with $m(P) = 3$. Suppose that P has a minimal element x_1 less than every non-minimal element in P; then, $pa(P) = pa(P_{x_1}) + 5$. Therefore, if P^* is a poset with an only minimum, then $pa(P^* \uplus \mathbf{1} \uplus \mathbf{1}) = pa(P^*_{x_1} \uplus \mathbf{1} \uplus \mathbf{1}) + 5$.

viii) $i(P) = i(P^\partial)$ and $pa(P) = pa(P^\partial)$.

Proof:

i) These are well-known properties of $e(P)$. See [23], [19].

ii) Consider $x \in \mathcal{M}(P)$ fixed and let us define

$$\begin{array}{rccc} F: & \mathcal{I}(P_x) & \rightarrow & \mathcal{I}(P) \\ & I & \mapsto & I \cup \{x\} \end{array}$$

As $x \in \mathcal{M}(P)$, if $I \in \mathcal{I}(P_x)$, it follows that $I \cup \{x\} \in \mathcal{I}(P)$, and F is injective, whence the first inequality holds.

Let us now consider the function $G : \mathcal{I}(P) \rightarrow \mathcal{I}(P_x) \times 2$ given by

$$G(I) = \begin{cases} (I, 0), & x \notin I \\ (I \setminus \{x\}, 1), & x \in I \neq \{x\} \\ (\emptyset, 1), & I = \{x\} \end{cases}$$

Remark that if $x \notin I$ then $I \in \mathcal{I}(P_x)$; on the other hand, as $x \in \mathcal{M}(P)$, if $x \in I$, then $I \setminus \{x\} \in \mathcal{I}(P_x)$; we conclude that G is well-defined. As G is injective, the second inequality holds.

iii) For a fixed antichain $\{x, y\}$ of two elements, the number of positioned antichains associated to $\{x, y\}$ is given by $i(P \setminus \updownarrow \{x, y\})$. Then, $pa(P) = \sum_{(x,y) \in \mathcal{A}_2(P)} i(P \setminus \updownarrow \{x, y\})$.

iv) This result has been already shown in the proof of Prop. 1.

v) The set $\mathcal{PA}(P_1 \uplus P_2)$ can be partitioned in three parts in terms of the antichain a: $a \subseteq P_1, a \subseteq P_2$ and $a = \{x, y\}, x \in P_1, y \in P_2$. Then, if $\mathcal{A}_2(P)$ denotes the set of antichains of two elements of P, iii) implies that $pa(P_1 \uplus P_2)$ is given by

$$\sum_{(x,y) \in \mathcal{A}_2(P_1)} i((P_1 \setminus \updownarrow a) \uplus P_2) + \sum_{(x,y) \in \mathcal{A}_2(P_2)} i(P_1 \uplus (P_2 \setminus \updownarrow a)) + \sum_{(x,y) \in P_1 \times P_2} i((P_1 \setminus \updownarrow \{x\}) \uplus (P_2 \setminus \updownarrow \{y\})).$$

On the other hand, note that $i((P_1 \setminus \updownarrow a) \uplus P_2) = i(P_1)i(P_2)$. Consequently, if $a \in P_1$, then $V = V_1 \cup V_2$, where $V_1 \in \mathcal{I}(P_1 \setminus \updownarrow a), V_2 \in \mathcal{I}(P_2)$. Thus, $(a, V) \in \mathcal{PA}(P_1)$ and $V_2 \in \mathcal{I}(P_2)$, whence $\sum_{(x,y) \in \mathcal{A}_2(P_1)} i((P_1 \setminus \updownarrow a) \uplus P_2) = pa(P_1)i(P_2)$. Similarly, $\sum_{(x,y) \in \mathcal{A}_2(P_2)} i(P_1 \uplus (P_2 \setminus \updownarrow a)) = i(P_1)pa(P_2)$.

vi) Let x_1, x_2 be the minimal elements of P. Observe that x_2 is the only element in P that is not related or equal to x_1. Then, every antichain of two elements a in P is related or contains x_1. Therefore, $P \setminus \updownarrow a = P_{x_1} \setminus \updownarrow a$ and $\mathcal{I}(P \setminus \updownarrow a) = \mathcal{I}(P_{x_1} \setminus \updownarrow a)$. Then, if (a, V) is a positioned antichain of P such that $x_1 \notin a$ then (a, V) is a positioned antichain of P_{x_1}. Also, if (a, V) is a positioned antichain of P_{x_1}, then (a, V) is a positioned antichain of P. Then, the positioned antichains in P are the same as P_{x_1} plus the positioned antichains with $x_1 \in a$. Since x_1 is less than every element apart from x_2, then the only position antichain with $x_1 \in a$ is $(a = \{x_1, x_2\}, V = \emptyset)$. Therefore, $pa(P) = pa(P_{x_1}) + 1$.

vii) Let x_1, x_2, x_3 be the minimal elements of P and let us study the different kinds of positioned antichains (a, V) in P and P_{x_1}.

- Suppose a contains some element in $\uparrow \widehat{x_1}$; then, $x_1 \notin V$ and $P \backslash \updownarrow a = P_{x_1} \backslash \updownarrow a$, whence V is an ideal of P_{x_1}, and then (a, V) is a positioned antichain for P_{x_1}. Reciprocally, any $(a, V) \in \mathcal{PA}(P_{x_1})$ satisfying $a \cup \uparrow \widehat{x_1} \neq \emptyset$ can be associated to $(a, V) \in \mathcal{PA}(P)$. Then,

$$\begin{array}{rccc} f: & \mathcal{I}(P \backslash \updownarrow a) & \to & \mathcal{I}(P_{x_1} \backslash \updownarrow a) \\ & V & \to & V \end{array}$$

is a bijective function.

- Suppose on the other hand $a \cap \uparrow \widehat{x_1} = \emptyset$. Observe that this case only arises if $a \subset \{x_1, x_2, x_3\}$, since x_1 is related with every non-minimal element. The possible positioned antichains are $(a = (x_1, x_2), V = \emptyset), (a = \{x_1, x_3\}, V = \emptyset), (a = \{x_1, x_2\}, V = \{x_3\}), (a = \{x_1, x_3\}, V = \{x_2\})$ and the positioned antichains for $a = \{x_2, x_3\}$. Note that the first four positioned antichains are not positioned antichains of P_{x_1}.

  Let us then turn to positioned antichains with $a = (x_2, x_3)$. We have two possible choices for V in this case: either $x_1 \in V$ or $V = \emptyset$. In the first case, note that

$$\begin{array}{rccc} f: & \mathcal{I}(P \backslash \updownarrow \{x_2, x_3\}) & \to & \mathcal{I}(P_{x_1} \backslash \updownarrow \{x_2, x_3\}) \\ & V & \to & V \setminus \{x_1\} \end{array}$$

  is a bijective function. Then, the number of ideals associated to the positioned antichain $a = \{x_2, x_3\}$ in P is the number of ideals associated to the positioned antichain $a = \{x_2, x_3\}$ in P_{x_1} plus one (the remaining case $V = \emptyset$). This positioned antichain plus the four ones above which were not in P_{x_1} give us the result $pa(P) = pa(P_{x_1}) + 5$.

viii) Note that $i(P) = i(P^\partial)$ because $A \in \mathcal{I}(P) \Leftrightarrow A^c \in \mathcal{F}(P) = \mathcal{I}(P^\partial)$.

Let us now prove that $pa(P) = pa(P^\partial)$. Let a be an antichain in P; then, a is an antichain in P^∂. On the other hand, the number of positioned antichains associated to this antichain in P is $i(P \backslash \updownarrow a)$, and the number of positioned antichains associated to a in P^∂ is $i(P^\partial \backslash \updownarrow a)$; since $P^\partial \backslash \updownarrow a = (P \backslash \updownarrow a)^\partial$, then $i(P^\partial \backslash \updownarrow a) = i((P \backslash \updownarrow a)^\partial) = i(P \backslash \updownarrow a)$. By iii) we get $pa(P) = pa(P^\partial)$. □

Lemma 6. *Let P be a finite poset with an only minimal element x; then, $pa(P) = pa(P_x)$.*

Proof: Obviously, $pa(P) \geq pa(P_x)$. Now, as $x \preceq y, \forall y \in P_x$, for any positioned antichain (a, V) in P, it follows that $x \notin a, x \notin V$. Consequently, any positioned antichain in P is also a positioned antichain in P_x, whence $pa(P) \leq pa(P_x)$ and thus, $pa(P) = pa(P_x)$. □

Lemma 7. *Let P be a finite poset such that $m(P) \geq 4$. Then, $pa(P) \leq \sum_{x \in \mathcal{M}(P)} pa(P_x)$.*

Proof: The set $\mathcal{PA}(P)$ (resp. $\mathcal{PA}(P_x)$) can be partitioned in three groups, $\mathcal{PA}_0(P), \mathcal{PA}_1(P), \mathcal{PA}_2(P)$ (resp. $\mathcal{PA}_0(P_x), \mathcal{PA}_1(P_x), \mathcal{PA}_2(P_x)$) attending the number of minimal elements of P in the antichain. We will show that for each of these three cases the result holds.

- **Case 1:** $a = \{a_1, a_2\}, a_1, a_2 \notin \mathcal{M}(P)$**.** Consider an element $x(a_1, a_2) \in \mathcal{M}(P)$ such that $x(a_1, a_2) \in \downarrow \hat{a}$. Then, $x(a_1, a_2) \notin a, x(a_1, a_2) \notin V$, whence $(a, V) \in \mathcal{PA}(P_{x(a_1, a_2)})$. Consequently,

$$pa_0(P) \leq \sum_{x \in \mathcal{M}(P)} pa_0(P_x).$$

- **Case 2:** $a = \{x, a_0\}, a_0 \notin \mathcal{M}(P), x \in \mathcal{M}(P)$**.** As in the previous case, there exists an element $x(a_0) \in \mathcal{M}(P), x(a_0) \neq x$ such that $x(a_0) \in \downarrow a$. Then, $x(a_0) \notin a, x(a_0) \notin V$, whence $(a, V) \in \mathcal{PA}(P_{x(a_0)})$. Consequently,

$$pa_1(P) \leq \sum_{x \in \mathcal{M}(P)} pa_1(P_x).$$

- **Case 3:** $a = \{x_i, x_j\}, x_i, x_j \in \mathcal{M}(P)$**.** Note that for fixed $\{x_i, x_j\}$ the number of possible positioned antichains in P (resp. P_x, for $x \in \mathcal{M}(P) \backslash \{x_i, x_j\}$) is given by $i(P \backslash \uparrow \{x_i, x_j\})$ (resp. $i(P_x \backslash \uparrow \{x_i, x_j\})$). Thus, by Lemma 5 iii),

$$pa_2(P) = \sum_{(x_i,x_j) \in \mathcal{M}(P)^2} i(P \backslash \uparrow \{x_i, x_j\}), \quad \sum_{x \in \mathcal{M}(P)} pa_2(P_x) = \sum_{x \in \mathcal{M}(P)} \sum_{(x_i,x_j) \in \mathcal{M}(P) \backslash \{x\}} i(P_x \backslash \uparrow \{x_i, x_j\}).$$

Now, defining $P^{ij} = P \backslash \uparrow \{x_i, x_j\}$, we have

$$\begin{aligned} \sum_{x \in \mathcal{M}(P)} \sum_{(x_i,x_j) \in \mathcal{M}(P) \backslash \{x\}} i(P_x \backslash \uparrow \{x_i, x_j\}) &= \sum_{x \in \mathcal{M}(P)} \sum_{(x_i,x_j) \in \mathcal{M}(P) \backslash \{x\}} i(P^{ij} \setminus \{x\}) \\ &= \sum_{(x_i,x_j) \in \mathcal{M}(P)} \sum_{x \in \mathcal{M}(P) \backslash \{x_i,x_j\}} i(P^{ij} \setminus \{x\}) \end{aligned}$$

Since there are at least four minimal elements in P, the last expression has at least two addends. If $x^*_{i,j}$ is the minimal element with least $i(P^{ij} \setminus \{x^*_{i,j}\})$, it follows from Lemma 5 ii)

$$\begin{aligned} \sum_{(x_i,x_j) \in \mathcal{M}(P)^2} \sum_{x \in \mathcal{M}(P) \backslash \{x_i,x_j\}} i(P^{ij} \setminus \{x\}) &\geq \sum_{(x_i,x_j) \in \mathcal{M}(P)^2} 2i(P^{ij} \setminus \{x^*_{i,j}\}) \\ &\geq \sum_{(x_i,x_j) \in \mathcal{M}(P)^2} i(P^{ij}) \\ &= \sum_{(x_i,x_j) \in \mathcal{M}(P)^2} i(P \backslash \uparrow \{x_i, x_j\}) \end{aligned}$$

Adding up these three cases, the result holds. □

It is not difficult to find examples showing that there are not similar results to Lemmas 6 and 7 when $m(P) = 2$ and $m(P) = 3$. However, there is a special case where a similar result holds.

Lemma 8. *Let P be a finite poset such that $m(P) = 2$ and P has at least two maximal elements which are both non-minimal elements. Then:*

$$pa(P) \leq pa(P_{x_1}) + pa(P_{x_2}).$$

Proof: The proof is quite similar to the previous one. The set $\mathcal{PA}(P)$ (and $\mathcal{PA}(P_{x_1}), \mathcal{PA}(P_{x_2})$) can be partitioned in three groups, $\mathcal{PA}_0(P), \mathcal{PA}_1(P), \mathcal{PA}_2(P)$ (resp. $\mathcal{PA}_0(P_{x_i}), \mathcal{PA}_1(P_{x_i}), \mathcal{PA}_2(P_{x_i})$) attending the number of minimal elements of P in the antichain. The first two cases can be treated as in the previous lemma; in particular, $(\{M_1, M_2\}, V = \emptyset)$ is linked to $(\{M_1, M_2\}, \emptyset)$ in, say, $\mathcal{PA}(P_{x_1})$, with $x_1 \in \downarrow \{M_1, M_2\}$.

Let us then deal with the case of $\mathcal{PA}_2(P)$. In this case, the only positioned antichain is given by $(a = \{x_1, x_2\}, V = \emptyset)$. Then, we can link this positioned antichain to $(a = \{M_1, M_2\}, V = \emptyset) \in \mathcal{PA}(P_{x_2})$, which has not been linked to any other positioned antichain. □

Lemma 9. *Let P be a finite poset such that $\mathcal{M}(P) = \{x_1, x_2, x_3\}$ and P has at least three maximal elements which are non-minimals; then,*

$$pa(P) \leq pa(P_{x_1}) + pa(P_{x_2}) + pa(P_{x_3}).$$

Proof: The proof is very similar to the previous proofs. The set $\mathcal{PA}(P)$ (and $\mathcal{PA}(P_{x_i}), x_i \in \mathcal{M}(P)$) can be partitioned in three groups, $\mathcal{PA}_0(P), \mathcal{PA}_1(P), \mathcal{PA}_2(P)$ (resp. $\mathcal{PA}_0(P_{x_i}), \mathcal{PA}_1(P_{x_i}), \mathcal{PA}_2(P_{x_i})$) attending the number of minimal elements of P in the antichain a. The first two cases can be treated as in Lemma 7. In particular, if we denote by M_1, M_2, M_3 three maximal non-minimal elements, the positioned antichain $(a = \{M_i, M_j\}, V = \emptyset)$ is linked to $(a, V) \in \mathcal{PA}(P_{x(M_i,M_j)})$ for a $x(M_i, M_j) \in \downarrow \{M_i, M_j\}$.

Let us then deal with the situation where $a = \{x_i, x_j\}$ and consider the positioned antichain (a, V). In this case, either $V = \emptyset$ or $x_k \in V$, where x_k is the minimal outside a. If $x_k \in V$, we link (a, V) to $(a, V \setminus \{x_k\}) \in \mathcal{PA}(P_{x_k})$. If $V = \emptyset$, we link $(a = \{x_i, x_j\}, V = \emptyset)$ to $(a = \{M_i, M_j\}, V = \emptyset) \in \mathcal{PA}(P_{x^*(M_i,M_j)})$ for $x^*(M_i, M_j) \neq x(M_i, M_j)$. □

Lemma 10. *Let P be a finite poset with 2 minimals x_1 and x_2 and such that $P \setminus \{x_1, x_2\}$ is neither the empty set nor a chain; then,*

$$\sum_{x \in P} i(P \setminus \updownarrow x) \leq (|P| - 1)e(P)$$

Proof: We are going to build an injective function F from $\mathcal{I}(P \setminus \updownarrow x) \times \{x\}$ to $\mathcal{L}(P) \times \{1, 2, \cdots, |P| - 1\}$. Define $f : P \to \{1, 2, \cdots, |P| - 1\}$ such that $f(x_1) = f(x_2) = 1$, $f(P \setminus \{x_1, x_2\}) = \{2, ..., |P| - 1\}$ (i.e. f is a bijection between $P \setminus \{x_1, x_2\}$ and $\{2, ..., |P| - 1\}$) and such that $f(i) < f(j)$ if $i \preceq j$.

Take $x \in P \setminus \{x_i, x_j\}$ and consider $(i, x) \in \mathcal{I}(P \setminus \updownarrow x) \times \{x\}$. We define $F(i, x) = (\epsilon, f(x))$, where ϵ is a linear extension given by $\epsilon := (i, \downarrow \hat{x}, x, R)$ where $R := P \setminus (i \cup \downarrow x)$ and such that the order in the elements of each part are given according to f and x_1 is placed before x_2. F is well-defined: note that $i \cap \downarrow x = \emptyset$ and there is no contradiction with the order if we place elements of i before elements of $\downarrow \hat{x}$; for if $y \in i, z \in \downarrow \hat{x}, z \preceq y$, then $z \in i$ as i is an ideal, whence $z \in i \cap \downarrow \hat{x} = \emptyset$, that is not possible. Similarly, there is no contradiction placing elements of R after i or $\downarrow \hat{x}$. Note that F is injective so far. Indeed, as f is bijective on $P \setminus \{x_1, x_2\}$, the value $f(x)$ provides element x; and then, i can be found as the elements placed before x and outside $\downarrow \hat{x}$.

Consider now $(i, x_k) \in \mathcal{I}(P \setminus \updownarrow x_k) \times \{x_k\}$. We define $F(i, x_k) = (\epsilon, 1)$. Let us then define ϵ:

- If $i \neq \emptyset$ (then i contains a minimal element) then we define $\epsilon = (i, x_k, R)$ where elements in i and in R are placed in increasing order according to f.

- Finally, $F(\emptyset, x_1) = (x_1, x_2, R^*)$ and $F(\emptyset, x_2) = (x_2, x_1, R^*)$, where R^* is a linear extension of $P \setminus \{x_1, x_2\}$ such that it does not fit to f. Note that this is always possible as $P \setminus \{x_1, x_2\} \neq \emptyset$ and it is not a chain. □

Now we can prove the principal theorem of this appendix.

Proof of Theorem 3: We are going to prove it by induction on $n = |P|$.

For $n = 1$ the only poset is 1 with $pa(1) = 0 < 1 = e(1)$.

Assume $pa(P) \leq e(P)$ holds if $|P| \leq n$ and let us prove the result for a poset with $n+1$ elements. We have several cases:

- **Case 1:** $m(P) = 1$. Applying the induction hypothesis and Lemma 6, if x is the minimum,

$$pa(P) = pa(P_x) \leq e(P_x) = e(P),$$

  whence the result.

- **Case 2:** $m(P) \geq 4$. Applying the induction hypothesis and Lemma 7

$$pa(P) \leq \sum_{x \in \mathcal{M}(P)} pa(P_x) \leq \sum_{x \in \mathcal{M}(P)} e(P_x) = e(P),$$

  whence the result.

- **Case 3:** $m(P) = 2$. We consider several cases in terms of the number of maximal elements that are not minimal in P. Let us denote by S this set.

  - If $S = \emptyset$, then $P = \overline{2}$ and $pa(\overline{2}) = 1 < 2 = e(\overline{2})$.
  - If $|S| = 1$, then either P has a maximum or $P = P^* \uplus \mathbf{1}$, where P^* has an only minimum. If P has a maximum, applying $pa(P) = pa(P^\partial)$ (Lemma 5 viii)), $e(P) = e(P^\partial)$ and the first case,
$$pa(P) = pa(P^\partial) \leq e(P^\partial) = e(P).$$
    Otherwise, if $P = P^* \uplus \mathbf{1}$, where P^* has an only minimum, we can apply Lemma 5 i), vi) and the induction hypothesis, whence

$$pa(P^* \uplus 1) = pa(P^*_{x_1} \uplus \mathbf{1}) + 1 \leq e(P^*_{x_1} \uplus \mathbf{1}) + 1 = (|P^*_{x_1}| + 1)e(P^*_{x_1}) + 1$$
$$\leq (|P^*_{x_1}| + 1)e(P^*_{x_1}) + e(P^*_{x_1}) = (|P^*_{x_1}| + 2)e(P^*_{x_1}) = (|P^*| + 1)e(P^*) = e(P^* \uplus \mathbf{1}).$$

- **Case 4:** $m(P) = 3$**.** In this last case we can suppose that P has exactly 3 maximal elements because otherwise we can apply Lemma 5 viii), $e(P) = e(P^\partial)$ and the corresponding case studied before to conclude:

$$pa(P) = pa(P^\partial) \leq e(P^\partial) = e(P).$$

  Then, let us suppose that P has three maximal elements; let us denote by S the set of maximal elements of P that are not minimal. We consider different cases in terms of $|S|$.

  - If $|S| = 0$, then $P = \overline{3}$ and $pa(\overline{3}) = 6 = e(\overline{3})$.

– If $|S| = 1$, then $P = P^* \uplus \mathbf{1} \uplus \mathbf{1}$, where P^* has just one minimal x_1. Then, applying Lemma 5 vii) and the induction hipothesis,

$$pa(P) = pa(P_{x_1}) + 5 \leq e(P_{x_1}) + 5.$$

We will prove that in this case $e(P_{x_1}) + 5 \leq e(P)$. To prove this, observe that if we take any linear extension of P_{x_1}, it can be obtained from a linear extension of P removing x_1. Then, the map

$$\begin{array}{cccc} G: & \mathcal{L}(P) & \rightarrow & \mathcal{L}(P_{x_1}) \\ & (\epsilon_1, ..., \epsilon_i, x_1, \epsilon_{i+2}, ..., \epsilon_n) & \rightarrow & (\epsilon_1, ..., \epsilon_i, \epsilon_{i+2}, ..., \epsilon_n) \end{array}$$

is a surjective function. Note that:

* $G(x_1, x_2, x_3, z, \ldots) = G(x_2, x_1, x_3, z, \ldots) = G(x_2, x_3, x_1, z, \ldots) = (x_2, x_3, z, \ldots)$.
* $G(x_1, x_3, x_2, z, \ldots) = G(x_3, x_1, x_2, z, \ldots) = G(x_3, x_2, x_1, z, \ldots) = (x_3, x_2, z, \ldots)$.
* $G(x_1, x_3, z, x_2, \ldots) = G(x_3, x_1, z, x_2, \ldots) = (x_3, z, x_2, \ldots)$.

Thus, $e(P) \geq e(P_{x_1}) + 5$ and the result holds.

– If $|S| = 2$, then $P = P^* \uplus \mathbf{1}$, where P^* is a poset with 2 minimal elements and two (different) maximal elements. Then, we can apply Lemma 5 v), Lemma 10 and the induction hypothesis to conclude

$$pa(P^* \uplus \mathbf{1}) = 2pa(P^*) + \sum_{x \in P^*} i(P^* \setminus \updownarrow x) \leq 2e(P^*) + \sum_{x \in P^*} i(P^* \setminus \updownarrow x)$$

$$\leq 2e(P^*) + (|P^*| - 1)e(P^*) = (|P^*| + 1)e(P^*) = e(P^* \uplus \mathbf{1}).$$

– If $|S| = 3$, we can use Lemma 9 and the induction hipothesis to conclude

$$pa(P) \leq pa(P_{x_1}) + pa(P_{x_2}) + pa(P_{x_3}) \leq e(P_{x_1}) + e(P_{x_2}) + e(P_{x_3}) = e(P).$$

Therefore we have proved the inductive step for any P with $n+1$ elements, and then by induction the result holds. □

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