Combinatorial structure of the polytope of 2-additive measures

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Abstract

In this paper we study the polytope of 2-additive measures, an important subpolytope of the polytope of fuzzy measures. For this polytope, we obtain its combinatorial structure, namely the adjacency structure and the structure of 2-dimensional faces, 3-dimensional faces, and so on. Basing on this information, we build a triangulation of this polytope satisfying that all simplices in the triangulation have the same volume. As a consequence, this allows a very simple and appealing way to generate points in a random way in this polytope, an interesting problem arising in the practical identification of 2-additive measures. Finally, we also derive the volume, the centroid, and some properties concerning the adjacency graph of this polytope.

Keywords: Fuzzy measures; 2-additive measures; Combinatorial structure; triangulation; random generation.

1 Introduction

Consider a finite set X of n elements, $X = \{x_1, ..., x_n\}$. Elements of X are criteria in the field of Multicriteria Decision Making, players in Cooperative Game Theory, and so on. We will denote subsets of X by $A, B, ...$ In order to avoid hard notation, we will often use $i_1 i_2 \cdots i_n$ for denoting the set $\{i_1, i_2, \cdots, i_n\}$, specially for singletons and pairs. We also define $\binom{X}{k}$ as the set of all k-element subsets of X.

Definition 1. *A* **fuzzy measure** *[34] (or* **capacity** *or* **non-additive measure***) over X is a set function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying*

- *Boundary conditions: $\mu(\emptyset) = 0, \mu(X) = 1$.*
- *Monotonicity: $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$.*

We will denote by $\mathcal{FM}(X)$ the set of all fuzzy measures on X; it can be easily seen that it is a bounded polyhedron, i.e. a polytope. Thus defined, fuzzy measures are a generalization of probability measures where additivity has been replaced by monotonicity. Fuzzy measures, together with Choquet integral [4], have been proved to be a powerful tool applying to many different fields, as Decision Making, Game Theory and Imprecise Probabilities among many others (see e.g. [13, 36] and the references therein). The key of this success relies on the fact that fuzzy measures are able to model situations where probability measures fail. For example, in the field of Multicriteria Decision Making, fuzzy measures allow to model situations of veto and favor, and can also model interactions between criteria [9, 17]; in Cooperative Game Theory, fuzzy measures are called normalized monotone games and constitute the basis of NTU-games [29]. This ability of modeling many situations have led to a deep study of fuzzy measures both from a theoretical and applied point of view (see e.g. [20, 14, 15, 11, 8]).

On the other hand, this wealth in terms of interpretation has to be paid with an increment of the complexity. Thus, while for a referential of n elements just $n-1$ coefficients suffice to define a probability measure, 2^n-2 coefficients are needed to define a fuzzy measure over the same referential. Then, the complexity grows in an exponential way and makes fuzzy measures infeasible in practice for big referentials. To cope with this problem, several alternatives have arisen; for example, some subsets of X may be forbidden; this situation is very common in Game Theory, where some coalitions of players may not be possible [12]. Another alternative, more usual in Multicriteria Decision Making, is to add additional constraints to the definition, thus allowing a reduction in the number of coefficients needed to define a fuzzy measure while keeping the wealth in terms of modeling; examples of subfamilies in this case are k-intolerant measures [22], k-maxitive measures [1], k-symmetric measures [26], and many others. Perhaps the most successful subfamily is the subfamily of k-additive measures [10], and inside this subfamily, the most appealing case is the case of 2-additive measures; as it will become clear below, 2-additive measures are able to model a great deal of situations while keeping a reduced complexity.

The aim of the paper is double: first, we study the structure of the set of fuzzy measures being 1-additive or 2-additive; this set is a polytope and we characterize its k-dimensional faces; we also derive some other properties concerning the geometrical structure of this polytope. Second, inspired by the results about adjacency, we develop a procedure to generate uniformly distributed random points in this set; for this, we obtain a triangulation of this polytope; as a consequence of the results for the algorithm, we also derive some other properties of this polytope, as the volume or the centroid. We finish the paper with a section where some results related to the adjacency graph of this polytope are presented.

There is no doubt about the mathematical appeal of these problems. We just want to remark here that sorting the combinatorial structure of a polytope is in general a complex problem [37] and this also applies for other problems derived from the combinatorial structure, as the volume [2]. For example, even if the definition of adjacency between two vertices provides a way to determine whether two vertices are adjacent, it is difficult in general to decide *at a glance* if two vertices satisfy this condition. The same can be said for higher order faces and this can be extended to characterization of vertices, and many other problems.

Similarly, finding a random way to generate points in a polytope is a difficult problem that has attracted attention of many researchers (see e.g. [7]) but has not been solved in general in a satisfactory way. Thus, developing an algorithm for random generation in a family of polytopes is an interesting problem from a theoretical point of view.

We add now a practical reason for this study, that is linked to the problem of identification of fuzzy measures. This problem arises when we have some sample data and we look for the fuzzy measure (possibly restricted to a subfamily) that best fits them. There are many procedures to deal with this problem [27]. Among them, in [5], a method for convex families of fuzzy measures based on genetic algorithms has been proposed. This algorithm is very fast and the simulations carried out suggest that it is stable with respect to the presence of noise. However, the mutation operator of this algorithm should be based on random generating points in the polytope [24]. This is the problem we tackle in Section 4.

The rest of the paper goes as follows. In next section, we will review the basic results and definitions that will be needed in the paper. Section 3 deals with the combinatorial structure of the set of fuzzy measures being 1-aditive or 2-additive; thus, we study some properties of the geometry of this polytope, namely the adjacency structure and k-dimensional faces. In Section 4, we obtain a random procedure to generate uniformly distributed points in this polytope; from this procedure, we also obtain as a corollary some other characteristics of this set. Section 5 deals with the adjacency graph of this polytope. We finish with the conclusions and open problems.

2 Basic results on 2-additivite measures

Let us start with the concept of k-additivity. This concept is based on the Möbius transform.

Definition 2. *[30] Let μ be a set function (not necessarily a fuzzy measure) on X. The* **Möbius transform (or inverse)** *of μ is another set function on X defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \ \forall A \subseteq X.$$

The Möbius transform is an alternative representation of fuzzy measures, in the sense that given the Möbius transform m of a fuzzy measure μ, this measure can be recovered through the *Zeta transform* [3]:

$$\mu(A) = \sum_{B \subseteq A} m(B).$$

As explained before, the Möbius transform can be applied to any set function; if related to a fuzzy measure, the Möbius transform can be characterized as follows.

Proposition 1. *[3] A set of 2^n coefficients $m(A)$, $A \subseteq N$, corresponds to the Möbius transform of a fuzzy measure if and only if*

1. $m(\emptyset) = 0, \ \sum_{A \subseteq N} m(A) = 1,$

2. $\sum_{i \in B \subseteq A} m(B) \geq 0$, *for all* $A \subseteq N$, $\forall i \in A$.

The Möbius transform corresponds to the *basic probability mass assignment* in Dempster-Shafer theory of evidence [32] and the Harsanyi *dividends* [18] in Cooperative Game Theory. The Möbius transform gives a measure of the importance of a coalition by itself, without taking account of its different parts. Thus, remark that it could be difficult for an expert to assess values to the interactions of say, 4 criteria, and interpret what these interactions mean. Then, it makes sense to restrict the range of allowed interactions to coalitions of a reduced number of criteria, i.e. no interactions among more than k criteria are permitted. This translates into the condition $m(A) = 0$ if $|A| > k$. Based on this fact, we arrive to the concept of k-additivity in a natural way.

Definition 3. *[10] A fuzzy measure μ is said to be k-**additive** if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$.*

From this definition, it follows that a probability measure is just a 1-additive measure; therefore, k-additive measures generalize probability measures and constitute a gradation between probability measures and general fuzzy measures (n-additive measures). For a k-additive measure, the number of coefficients is reduced to

$$\sum_{i=1}^{k} \binom{n}{i} - 1,$$

a middle term between $n-1$ (probabilities) and $2^n - 2$ (general fuzzy measures). We will denote by $\mathcal{FM}^k(X)$ the set of all fuzzy measures being *at most* k-additive. Specially appealing is the 2-additive case, that allows to model interactions between two criteria, that are the most important interactions, while keeping a reduced (indeed quadratic) complexity. Moreover, simple expressions for μ in terms of the Möbius transform, an the corresponding Choquet integral can be obtained (see [10]). Indeed, applying the monotonicity conditions on the Möbius transform, it can be seen that μ can be recovered via the values of μ on singletons and all pairs except one of them, say $\{n-1, n\}$ through the following equations:

$$\mu(\{n-1, n\}) = 1 - \sum_{ij \in \binom{X}{2}} \mu(ij) + (n-2) \sum_{i=1}^{n} \mu(i) + \mu(n) + \mu(n-1), \tag{1}$$

where $\mathcal{S} = \{1, 2 \ldots, n, 12, 13, \ldots, (n-2)n\}$.

$$\mu(A) = \sum_{B \subset A} (-1)^{|A \setminus B|+1} \mu(B) = \sum_{ij \in A} \mu(ij) - (|A| - 2) \sum_{i \in A} \mu(i), \ \forall |A| > 2. \tag{2}$$

3 Combinatorial structure of $\mathcal{FM}^2(X)$

In this section we tackle the problem of obtaining the combinatorial structure of $\mathcal{FM}^2(X)$, that is, its k-dimensional faces. For the different concepts relating polytopes appearing in this section, see [37]. It can be easily seen that $\mathcal{FM}^2(X)$ is a convex polyhedron in $\mathbb{R}^{2^n-2}$, i.e., a polytope, as it is the intersection of the polytope $\mathcal{FM}(X)$ and the hyperplanes $m(A) = 0$, $|A| > 2$. Then, it can be characterized in terms of its vertices. The vertices of $\mathcal{FM}^2(X)$ have been obtained in [25] and are given in next proposition.

Proposition 2. *The set of vertices of $\mathcal{FM}^2(X)$ are given by the $\{0,1\}$-valued fuzzy measures in $\mathcal{FM}^2(X)$, i.e., u_i, u_{ij}, μ_{ij}, that are defined by*

$$u_A(B) := \begin{cases} 1 & \text{if } A \subseteq \mathrm{B} \\ 0 & \text{otherwise} \end{cases}, \quad \mu_{ij}(B) := \begin{cases} 1 & \text{if } i \in \mathrm{B} \text{ or } j \in \mathrm{B} \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathcal{FM}^2(X)$ has n^2 vertices.

This result shows an important difference between 2-additive measures and general k-additive measures, $k > 2$, as it has been proved in [25] that there are vertices in $\mathcal{FM}^k(X), k > 2$ that are not $\{0,1\}$-valued.

Recall that two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are **combinatorially equivalent** if there is a one-to-one correspondence ϕ between the set of all faces of $\mathcal{P}$ and the set of all faces of $\mathcal{Q}$, such that ϕ is inclusion-preserving, i.e. two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathcal{P}$ satisfy $\mathcal{F}_1 \subset \mathcal{F}_2 \Leftrightarrow \phi(\mathcal{F}_1) \subset \phi(\mathcal{F}_2)$ in $\mathcal{Q}$.

For $\mathcal{FM}^k(X)$, it is convenient in many situations to use the equivalent Möbius transform. For this reason, let us define a map that will be of aid in the following:

$$\begin{array}{rccc} \mathsf{m}: & \mathcal{FM}(X) & \rightarrow & \mathcal{M}(X) \\ & \mu & \hookrightarrow & m_\mu \end{array}$$

We will denote $\mathcal{M}^2(X) := \mathsf{m}(\mathcal{FM}^2(X))$. As m is a nonsingular linear application, we conclude that $\mathcal{M}^2(X)$ is a convex polytope that is combinatorially equivalent to $\mathcal{FM}^2(X)$ (see [37]). Thus, this function maps k-dimensional faces into k-dimensional faces and keeps adjacency and any other result concerning the combinatorial structure. Note however that it does not keep the volume nor distances.

This way, we can study k-dimensional faces of $\mathcal{M}^2(X)$ (a simpler problem, as it will be seen below) and apply m^{-1} to get the same conclusions about the k-dimensional faces of $\mathcal{FM}^2(X)$. In particular, m maps vertices into vertices; hence, the vertices of $\mathcal{M}^2(X)$ are:

For u_i : we get m_i defined as $m_i(i) = 1$ and $m_i(A) = 0, \forall A \neq i$.

For u_{ij} : we get m_{ij} defined as $m_{ij}(ij) = 1$ and $m_{ij}(A) = 0, \forall A \neq ij$.

For μ_{ij} : we get $\overline{m}_{ij}$ defined as $\overline{m}_{ij}(ij) = -1, \overline{m}_{ij}(i) = 1, \overline{m}_{ij}(j) = 1$ and $\overline{m}_{ij}(A) = 0, \forall A \notin \{i, j, ij\}$.

We start studying the dimension of $\mathcal{FM}^2(X)$. Recall that the **dimension** of a polytope P is defined as the dimension of the smallest affine subspace containing its vertices, denoted aff(P) (for more details, see [37]).

Lemma 1. $dim(\mathcal{FM}^2(X)) = \binom{n}{2} + n - 1.$

Proof. It is clear that $dim(\mathcal{FM}(X)) = 2^n - 2$. Hence, $dim(\mathcal{M}(X)) = 2^n - 2$. Now, remark that

$$\mathcal{M}^2(X) = \mathcal{M}(X) \bigcap_{|A|\geq 3} \{m : m(A) = 0\}.$$

Therefore, $dim(\mathcal{M}^2(X)) = 2^n - 2 - \sum_{i=3}^{n} \binom{n}{i} = \binom{n}{2} + n - 1$. □

In what follows, we will study the properties of $\mathcal{FM}^2(X)$ as a polytope in $\mathbb{R}^{\binom{n}{2}+n-1}$. We start showing that, although the vertices of $\mathcal{FM}^2(X)$ are $\{0, 1\}$-valued, it is not an order polytope. These polytopes are defined in terms of a partial ordered set (brief poset) as follows:

Definition 4. *[33] Given a poset $(P, \preceq)$ (or P for short) with n elements, it is possible to associate to P a polytope $\mathcal{O}(P)$ over $\mathbb{R}^p$, called the* **order polytope** *of P, formed by the p-uples f of real numbers indexed by the elements of P satisfying*

- $0 \leq f(a) \leq 1$ *for every a in P,*
- $f(a) \leq f(b)$ *whenever $a \preceq b$ in P.*

For example, it can be seen [6] that $\mathcal{FM}(X)$ is an order polytope whose subjacent poset is the Boolean set on X, i.e. the poset $\mathcal{P}(X)\backslash\{\emptyset, X\}$ with the order $A \preceq B \Leftrightarrow A \subseteq B$.

Our interest in order polytopes comes from the fact that there are known results characterizing the vertices and many other properties of the polytope $\mathcal{O}(P)$ in terms of the structure of the subjacent poset P. For example, it can be seen [33] that the vertices of $\mathcal{O}(P)$ are the characteristic functions of filters of P. Similarly, it can be seen [6] that two vertices of an order polytope whose associated filters are F_1 and F_2 are adjacent vertices if and only if the symmetric difference $F_1 \Delta F_2 := (F_1\backslash F_2) \uplus (F_2\backslash F_1)$ is a connected subposet of P.

Proposition 3. *If $|X| > 2$, the polytope $\mathcal{FM}^2(X)$ is not an order polytope.*

Proof. Assume there is a poset P such that $\mathcal{O}(P) = \mathcal{FM}^2(X)$. Then, P is a filter for P and thus there is a maximum vertex with all coordinates equal to 1, i.e. there is a vertex $\mu \in \mathcal{FM}^2(X)$ such that $\mu(A) \geq \mu'(A), \forall A \subseteq X, \forall \mu' \in \mathcal{FM}^2(X)$. However, if $|X| > 2$, there is not $\mu \in \mathcal{FM}^2(X)$ dominating say μ_{12}, and this measure does not dominate all 2-additive measures. Therefore, we get a contradiction and the result holds. If $|X| = 2$, then $\mathcal{FM}^2(X) = \mathcal{FM}(X)$ which is an order polytope. □

As $\mathcal{FM}^2(X)$ is not an order polytope, we cannot apply the results of [6, 33] to sort out the combinatorial structure of this polytope and we have to look for another way to solve the problem.

Remember that a **face** of a polytope $\mathcal{P}$ is defined as a subset $\mathcal{F} \subseteq \mathcal{P}$ such that there exists a vector $\boldsymbol{\alpha}$ and a constant $c \in \mathbb{R}$ such that

$$\boldsymbol{\alpha}^T \cdot \boldsymbol{x} \leq c, \; \forall \boldsymbol{x} \in \mathcal{P} \quad \text{and} \quad \mathcal{F} = \mathcal{P} \cap \{\boldsymbol{\alpha}^T \cdot \boldsymbol{x} = c\}.$$

We will denote the face defined via $\boldsymbol{\alpha}$ and c by $\mathcal{F}_{\boldsymbol{\alpha},c}$. Equivalently, a face $\mathcal{F}$ is also characterized by the vertices of $\mathcal{P}$ in $\mathcal{F}$.

Given $\mathcal{C}$ a set of vertices of $\mathcal{P}$, we will denote by $\text{Conv}(\mathcal{C})$ the convex hull of $\mathcal{C}$. Note that if $\mathcal{C}$ is the set of vertices of a face $\mathcal{F}$, then $\mathcal{F} = \text{Conv}(\mathcal{C})$. When this is the case, we will denote by $\mathcal{F}_{\mathcal{C}}$ the face defined by vertices in $\mathcal{C}$. On the other hand, remark that it could be that a family $\mathcal{C}$ of vertices does not determine a face. Then, the problem of determining all the faces of a polytope consists in characterizing the conditions for a subset of vertices $\mathcal{C}$ to determine a face. Next lemma shows the basic result for characterizing the faces of $\mathcal{FM}^2(X)$.

Lemma 2. *Let $\mathcal{F}$ be a face of $\mathcal{FM}^2(X)$. Then, $u_{ij}, \mu_{ij} \in \mathcal{F}$ if and only if $u_i, u_j \in \mathcal{F}$.*

Proof. We are going to show the result for $\mathcal{M}^2(X)$. Let $\mathcal{F}_{\boldsymbol{\alpha},c}$ be the face of $\mathcal{M}^2(X)$ defined by $\boldsymbol{\alpha} = \{\alpha_i, \alpha_{ij} : i, j \in X\}$ and c. By definition of face $\boldsymbol{\alpha} \cdot m \leq c$, $\forall m \in \mathcal{M}^2(X)$ and $\boldsymbol{\alpha} \cdot m = c$, $\forall m \in \mathcal{F}_{\boldsymbol{\alpha},c}$.

$\Rightarrow$) Since $m_{ij}, \overline{m}_{ij} \in \mathcal{F}_{\boldsymbol{\alpha},c}$, we get

$$\left.\begin{array}{r} \alpha_{ij} = c \\ -\alpha_{ij} + \alpha_i + \alpha_j = c \end{array}\right\} \Rightarrow \alpha_i + \alpha_j = 2c.$$

If $m_i \notin \mathcal{F}_{\boldsymbol{\alpha},c}$, then, $\alpha_i < c$, and thus $\alpha_j > c$, implying $m_j \notin \mathcal{M}^2(X)$, a contradiction. Thus, $m_i \in \mathcal{F}_{\boldsymbol{\alpha},c}$, so that $\alpha_i = c$, and hence $\alpha_j = c$, thus concluding $m_j \in \mathcal{F}_{\boldsymbol{\alpha},c}$.

$\Leftarrow$) Since $m_i, m_j \in \mathcal{F}_{\boldsymbol{\alpha},c}$, we get $\alpha_i = c, \alpha_j = c$. If $m_{ij} \notin \mathcal{F}_{\boldsymbol{\alpha},c}$, then $\alpha_{ij} < c$; but this implies that $-\alpha_{ij} + \alpha_i + \alpha_j > c$, and hence $\overline{m}_{ij} \notin \mathcal{M}^2(X)$, a contradiction. Thus, $m_{ij} \in \mathcal{F}_{\boldsymbol{\alpha},c}$ and hence $\alpha_{ij} = c$. Consequently, $-\alpha_{ij} + \alpha_i + \alpha_j = c$ and $\overline{m}_{ij} \in \mathcal{F}_{\boldsymbol{\alpha},c}$. □

We are now in a position to present the main result of this section, in which we give a complete characterization of the faces of $\mathcal{FM}^2(X)$. In it, we show that the necessary condition of Lemma 2 is also sufficient.

Theorem 1. ***Combinatorial structure of*** $\mathcal{FM}^2(X)$. *Let $\mathcal{C}$ be a collection of vertices of $\mathcal{FM}^2(X)$. Then the following are equivalent:*

i) $\text{Conv}(\mathcal{C})$ *is a face of* $\mathcal{FM}^2(X)$.

ii) $u_i, u_j \in \mathcal{C} \Leftrightarrow u_{ij}, \mu_{ij} \in \mathcal{C}$.

Proof. *i)* $\Rightarrow$ *ii)* This is Lemma 2.

ii) $\Rightarrow$ *i)* Consider a set of vertices $\mathcal{C}$ satisfying ii) and consider the corresponding vertices in $\mathcal{M}^2(X)$. Let us define

$$H := \{i \in X : u_i \in \mathcal{C}\},\ I := \{ij \in \binom{X}{2} : u_{ij}, \mu_{ij} \in \mathcal{C}\},$$

$$A := \{ij \in \binom{X}{2} : u_{ij} \in \mathcal{C}, \mu_{ij} \notin \mathcal{C}\},\ B := \{ij \in \binom{X}{2} : \mu_{ij} \in \mathcal{C}, u_{ij} \notin \mathcal{C}\}.$$

Now, we define the following halfspace

$$2\sum_{i \in H} m(i) + 2\sum_{ij \in A \cup I} m(ij) + \sum_{ij \notin A \cup B \cup I,\ i \in H, j \notin H} m(ij) - 2\sum_{ij \in B, i \notin H, j \notin H} m(ij) \leq 2.$$

Let us show that this halfspace defines a face in $\mathcal{M}^2(X)$ such that the corresponding face in $\mathcal{FM}^2(X)$ has $\mathcal{C}$ as vertices. First, let us see that $\mathcal{M}^2(X)$ is in the halfspace. To show this, it suffices to check that all vertices in $\mathcal{M}^2(X)$ are in the halfspace. By ii), we have $ij \in I \Leftrightarrow i, j \in H$, thus avoiding also $ij \in B, i, j \in H$ and $ij \in A, i, j \in H$.

For m_i, $i \in H$: we get $2 \leq 2$.

For m_i, $i \notin H$: we get $0 \leq 2$.

For m_{ij}, $ij \in I$: we get $2 \leq 2$.

For m_{ij}, $ij \in A$: we get $2 \leq 2$.

For m_{ij}, $ij \in B$: we get $0 \leq 2$ if $i \in H, j \notin H$ and $-2 \leq 2$ if $i \notin H, j \notin H$.

For m_{ij}, $ij \notin A \cup B \cup I$: we get $1 \leq 2$ if $i \in H, j \notin H$ and $0 \leq 2$ if $i \notin H, j \notin H$.

For $\overline{m}_{ij}$, $ij \in I$: we get $2 \leq 2$.

For $\overline{m}_{ij}$, $ij \in A$: we get $0 \leq 2$ if $i \in H, j \notin H$ and $-2 \leq 2$ if $i \notin H, j \notin H$.

For $\overline{m}_{ij}$, $ij \in B$: we get $2 \leq 2$ if $i \in H, j \notin H$ and $2 \leq 2$ if $i \notin H, j \notin H$.

For $\overline{m}_{ij}$, $ij \notin A \cup B \cup I$: we get $1 \leq 2$ if $i \in H, j \notin H$ and $0 \leq 2$ if $i \notin H, j \notin H$.

Note further that equality holds exactly for the vertices in $\mathcal{M}^2(X)$ whose image by m^{-1} is in $\mathcal{C}$. Therefore, $\mathcal{C}$ defines a face in $\mathcal{FM}^2(X)$ and the result holds. □

From this theorem, we can derive in an easy and fast way the adjacency structure of $\mathcal{FM}^2(X)$.

Corollary 1. *Let μ_1 and μ_2 be two different vertices of $\mathcal{FM}^2(X)$. Then, μ_1 and μ_2 are adjacent vertices in $\mathcal{FM}^2(X)$ except if $\mu_1 = u_i$, $\mu_2 = u_j$ or $\mu_1 = u_{ij}, \mu_2 = \mu_{ij}$.*

Proof. It suffices to remark that two vertices are adjacent if and only if they form a 1-dimensional face, i.e. an edge. Let $\mathcal{C}$ be a collection of two vertices of $\mathcal{FM}^2(X)$. Applying Theorem 1, it follows that $\mathcal{C}$ defines an edge if and only if $\mathcal{C} \neq \{u_i, u_j\}$ or $\mathcal{C} \neq \{u_{ij}, \mu_{ij}\}$. □

Definition 5. *Consider $n+1$ affinely independent points in $\mathbb{R}^m$, $m \geq n$, i.e. $n+1$ points $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of $\mathbb{R}^m$ where the vectors $\boldsymbol{x}_1 - \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n - \boldsymbol{x}_0$ are linearly independent. The convex hull of these points is called a* **simplex**.

This notion is a generalization of the notion of triangle for the m-dimensional space. Recall that all vertices of a simplex are adjacent to each other (although there are non-simplicial polytopes in which every two vertices are adjacent, see [37]).

Definition 6. *Let $\mathcal{P}$ be a convex polytope and $\boldsymbol{x}$ be a non-collinear point, i.e $\boldsymbol{x} \notin aff(\mathcal{P})$. Point $\boldsymbol{x}$ is called* **apex**. *We define a* **pyramid** *with base $\mathcal{P}$ and apex $\boldsymbol{x}$, denoted by $pyr(\mathcal{P}, \boldsymbol{x})$, as the polytope whose vertices are the ones of $\mathcal{P}$ and $\boldsymbol{x}$. Observe that $\boldsymbol{x}$ is adjacent to every vertex in $\mathcal{P}$.*

Remark that if we consider $\boldsymbol{y} \notin \text{aff}(\text{pyr}(\mathcal{P}, \boldsymbol{x}))$ then $\boldsymbol{y}$ is an apex for $\text{pyr}(\mathcal{P}, \boldsymbol{x})$, and we can define a new pyramid $\text{pyr}(\text{pyr}(\mathcal{P}, \boldsymbol{x}), \boldsymbol{y})$, denoted $\text{cpyr}(\mathcal{P}, \{\boldsymbol{x}, \boldsymbol{y}\})$. In general, we can iterate this process to define a **consecutive pyramid** with apexes $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$, denoted by $\text{cpyr}(\mathcal{P}, \mathcal{A})$.

From Corollary 1 and Theorem 1 it is also possible to determine whether a face is a simplex. Next theorem describe the geometry of each face. Before stating the result, let us define some concepts.

Theorem 2. *Let $\mathcal{F}_{\mathcal{C}}$ be a face of $\mathcal{FM}^2(X)$ and let us consider the following sets*

$$\mathcal{U} := \{i \in X : u_i \in \mathcal{C}\}, \; \mathcal{V} := \left\{ ij \in \binom{X}{2} : |\{u_{ij}, \mu_{ij}\} \cap \mathcal{C}| = 1 \right\}.$$

Then, the following holds:

i) If $|\mathcal{U}| \leq 1$, then $\mathcal{F}_{\mathcal{C}}$ is a simplicial face of dimension $|\mathcal{C}| - 1$.

ii) If $|\mathcal{U}| > 1$, then $\mathcal{F}_{\mathcal{C}}$ is a non-simplicial face of dimension $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| - 1$. Moreover, if $\mathcal{V} = \emptyset$, then $\mathcal{F}_{\mathcal{C}} = \mathcal{FM}^2(\mathcal{U})$. Otherwise, $\mathcal{F}_{\mathcal{C}} = cpyr(\mathcal{FM}^2(\mathcal{U}), \mathcal{V})$.

Proof. i) Since $|\mathcal{U}| \leq 1$ and $\mathcal{F}_{\mathcal{C}}$ is a face, we conclude from Theorem 1 that $\{u_{ij}, \mu_{ij}\} \not\subseteq \mathcal{C}, \forall ij \in \binom{X}{2}$. We are going to show that vertices in $\mathcal{C}$ are affinely independent and therefore they form a simplex. To show this, we are going to work in $\mathcal{M}^2(X)$ with Möbius coordinates. Let us write $\mathsf{m}(\mathcal{C}) = \{v_0, \ldots, v_s\}$, with $s \geq 1$, as otherwise the result trivially holds. We will show that no $v_i - v_0$ can be written as a convex combination of the other $v_j - v_0, \; j \neq i$. We consider two cases:

- Consider $m_{ij} \in \mathsf{m}(\mathcal{C})$ and suppose we can write $m_{ij} - v_0$ as a convex combination of the other $v_j - v_0$ (the case for $\overline{m}_{ij}$ follows exactly the same reasoning). Without loss of generality, let us denote $v_1 = m_{ij}$. Remark that $m_{ij}(ij) \neq 0 = v_k(ij), \; \forall k \neq 1$ (as $\overline{m}_{ij} \notin \mathsf{m}(\mathcal{C})$). Thus, $(v_1 - v_0)(ij) \neq 0 = (v_k - v_0)(ij), \; \forall k > 1$, and hence we conclude that $v_1 - v_0$ cannot be written as a linear combination of the vectors $v_k - v_0, \; k > 1$.
- Let us now suppose that we can write the (possibly) only $m_i \in \mathsf{m}(\mathcal{C})$ in the way that $m_i - v_0$ is a convex combination of the other $v_j - v_0$. Let us assume without loss of generality $v_1 = m_i$. Then

$$v_1 - v_0 = \sum_{i=2}^{s} \alpha_i (v_i - v_0),$$

for some α_i not all of them null, say $\alpha_2 \neq 0$. Then, we can rewrite the previous expression, thus obtaining

$$v_2 - v_0 = \sum_{k=3}^{s} \frac{-\alpha_k}{\alpha_2} (v_k - v_0) + \frac{1}{\alpha_2} (v_1 - v_0).$$

As v_2 is either m_{kr} or $\overline{m}_{kr}$, this is a contradiction with the previous case.

Then, the vertices of $\mathcal{C}$ are affinely independent and $\mathcal{F}_{\mathcal{C}}$ is a simplicial face of dimension $|\mathcal{C}| - 1$.

ii) Since $|\mathcal{U}| > 1$, there are two vertices u_i and u_j that are not adjacent to each other by Corollary 1. Therefore, $\mathcal{F}_{\mathcal{C}}$ is not a simplex.

If $\mathcal{V} = \emptyset$, we conclude from Lemma 2 that

$$\mathcal{C} = \{u_i : i \in \mathcal{U}\} \cup \left\{ u_{ij}, \mu_{ij} : ij \in \binom{\mathcal{U}}{2} \right\}.$$

Hence, $\mathcal{F}_{\mathcal{C}} = \mathcal{FM}^2(\mathcal{U})$ and the dimension is $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| - 1$.

Let us suppose now that $\mathcal{V} \neq \emptyset$. By Lemma 2, we know that

$$\{u_i : i \in \mathcal{U}\} \cup \left\{ u_{ij}, \mu_{ij} : ij \in \binom{\mathcal{U}}{2} \right\} \subset \mathcal{C}.$$

Therefore, $\mathcal{FM}^2(\mathcal{U}) \subset \mathrm{Conv}(\mathcal{C})$. Take u_{ij} such that $ij \in \mathcal{V}$ (the case for μ_{ij} is completely similar) or, in Möbius coordinates, m_{ij} with $ij \in \mathcal{V}$. Note that for all vertices $\mu \in \mathcal{FM}^2(\mathcal{U})$, it is $m_\mu(ij) = 0$ and thus, they are non-collinear with m_{ij}. Then, when we add u_{ij} we get $\mathrm{pyr}(\mathcal{FM}^2(\mathcal{U}), u_{ij})$. The dimension is the dimension of the base plus 1. If we repeat this procedure with the other elements of $\mathcal{V}$ and we get the consecutive pyramid $\mathrm{cpyr}(\mathcal{FM}^2(\mathcal{U}), \mathcal{V})$ with dimension $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| - 1$.

□

The ***f*-vector** of a polytope is a vector $(f_0, \cdots, f_{d-1})$ where f_i is the number of i-dimensional faces of the polytope.

In next result, we compute the f-vector of $\mathcal{FM}^2(X)$.

Theorem 3. *Let fs_k and fns_k be the number of simplicial and non-simplicial k-dimensional faces of $\mathcal{FM}^2(X)$, respectively. Then:*

i) The number of simplicial k-dimensional faces is given by:

$$fs_k = \begin{cases} 2^{k+1}\binom{\binom{n}{2}}{k+1} + n2^k\binom{\binom{n}{2}}{k} & \text{if } k \leq \binom{n}{2} - 1 \\ n2^{\binom{n}{2}} & \text{if } k = \binom{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

ii) If we denote $p(j) := \binom{j}{2} + j$, and by $s(k)$ the maximum value of j such that $k + 1 - p(j) \geq 0$, then the number of non-simplicial k-dimensional faces is given by:

$$\sum_{j=2}^{s(k)} 2^{k+1-p(j)} \binom{n}{j} \binom{\binom{n}{2} - \binom{j}{2}}{k+1-p(j)}.$$

Finally, $f_k = fs_k + fns_k$.

Proof. *i)* Applying Theorem 2 and following the same notation, we have two kinds of simplicial faces, namely the ones with $|\mathcal{U}| = 0$ and the ones where $|\mathcal{U}| = 1$.

For the first case, as the dimension is the number of vertices minus 1, we need to select $k+1$ vertices derived from $\mathcal{V}$. Note that for a chosen pair $ij \in \mathcal{V}$, either u_{ij} or either μ_{ij} are in the face. Therefore, the number of possible faces is given by

$$2^{k+1}\binom{\binom{n}{2}}{k+1}.$$

If $|\mathcal{U}| = 1$ we need to select k vertices derived from $\mathcal{V}$. As we have n possible choices for the vertex in $\mathcal{U}$, we conclude that the number of such faces is given by

$$n2^k\binom{\binom{n}{2}}{k}.$$

Thus,

$$fs_k = \begin{cases} 2^{k+1}\binom{\binom{n}{2}}{k+1} + n2^k\binom{\binom{n}{2}}{k} & \text{if } k \leq \binom{n}{2} - 1 \\ n2^{\binom{n}{2}} & \text{if } k = \binom{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

ii) In order to get a non-simplicial k-dimensional face, applying Theorem 2, we need $|\mathcal{U}| \geq 2$; now, we look for the number of possibilities of $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| = k+1$. Remark that the number of pairs suitable for $\mathcal{V}$ is at most

$$\binom{n}{2} - \binom{|\mathcal{U}|}{2}.$$

Then, the number of possible faces is the number of combinations of $\mathcal{U}$ and $\mathcal{V}$ in these conditions. In addition, consider that we must choose between u_{ij} and μ_{ij} for each pair in $\mathcal{V}$. If we denote $j := |\mathcal{U}|$, $p(j) = \binom{j}{2} + j$, and by $s(k)$ the maximum value of j such that $k + 1 - p(j) \geq 0$, then the number of non-simplicial faces is

$$\sum_{j=2}^{s(k)} \binom{n}{j} 2^{k+1-p(j)} \binom{\binom{n}{2} - \binom{j}{2}}{k+1-p(j)},$$

where we are assuming that $\binom{a}{b} = 0$ if $a < b$.

□

4 A random procedure for generating points in $\mathcal{FM}^2(X)$

Inspired by the adjacency structure of $\mathcal{FM}^2(X)$ obtained previously, in this section we are going to develop a procedure for generating random points uniformly distributed in $\mathcal{FM}^2(X)$. Generating points in a polytope is a complex problem and several methods, not completely satisfactory, have been presented to cope with this problem [7, 21]. Among them, we have the triangulation method [7]. The triangulation method takes advantage of the fact that random generation in simplices is very simple and fast [31].

The triangulation method is based on the decomposition of the polytope into simplices such that any pair of simplices intersects in a (possibly empty) common face. Once the decomposition is obtained, we assign to each simplex a probability proportional to its volume; next, these probabilities are used for selecting one of the simplices; finally, a random m-uple in the simplex is generated.

The main drawback of this method is that in general it is not easy to split a polytope into simplices. Moreover, even if we are able to decompose the polytope in a suitable way, we have to deal with the problem of determining the volume of each simplex in order to randomly select one of them. Computing the volume of a polytope is a complex problem and only partial results are known. However, in the case of simplices, the volume is given in next result.

Lemma 3. *[16] Let Δ be a k-dimensional simplex in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_{k+1}$. Then, the k-dimensional volume of Δ is given by:*

$$Vol_k(\Delta) = \sqrt{\frac{(-1)^{k+1}}{2^k (k!)^2} \det(CM_\Delta)},$$

where $\det(CM_\Delta)$ *is the Cayley-Menger determinant defined as*

$$\det(CM_\Delta) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & d^2_{1,2} & \cdots & d^2_{1,k} & d^2_{1,k+1} \\ 1 & d^2_{2,1} & 0 & \cdots & d^2_{2,k} & d^2_{2,k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & d^2_{k+1,1} & d^2_{k+1,2} & \cdots & d^2_{k+1,k} & 0 \end{vmatrix}$$

being $d^2_{i,j}$ the square of the distance between vertices v_i and v_j.

The triangulation method is specially appealing for order polytopes, as it is easy to decompose the polytope in simplices having the same volume [23]. However, as we have seen in Proposition 3, $\mathcal{FM}^2(X)$ is not an order polytope and thus, we have to look for another way to split the polytope into simplices. This is the task we achieve in this section.

To develop an algorithm to generate random points in $\mathcal{FM}^2(X)$, we will profit its combinatorial structure, and more concretely the adjacency structure developed in the previous section. We will use the fact that

$$u_i + u_j = u_{ij} + \mu_{ij}.$$

Lemma 4. *Given $\mu \in \mathcal{FM}^2(X)$, it is possible to write μ as a unique convex combination of vertices of $\mathcal{FM}^2(X)$ in a way such that either u_{ij} or either μ_{ij} has null coefficient, for all pairs $ij \in \binom{X}{2}$.*

Proof. Given a measure $\mu \in \mathcal{FM}^2(X)$, let us write μ as

$$\mu = \sum_{i=1}^{n} \alpha_i u_i + \sum_{ij} \alpha_{ij} u_{ij} + \sum_{ij} \beta_{ij} \mu_{ij}, \quad \sum_{i=1}^{n} \alpha_i + \sum_{ij} \alpha_{ij} + \sum_{ij} \beta_{ij} = 1.$$

Notice that this convex combination might not be unique to represent μ. Now, if say $\alpha_{ij} > \beta_{ij}$, we apply

$$\alpha_{ij} u_{ij} + \beta_{ij} \mu_{ij} = \beta_{ij} u_i + \beta_{ij} u_j + (\alpha_{ij} - \beta_{ij}) u_{ij}.$$

Similarly, if $\alpha_{ij} < \beta_{ij}$, we apply

$$\alpha_{ij} u_{ij} + \beta_{ij} \mu_{ij} = \alpha_{ij} u_i + \alpha_{ij} u_j + (\beta_{ij} - \alpha_{ij}) \mu_{ij}.$$

Hence, we have proved the first part. Assume now that two such representations are possible; then,

$$\mu = \sum_{i=1}^{n} \alpha_i u_i + \sum_{ij} \alpha_{ij} u_{ij} + \sum_{ij} \beta_{ij} \mu_{ij} = \sum_{i=1}^{n} \alpha'_i u_i + \sum_{ij} \alpha'_{ij} u_{ij} + \sum_{ij} \beta'_{ij} \mu_{ij}.$$

Now, for a pair ij, its Möbius transform is given by

$$m(ij) = \alpha_{ij} - \beta_{ij} = \alpha'_{ij} - \beta'_{ij}.$$

But as one α_{ij}, β_{ij} (resp. $\alpha'_{ij}, \beta'_{ij}$) vanishes by hypothesis and they are all non-negative, this implies $\alpha_{ij} = \alpha'_{ij}$, $\beta_{ij} = \beta'_{ij}$, whence the result. □

Let us consider $\binom{X}{2}$ and for each pair $ij \in \binom{X}{2}$, either u_{ij} or μ_{ij} is assigned. We define $\mathcal{A}^-$ as the subset of pairs of $\binom{X}{2}$ where u_{ij} is selected and $\mathcal{A}^+$ the set of pairs for which μ_{ij} is selected. There are $2^{\binom{n}{2}}$ different $\mathcal{A}^-$, so that there are $2^{\binom{n}{2}}$ different $\mathcal{A}^-, \mathcal{A}^+$. For fixed $\mathcal{A}^-, \mathcal{A}^+$, we define $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ as the convex hull of $\{u_i : i \in X\} \cup \{u_{ij} : ij \in \mathcal{A}^-\} \cup \{\mu_{ij} : ij \in \mathcal{A}^+\}$. In other words, $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ consists of all fuzzy measures μ in $\mathcal{FM}^2(X)$ such that the unique representation of μ in terms of Lemma 4 is such that $ij \in \mathcal{A}^-$ if u_{ij} appears in the representation and $ij \in \mathcal{A}^+$ if it is μ_{ij} who appears in the representation. Remark that $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a polytope whose vertices are

$$\{u_i : i \in X\} \cup \{u_{ij} : ij \in \mathcal{A}^-\} \cup \{\mu_{ij} : ij \in \mathcal{A}^+\}. \tag{3}$$

We have $2^{\binom{n}{2}}$ different subsets $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, one for each possible $\mathcal{A}^-, \mathcal{A}^+$.

In next results we will show that it is possible to derive an appealing algorithm for random generation in $\mathcal{FM}^2(X)$ from these subsets applying triangulation methods. For this, we will prove in Theorem 4 below that $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ for different choices of $\mathcal{A}^-, \mathcal{A}^+$ provide a triangulation of $\mathcal{FM}^2(X)$. Next, we will show in Proposition 4 that all of them share the same volume. In these conditions, in order to generate a fuzzy measure in $\mathcal{FM}^2(X)$ in a random fashion, it just suffices to select randomly one of the possible $\mathcal{A}^-, \mathcal{A}^+$ and then generate a point in the corresponding simplex $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$. Before proving this, let us define formally what is meant by a triangulation.

Definition 7. *Let $\mathcal{P}$ be a polytope whose vertices define the set of vectors $\mathcal{A}$. A triangulation is a collection Δ of simplices all of whose vertices are points in $\mathcal{A}$ that satisfies the following properties:*

i) The union of all these simplices equals $\mathcal{P} = Conv(\mathcal{A})$.

ii) If $\mathcal{F}, \mathcal{F}' \in \Delta \Rightarrow \mathcal{F} \cap \mathcal{F}'$ is a (possibly empty) common face of $\mathcal{F}$ and $\mathcal{F}'$.

Theorem 4. *Let Δ be the collection of all the polytopes $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ where $\{\mathcal{A}^-, \mathcal{A}^+\}$ is any possible decomposition of $\binom{X}{2}$. Then, Δ is a triangulation of $\mathcal{FM}^2(X)$.*

Proof. By Lemmma 4 the union of these polytopes is $\mathcal{FM}^2(X)$,

$$\mathcal{FM}^2(X) = \bigcup_{\mathcal{A}^-,\mathcal{A}^+} \mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X).$$

Let us show that each $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a simplex. For this, we have to prove that the vertices of every $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ are affinely independent and this is equivalent to prove that the Möbius transform of these vertices form an affinely independent set.

As the vertices of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ are given in Eq. (3), it follows that the vertices of $\mathsf{m}(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X))$ are

$$\{m_i : i = 1, ..., n\} \cup \{m_{ij} : ij \in \mathcal{A}^-\} \cup \{\overline{m}_{ij} : ij \in \mathcal{A}^+\}.$$

Let us rename these vertices as $\{v_0, ..., v_s\}$ and let us assume that $v_0 = m_1$. We will show that no $v_i - v_0$ can be written as a convex combination of the other $v_j - v_0,\ j \neq i$. We consider two cases:

- Consider a pair $ij \in \mathcal{A}^-$ and suppose we can write $m_{ij} - v_0$ as a convex combination of the other $v_j - v_0$ (the case $ij \in \mathcal{A}^+$ is completely symmetric). Without loss of generality, let us denote $v_1 = m_{ij}$. Remark that by construction $m_{ij}(ij) \neq 0 = v_k(ij),\ \forall k \neq 1$ as $\overline{m}_{ij}$ is not a vertex of $\mathsf{m}(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X))$. Thus, $(v_1 - v_0)(ij) \neq 0 = (v_k - v_0)(ij),\ \forall k > 1$, and hence we conclude that $v_1 - v_0$ cannot be written as a linear combination of the vectors $v_k - v_0,\ k > 1$.

- Let us now consider $m_i, i \neq 1$, and let us assume without loss of generality $v_1 = m_i$. Suppose

  $$v_1 - v_0 = \sum_{i=2}^{s} \alpha_i (v_i - v_0),$$

  for some α_i not all of them null. Besides, as $m_j(i) = 0,\ \forall j \neq i$, it follows that there exists, say v_2, corresponding to $\overline{m}_{ij}$ and such that $\alpha_2 \neq 0$. Then, we can rewrite the previous expression, thus obtaining

  $$v_2 - v_0 = \sum_{k=3}^{s} \frac{-\alpha_k}{\alpha_2}(v_k - v_0) + \frac{1}{\alpha_2}(v_1 - v_0).$$

  But this a contradiction with the previous case.

Thus, we have shown that the vertices of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ are affinely independent and then they form a simplex.

Besides, as the number of vertices in $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is $\binom{n}{2}+n$, it follows that this set is a simplex of dimension $\binom{n}{2}+n-1$. As the dimension of $\mathcal{FM}^2(X)$ is $\binom{n}{2}+n-1$, we conclude that $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ is a full-dimensional simplex in $\mathcal{FM}^2(X)$.

It just rests to show that the intersection of two of these simplices is a (possibly empty) common face. Consider $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$ and $\mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$ and suppose that they have non-empty intersection. Let us denote by $\mathcal{CV}$ the common vertices of these two simplices, i.e.,

$$\mathcal{CV} := \{u_i : i \in X\} \cup \{u_{ij} : ij \in \mathcal{A}_1^- \cap \mathcal{A}_2^-\} \cup \{\mu_{ij} : ij \in \mathcal{A}_1^+ \cap \mathcal{A}_2^+\}.$$

Lemma 4 shows that any fuzzy measure in $\mathcal{FM}^2(X)$ can be written as a unique convex combination such that either the coefficient of u_{ij} or the coefficient of μ_{ij} vanishes. Consequently,

$$\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X) \cap \mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X) = \mathrm{Conv}(\mathcal{CV}).$$

It follows that $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X) \cap \mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$ is a simplicial face of $\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$ and $\mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$. $\square$

Next step is to prove that all the simplices obtained with the triangulation developed above share the same volume.

Proposition 4. *All simplices $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ have the same $\left(\binom{n}{2}+n-1\right)$-dimensional volume.*

Proof. First, let us recall that the volume of a simplex only depends on the distances between each pair of vertices by Lemma 3. Now consider two decompositions $\{\mathcal{A}_1^-, \mathcal{A}_1^+\}$ and $\{\mathcal{A}_2^-, \mathcal{A}_2^+\}$ and let us define the following bijection between the two sets of vertices associated to each simplex:

$$\begin{array}{rccc} F: & \mathcal{V}\left(\mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)\right) & \rightarrow & \mathcal{V}\left(\mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)\right) \\ & \mu & \hookrightarrow & F(\mu) \end{array}$$

such that

$$F(u_i) = u_i, \forall i \in X,\ F(u_{ij}) = \begin{cases} u_{ij} & \text{if } ij \in \mathcal{A}_2^- \\ \mu_{ij} & \text{if } ij \in \mathcal{A}_2^+ \end{cases},\ F(\mu_{ij}) = \begin{cases} u_{ij} & \text{if } ij \in \mathcal{A}_2^- \\ \mu_{ij} & \text{if } ij \in \mathcal{A}_2^+ \end{cases}$$

Now, let us prove that for any pair of vertices $v_1, v_2 \in \mathcal{FM}^2_{\mathcal{A}_1^-,\mathcal{A}_1^+}(X)$ we get $d(v_1, v_2) = d(F(v_1), F(v_2))$ and then $F(v_1), F(v_2) \in \mathcal{FM}^2_{\mathcal{A}_2^-,\mathcal{A}_2^+}(X)$ are vertices with the same distance as v_1, v_2. We study all possible cases:

$$d^2(u_i,u_j) = \sum_{A\subseteq X}(u_i(A)-u_j(A))^2 = \sum_{A\subseteq X} u_i(A)+\sum_{A\subseteq X} u_j(A)-2\cdot\sum_{A\subseteq X} u_i(A)u_j(A) = 2^{n-1}+2^{n-1}-2\cdot 2^{n-2} = 2^{n-1}.$$

Similar computations lead to

$$d^2(u_i, u_{ij}) = 2^{n-2},\ d^2(u_i, \mu_{ij}) = 2^{n-2},\ d^2(u_i, u_{kr}) = 2^{n-1},\ d^2(u_i, \mu_{kr}) = 2^{n-1},$$

$$d^2(u_{ij}, u_{ik}) = 2^{n-2},\ d^2(\mu_{ij}, \mu_{ik}) = 2^{n-2},\ d^2(u_{ij}, \mu_{ik}) = 2^{n-1},\ d^2(\mu_{ij}, u_{ik}) = 2^{n-1},$$

$$d^2(u_{ij}, u_{kr}) = 3 \cdot 2^{n-3},\ d^2(\mu_{ij}, \mu_{kr}) = 3 \cdot 2^{n-3},\ d^2(u_{ij}, \mu_{kr}) = 5 \cdot 2^{n-3}.$$

Thus, $d^2(\mu_1, \mu_2) = d^2(F(\mu_1), F(\mu_2)), \forall \mu_1, \mu_2 \in \mathcal{V}\left(\mathcal{FM}^2_{\mathcal{A}_1^-, \mathcal{A}_1^+}(X)\right)$ and hence the result holds. $\square$

As a consequence, we can apply the triangulation method as follows: As all subsets $\mathcal{FM}^2_{\mathcal{A}^-, \mathcal{A}^+}(X)$ have the same volume, we need to select one of these subsets at random. For this, it suffices to choose at random for each pair ij if it is included in $\mathcal{A}^-$ or $\mathcal{A}^+$.

Next step is to generate a point in the selected $\mathcal{FM}^2_{\mathcal{A}^-, \mathcal{A}^+}(X)$. Generation in simplices is easy and can be found e.g. in [31], page 71. For the sake of completeness, we explain the procedure in the following. Let us reuse the notation $p(n) = \binom{n}{2} + n$. Since the dimension of each simplex is $p(n) - 1$ we are going to use just the first $p(n) - 1$ coordinates. In other words, we work with the projection of $\mathcal{FM}^2_{\mathcal{A}^-, \mathcal{A}^+}(X)$ into the subspace consisting in intersecting the hyperplanes $X_A = 0$ for $|A| > 2$ and $X_{(n-1)n} = 0$. We call this projection $\pi : \mathbb{R}^{2^n} \to \mathbb{R}^{p(n)-1}$.

1) Start sampling a uniformly distributed random point in the simplex

$$\mathcal{H}_n = \{\boldsymbol{U} \in [0,1]^{p(n)-1} \ : U_1 \geq U_2 \geq \cdots \geq U_{p(n)-1}\}.$$

   For this, generate an independent and identically distributed sample $\widehat{U}_1 \ldots, \widehat{U}_{p(n)-1}$ with distribution $U(0,1)$. Then sort the $\widehat{U}_i$ to give the order statistics with the reverse order $U_1 \geq U_2 \geq \ldots \geq U_{p(n)-1}$. This generates a uniformly distributed vector U in $\mathcal{H}_n$. Note that the vertices of $\mathcal{H}_n$ are $(0, 0, \ldots, 0, 0), (1, 0, \ldots, 0, 0), (1, 1, \ldots, 0, 0), \ldots, (1, 1, \ldots, 1, 0)$ and $(1, 1, \ldots, 1, 1)$.

2) Apply the affine transformation $\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$ which maps $\mathcal{H}_n$ into the desired simplex $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-, \mathcal{A}^+}(X)\right)$ associated to the partition $\mathcal{A}^-, \mathcal{A}^+$. Note that if $h(\boldsymbol{u})$ is the density function of $\boldsymbol{U}$, then the density $g(\boldsymbol{x})$ of $\boldsymbol{X}$ would be

$$g(\boldsymbol{x}) = h(\boldsymbol{u}) |\det(\boldsymbol{A})|^{-1}.$$

   Consequently, if $h(\boldsymbol{u})$ is uniform in $\mathcal{H}_n$, then $g(\boldsymbol{x})$ is uniform in $\mathcal{FM}^2_{\mathcal{A}^-, \mathcal{A}^+}(X)$, because $|\det(\boldsymbol{A})|$ is a constant value.

3) Finally, observe that our simplex $\mathcal{FM}^2_{\mathcal{A}^-, \mathcal{A}^+}(X)$ is a $(p(n) - 1)$-dimensional simplex in $\mathbb{R}^{2^n}$. We should recover the rest of coordinates to get the final vector $\boldsymbol{X}^* \in \mathbb{R}^{2^n}$. Obviously $\boldsymbol{X}^*(A) = \boldsymbol{X}(A)$ for the first $p(n) - 1$ coordinates. By using Eqs. (1) and (2), we recover the rest of coordinates.

It just remains to give a full description of the affine transformation $\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$. This affine transformation maps the vertices of $\mathcal{H}_n$ into the vertices of $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)\right)$. Recall that the vertices of $\pi\left(\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)\right)$ are the first $p(n)-1$ coordinates of u_i and v_{ij} where $v_{ij} = u_{ij}$ if $ij \in \mathcal{A}^-$ and $v_{ij} = \mu_{ij}$ if $ij \in \mathcal{A}^+$. We denote the restriction of a vector v to the first $p(n)-1$ coordinates with an overline, $\overline{v}$. Indeed, we are going to denote the vertices by $v_1, v_2, \ldots, v_{p(n)}$ in the natural order, that is $u_1, u_2, \ldots, v_{12}, v_{13}, \ldots, v_{(n-1)n}$. We also identify $\boldsymbol{V}_0 = \overline{u}_1$. Now consider the matrix having as i-th column the vector $v_{i+1} - v_i$, that is

$$\boldsymbol{A} = \left[\overline{u}_2 - \overline{u}_1 \mid \overline{u}_3 - \overline{u}_2 \mid \cdots \mid \overline{v}_{12} - \overline{u}_n \mid \cdots \mid \overline{v}_{(n-1)n} - \overline{v}_{(n-2)n}\right].$$

It is easy to see that with these choices $\boldsymbol{X} = \boldsymbol{A} \cdot \boldsymbol{U} + \boldsymbol{V}_0$. Therefore,

$$\boldsymbol{X}_k = \overline{u}_{1k} + \sum_{j=2}^{p(n)} (\overline{v}_{jk} - \overline{v}_{(j-1)k}) \boldsymbol{U}_{j-1}$$

where $\overline{v}_{jk}$ is the k-th position of the vertex $\overline{v}_j$. Thus, we are in conditions to present our algorithm for generating random points in $\mathcal{FM}^2(X)$.

ALGORITHM FOR SAMPLING IN $\mathcal{F}M^2(X)$

1. Choose randomly between u_{ij} and μ_{ij} for any pair of elements $ij \in \binom{X}{2}$ to get the partition $\mathcal{A}^-, \mathcal{A}^+$.
2. Generate an iid sample $\widehat{U}_1 \ldots, \widehat{U}_{p(n)-1}$ with distribution $U(0,1)$. Then sort the $\widehat{U}_i$ with the reverse order to get U, s.t. $U_1 \geq U_2 \geq \ldots \geq U_{p(n)-1}$.
3. Apply the linear map
$$\boldsymbol{X}_k = \overline{u}_{1k} + \sum_{j=2}^{p(n)} (\overline{v}_{jk} - \overline{v}_{(j-1)k}) \boldsymbol{U}_{j-1},$$
$\forall k \in \{1, \ldots, p(n)-1\}$.
4. Recover the rest of coordinates by using Eqs. (1) and (2).

If we work just with the first $p(n)-1$ coordinates the last algorithm has quartic complexity as we show in the next result.

Proposition 5. *The computational complexity of the sampling algorithm for 2-additive measures is $O(n^4)$.*

Proof. We compute the complexity of each part:

1. We should choose if each pair ij is associated to u_{ij} or μ_{ij}. Then, we compute a vector of $\binom{n}{2}$ values 0 and 1. The complexity is $O(\binom{n}{2}) = O(n^2)$.

2. We generate an iid sample $\widehat{U}_1 \ldots, \widehat{U}_{p(n)-1}$ with distribution $U(0,1)$. Then sort the U_i with the reverse order $U_1 \geq U_2 \geq \ldots \geq U_{p(n)-1}$. Since the complexity is linear for sampling and for sorting this step needs $O(\binom{n}{2}+n) = O(n^2)$ computations.

3. Apply the transformation $\boldsymbol{X}_k = \overline{u}_{1k} + \sum_{j=2}^{p(n)} (\overline{v}_{jk} - \overline{v}_{(j-1)k})\boldsymbol{U}_{j-1}$. In this step, we multiply a $(p(n)-1) \times (p(n)-1)$ matrix by a vector. Then, the complexity is $O\left(\left(\binom{n}{2}+n\right)^2\right) = O(n^4)$.

4. The last step is not be necessary because we are working just with the first $p(n)-1$ coordinates.

Therefore, the complexity is $O(n^4)$. □

Obviously, if we work with all the $2^n - 2$ coordinates, the complexity increases to $O(2^n)$, because we need to recover the value for each subset of X.

We finish this section with two results that can be derived from the proposed triangulation. The first one refers to the volume of $\mathcal{FM}^2(X)$. For this, as we have computed the distance between each couple of vertices, we can apply Lemma 3 to obtain the volume of one of the simplices, and multiply this value by $2^{\binom{n}{2}}$ to obtain $Vol(\mathcal{FM}^2(X))$.

Corollary 2.

$$Vol(\mathcal{FM}^2(X)) = 2^{\frac{(n-1)(n-2)}{4}} \frac{\sqrt{|\det(CM_\Delta)|}}{\left[\binom{n}{2}+n-1\right]!},$$

where Δ is any simplex of the triangulation.

The volumes for the first values of n are given in next table.

$\lvert X\rvert$	2	3	4	5
$Vol(\mathcal{FM}^2(X))$	1	0.1632	0.0298	0.0001

Another consequence of the triangulation proposed in this section is that it allows a simple way of computing the **center of gravity** or **centroid** of $\mathcal{FM}^2(X)$, i.e. the mean position of all the points in all of the coordinate directions. From a mathematical point of view, the centroid of $\mathcal{P}$ is given by the integral

$$C = \frac{\int x I_{\mathcal{P}}(x)dx}{\int I_{\mathcal{P}}(x)dx},$$

where $I_{\mathcal{P}}(x)$ is the characteristic function of $\mathcal{P}$. At this point, note that computing the center of gravity of a polytope is a difficult problem and usually, complicated methods and formulas are given. Only for special cases, the center of gravity has been obtained. One of this cases is the case of simplices, for which the following can be shown.

Lemma 5. *[19] Consider an n-dimensional simplex whose vertices are $v_0, \ldots, v_n$; then, considering the vertices as vectors, the centroid of the simplex is*

$$C = \frac{1}{n+1}\sum_{i=0}^{n} v_i.$$

Now, for a given polytope and a decomposition, the following can be shown.

Lemma 6. *Let $\mathcal{P}$ be a polytope and $\{\mathcal{P}_1, ..., \mathcal{P}_r\}$ a partition of $\mathcal{P}$. Suppose that the centroid of $\mathcal{P}_i$ is C_i, $i = 1, .., r$ and let us denote $Vol(\mathcal{P}_i) = V_i$. Then, the centroid of $\mathcal{P}$ is given by*

$$C = \frac{\sum_{i=1}^r C_i V_i}{\sum_{i=1}^r V_i}.$$

Proof. We have

$$C = \frac{\int x I_{\mathcal{P}}(x)dx}{\int I_{\mathcal{P}}(x)dx} = \frac{\sum_{i=1}^r \int x I_{\mathcal{P}_i}(x)dx}{\sum_{i=1}^r \int I_{\mathcal{P}_i}(x)dx} = \frac{\sum_{i=1}^r V_i \frac{\int x I_{\mathcal{P}_i}(x)dx}{V_i}}{\sum_{i=1}^r V_i} = \frac{\sum_{i=1}^r C_i V_i}{\sum_{i=1}^r V_i}.$$

□

Applying the triangulation proposed in this section, the following can be shown.

Proposition 6. *The centroid of $\mathcal{FM}^2(X)$ is given by $\overline{\mu}$ given by*

$$\overline{\mu}(B) = \frac{|B|}{n}.$$

Proof. Consider one of the simplices of the triangulation proposed in this section; this simplex is defined via the sets $\mathcal{A}^-, \mathcal{A}^+$; besides, we have shown that all these simplices have the same volume. As we have $2^{\binom{n}{2}}$ subsets $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, by Lemma 6,

$$\overline{\mu} = \frac{1}{2^{\binom{n}{2}}} \sum_{\mathcal{A}^-,\mathcal{A}^+} \overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+},$$

where $\overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+}$ is the centroid of $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$. On the other hand, for a given $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, as it is a simplex, we know by Lemma 5 that

$$\overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+} = \frac{1}{\binom{n}{2} + n} \left[\sum_{i=1}^n u_i + \sum_{ij \in \mathcal{A}^-} u_{ij} + \sum_{ij \in \mathcal{A}^+} \mu_{ij} \right].$$

Next, for a given $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$, let us consider $\mathcal{FM}^2_{\mathcal{A}^+,\mathcal{A}^-}(X)$, i.e. the simplex such that $u_{ij} \in \mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X) \Leftrightarrow \mu_{ij} \in \mathcal{FM}^2_{\mathcal{A}^+,\mathcal{A}^-}(X)$ and $\mu_{ij} \in \mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X) \Leftrightarrow u_{ij} \in \mathcal{FM}^2_{\mathcal{A}^+,\mathcal{A}^-}(X)$. Then, the sum of the center of gravity of these two simplices altogether is

$$\mu_{aux} = \frac{1}{\binom{n}{2} + n} \left[2 \sum_{i=1}^n u_i + \sum_{ij} u_{ij} + \sum_{ij} \mu_{ij} \right] = \frac{1}{\binom{n}{2} + n} \left[(n+1) \sum_{i=1}^n u_i \right] = 2 \frac{\sum_{i=1}^n u_i}{n}.$$

Then,

$$\overline{\mu} = \frac{1}{2^{\binom{n}{2}}} \sum_{\mathcal{A}^-,\mathcal{A}^+} \overline{\mu}_{\mathcal{A}^-,\mathcal{A}^+} = \frac{2^{\binom{n}{2}-1}}{2^{\binom{n}{2}}} \mu_{aux} = \frac{\sum_{i=1}^n u_i}{n} \Rightarrow \overline{\mu}(B) = \frac{|B|}{n}.$$

□

5 The adjacency graph of $\mathcal{FM}(X)$

We finish the paper presenting some properties of the adjacency graph of this polytope. Given a polytope $\mathcal{P}$, we define its **associated graph** $G(\mathcal{P})$ (also called adjacency graph or 1-skeleton) as the graph whose vertices are the vertices of $\mathcal{P}$ and two nodes are adjacent if the corresponding vertices are adjacent in $\mathcal{P}$.

For example, any n-dimensional simplex has the complete graph as associated graph, because all vertices are adjacent to each other. In the next figures we can compare the graph of $\mathcal{FM}^2(X)$ with the graph of a (n^2-1)-dimensional simplex. We observe that when the size of X grows, $G(\mathcal{FM}^2(X))$ tends to be very similar to the complete graph.

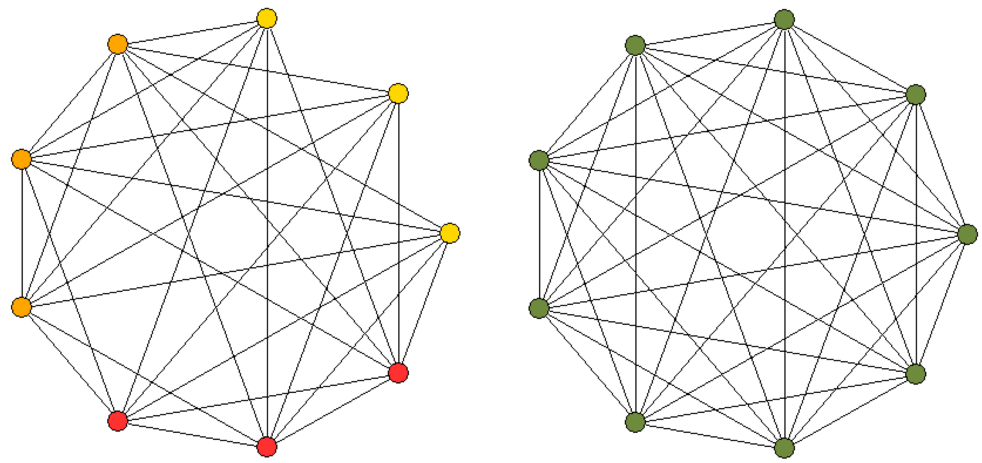

Figure 1: Adjacency graph of $\mathcal{FM}^2(X), |X| = 3$ (left) and simplex (right).

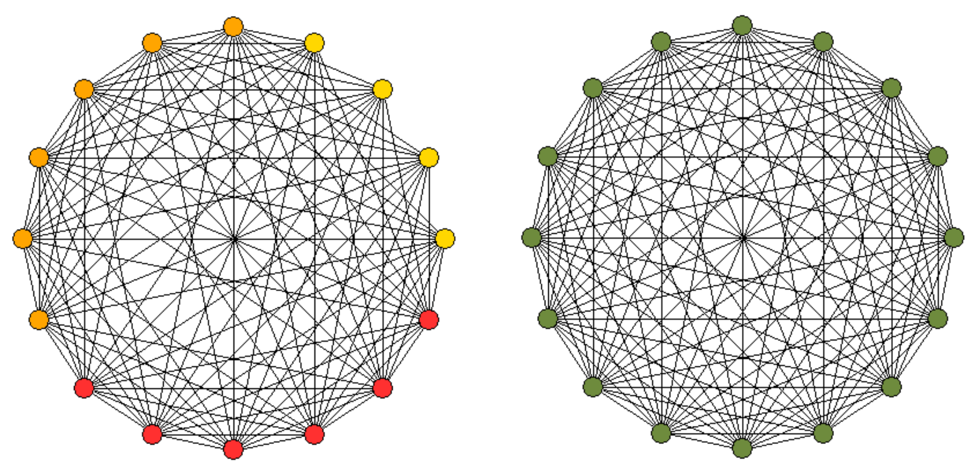

Figure 2: Adjacency graph of $\mathcal{FM}^2(X), |X| = 4$ (left) and simplex (right).

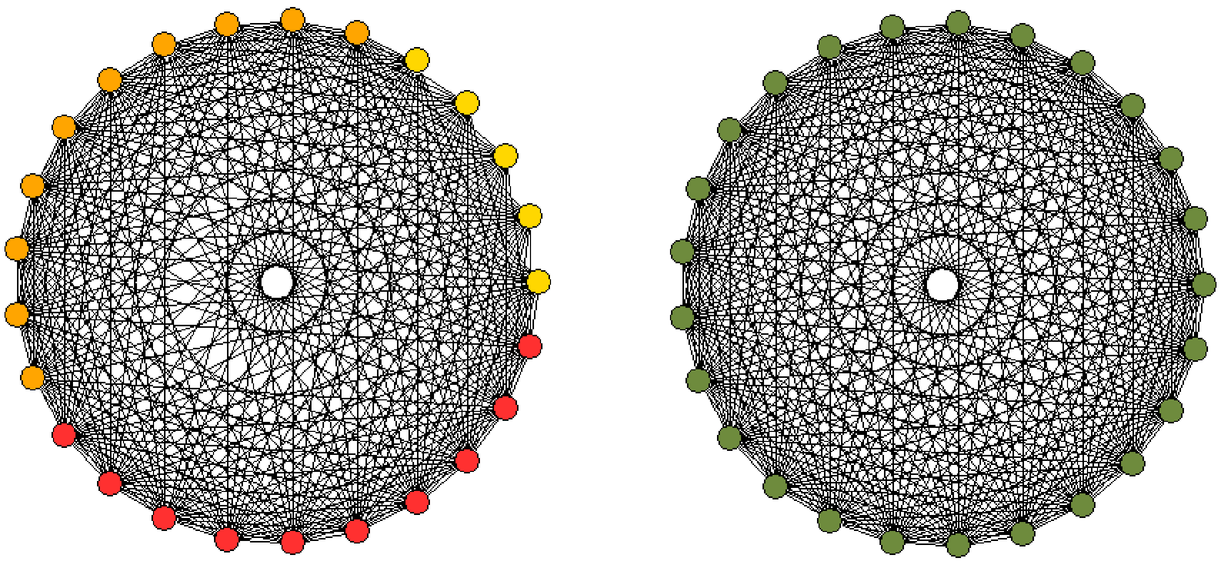

Figure 3: Adjacency graph of $\mathcal{FM}^2(X), |X| = 5$ (left) and simplex (right).

In these figures, the yellow vertices are u_i, the orange ones u_{ij}, and the red ones μ_{ij}.

The **distance** between two vertices of a polytope $\mathcal{P}$ is defined as the shortest path connecting the corresponding nodes in $G(\mathcal{P})$. The **diameter** of a polytope, $diam(\mathcal{P})$, is defined as the longest distance between any pair of vertices.

A straight consequence of Theorem 1 is the following.

Corollary 3. *The diameter of $\mathcal{FM}^2(X)$ is 2.*

Proof. By Corollary 1, the distance between two vertices is 1 except for u_i, u_j and u_{ij}, μ_{ij}. But in these cases, we have the paths $u_i - \mu_{ij} - u_j$ and $u_{ij} - u_i - \mu_{ij}$. $\square$

One important feature of a graph is the chromatic number [35]. The **chromatic number**, $\chi(G)$, of a graph G is the smallest number of colors needed to color the vertices of G so that no adjacent vertices share the same color.

Theorem 5. *Let $|X| = n$. Then $\chi\left(G(\mathcal{FM}^2(X))\right) = \binom{n}{2} + 1$.*

Proof. Start observing that all the vertices u_{ij} are adjacent to each other, so that we need $\binom{n}{2}$ colors. Since μ_{ij} is not adjacent to u_{ij} we can use for μ_{ij} the same color as u_{ij}. This way we can color all the vertices μ_{ij} using the same $\binom{n}{2}$ colors. Finally, as the u_i vertices are not related to each other but are related with the rest of vertices we need one last color. $\square$

Figure 4 shows a graph coloring for $|X| = 4$.

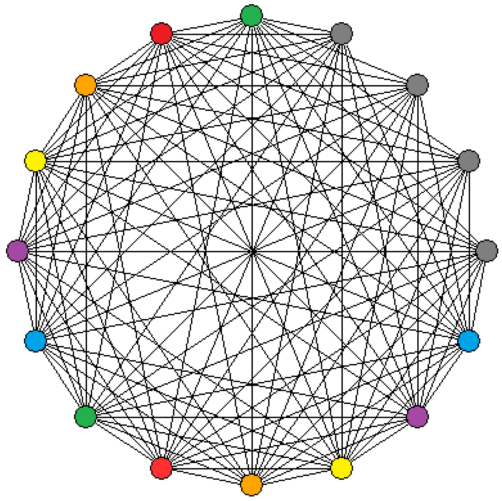

Figure 4: $\mathcal{FM}^2(X), |X| = 4$ graph coloring.

Let us analyze the Hamiltonicity of these graphs. Recall that a **Hamiltonian path** is a path that visits each vertex exactly once. Also, a graph is **Hamilton connected** if there exists a Hamiltonian path between each pair of vertices.

To see that $G(\mathcal{FM}^2(X)$ is Hamilton connected we need some previous results.

We say that a polytope is **combinatorial** [28] if its vertices are $\{0, 1\}$-valued and for each pair of non-adjacent vertices u, v there exist two other vertices w, h such that $u + v = w + h$.

Lemma 7. *$\mathcal{FM}^2(X)$ is a combinatorial polytope.*

Proof. By Proposition 2 all the vertices are $\{0, 1\}$-valued. Moreover, by Corollary 1, the only pairs that are not adjacent are (u_i, u_j) and (u_{ij}, μ_{ij}), and they satisfy $u_i + u_j = u_{ij} + \mu_{ij}$. □

For combinatorial polytopes we can use the following result.

Proposition 7. *[28] Let $\mathcal{P}$ be a combinatorial polytope. Then, $G(\mathcal{P})$ is either Hamilton connected or the graph of a hypercube.*

Theorem 6. *Let $|X| > 2$. Then $G(\mathcal{FM}^2(X))$ is Hamilton connected.*

Proof. It suffices to note that a hypercube has no complete subgraphs, i.e. it has no simplicial faces of dimension greater than 1. As $G(\mathcal{FM}^2(X))$ has such faces if $|X| > 2$ by Theorem 2, the result follows. □

Finally, let us study the planarity of this graph. A **planar graph** is a graph that can be drawn on the plane in such a way that its edges do not cross each other. By the four color theorem, every planar graph should have a chromatic number lower or equal than 4, i.e. $\chi(G) \leq 4$.

The **complete bipartite graph** $K_{n,m}$ has $n + m$ vertices and edges joining every vertex of the first n vertices to every vertex of the last m vertices.

A **minor** of a graph is a subgraph which can be obtained by deleting edges and vertices and by contracting edges. Edge contraction removes an edge from the graph while simultaneously merging the endpoints.

Theorem 7. *[35]* ***Wagner's theorem.*** *Let G be a finite graph. Then G is planar if and only if its minors include neither the complete graph of five elements K_5 nor the complete bipartite graph $K_{3,3}$.*

Theorem 8. *Let $|X| > 2$. Then $G(\mathcal{FM}^2(X))$ is not planar.*

Proof. If $|X| > 2$, we consider the minor formed by deleting every vertex but $u_1, u_2, u_3, u_{12}, u_{13}, u_{23}$ and deleting the edges between u_{ij} and u_{ik}. Hence, we obtain the complete bipartite graph $K_{3,3}$ and therefore $G(\mathcal{FM}^2(X))$ is not planar. □

6 Conclusions and open problems

In this paper we have presented a deep study about the polytope $\mathcal{FM}^2(X)$. This polytope is an interesting polytope that is halfway between probabilities and general fuzzy measures in terms of complexity and capacity of modeling.

First, we have obtained a result that characterizes whether a set of vertices of $\mathcal{FM}^2(X)$ defines a face. From this result, we have derived in particular the adjacency structure of this polytope and we have computed the number of faces of a given dimension, drawing a distinction between simplicial and non-simplicial faces.

Second, we have obtained a triangulation of $\mathcal{FM}^2(X)$ in simplices of the same volume. From this result, we have derived a method for random generation of 2-additive measures; this method is fast and simple, and it is appealing from an intuitive point of view. Besides, this triangulation allows us to provide a method to compute the volume of this polytope, a complicated problem when dealing with general polytopes. Finally, we have also obtained the center of gravity.

We have also presented some properties of the adjacency graph, showing that $\mathcal{FM}^2(X)$ is not planar, but is Hamilton connected and combinatorial. We have also shown that the diameter of this graph is 2 and found its chromatic number.

Next step is to try to translate these results for general $\mathcal{FM}^k(X)$, $k \geq 2$. For this, we have to face a different situation that leads to new problems. The most apparent one comes from the fact that for $k \geq 3$ there are vertices that are not $\{0, 1\}$-valued [25]. Moreover, these vertices have not been fully described in a suitable way. Thus, this seems a complicated problem for which more research is needed.

Following this line, another interesting problem appears if we restrict to the convex closure of $\{0, 1\}$-vertices in $\mathcal{FM}^k(X)$; this leads to a subpolytope of $\mathcal{FM}^k(X)$ and it seems interesting to study if the results obtained in this paper still hold in this case.

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