

Axiomatic structure of k -additive capacities*

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May 12, 2004

Abstract

In this paper we deal with the problem of axiomatizing the preference relations modelled through Choquet integral with respect to a k -additive capacity, i.e. whose Möbius transform vanishes for subsets of more than k elements. Thus, k -additive capacities range from probability measures ($k = 1$) to general capacities ($k = n$). The axiomatization is done in several steps, starting from symmetric 2-additive capacities, a case related to the Gini index, and finishing with general k -additive capacities. We put an emphasis on 2-additive capacities. Our axiomatization is done in the framework of social welfare, and complete previous results of Weymark, Gilboa and Ben Porath, and Gajdos.

Classification code: C43, D31, D63.

Keywords: axiomatic, capacities, k -additivity.

*This paper is an extended and revised version of [Miranda and Grabisch (2000)], presented at IPMU2000 Conference.

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1 Introduction

Since the pioneering works of Schmeidler [Schmeidler (1986), Schmeidler (1989)] and of many followers, capacities (otherwise called non-additive measures by Denneberg [Denneberg (1994)] or fuzzy measures by Sugeno [Sugeno (1974)]) and the Choquet integral [Choquet (1953)] have become an important tool in decision making, as a generalization of probability measures and expected value. The main concept underlying models based on capacities is the one of *comonotonic* functions (or acts, alternatives), i.e. functions f, g such that $(f(x) - f(x'))(g(x) - g(x')) \geq 0$, for any x, x' . For such functions, the Choquet integral becomes a linear operator, and many axiomatic characterizations of decision making models based on capacities contain axioms restricted to comonotonic acts, hence weakening corresponding axioms of probability-based models. This is the case for the axiomatic proposed by Schmeidler [Schmeidler (1986)], where the famous independence axiom is restricted to comonotonic acts (see also e.g. Chateauneuf [Chateauneuf (1994)], Wakker [Wakker (1989)]).

Capacities have also been widely used in decision under multiple criteria. Let X be a set of n criteria. A capacity $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ is defined on X , and for any subset of criteria $A \subseteq X$, roughly speaking $\mu(A)$ represents the importance of the set A of criteria for the decision making problem into consideration. Avoiding intricacies, an alternative can be viewed as a function $f : X \rightarrow \mathbb{R}$, where $f(x)$ stands for the score of alternative f w.r.t. criterion $x \in X$. Then the Choquet integral of f represents the overall score of alternative f , taking into account the importance of criteria, modelled by μ . The axiomatic justification and building of such a model has been given in [Grabisch *et al.* (2003), Labreuche and Grabisch (2003)]. The major interest of such models is that they enable a proper representation of interaction between criteria, a topic of high importance in applications (see e.g. [Grabisch *et al.* (2002)]).

This flexibility of capacity-based models has to be however paid by an exponential complexity, an important drawback in practice, since a capacity on a set of n elements needs 2^n real values to be defined. A solution to this problem is to work with some sub-families of capacities, requiring less coefficients to be defined. Among them, the sub-families of *k-additive measures* (which may be called also *k-additive capacities*), proposed by Grabisch [Grabisch (1996b), Grabisch (1997b)] have the interest to be nested, starting with classical (additive) measures ($k = 1$), and ending with non-additive measures in their full generality ($k = n$). The idea in fact stems from pseudo-Boolean functions $f : \{0, 1\}^n \rightarrow \mathbb{R}$, which are another view of set functions [Hammer and Holzman (1992)]. It is known that they can be expressed under a polynomial form of degree n , involving at most 2^n terms. Hence, a k -additive measure μ is simply a non-additive measure whose corresponding pseudo-Boolean function has a polynomial development of degree at most k . Interestingly enough, this amounts to say that the Möbius transform of μ vanishes for subsets of more than k elements, and also that there is no interaction among subsets of more than k elements [Grabisch (1997b)]. k -additive measures (and especially 2-additive measures, since being more general than additive ones, while remaining simple) have been successfully used in multicriteria decision making (see e.g. [Grabisch *et al.* (2002)]).

One question remains however open: what about the axiomatic characterization of k -additive measures? In decision making, and especially in decision under uncertainty or risk, the main issue is to know the properties of preference structures underlying a given mathematical model. k -

additive measures have proven their usefulness in practice, but it remains to know what precisely they imply or allow for the representation of the preference of the decision maker. Our aim in this paper is to fill this gap. We will see that our axiomatic has its roots in social welfare theory, and is related to previous works by Weymark [Weymark (1981)], and Ben Porath and Gilboa [Ben Porath and Gilboa (1994)].

The paper is organized as follows: We introduce some basic concepts in Section 2. Then, in Section 3 we deal with the problem of characterizing the preference relation when μ is a symmetric capacity; from this starting result, we characterize 2-additive symmetric capacities (Section 4) and k -additive symmetric capacities (Section 5).

We deal with the same problem in Sections 6, 7 and 8 removing the symmetry condition. We will prove that results from Section 3 can be straightforwardly applied to the general case, just removing the symmetry axiom. However, for the 2-additive and the k -additive cases (Sections 7 and 8), we will be forced to modify the axioms obtained in Sections 4 and 5, respectively.

Finally, in Section 9 we give some conclusions.

2 Basic concepts and notations

In this paper, the universal set $X = \{1, \dots, n\}$ denotes a finite set of n elements (states of nature, criteria, individuals, etc). The set of all subsets of X is denoted $\mathcal{P}(X)$. Subsets of X are denoted by A, B, \dots . We will sometimes write $i_1 \cdots i_k$ instead of $\{i_1, \dots, i_k\}$ in order to avoid heavy notation, specially with singletons and subsets of two elements. We start by recalling some definitions.

Definition 1 *A capacity [Choquet (1953)] or non-additive measure [Denneberg (1994)] or fuzzy measure [Sugeno (1974)] over X is a mapping $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ such that*

- $\mu(\emptyset) = 0, \mu(X) = 1$ (boundary conditions).
- $\forall A, B \in \mathcal{P}(X)$ such that $A \subseteq B$, we have $\mu(A) \leq \mu(B)$ (monotonicity).

As a special case of capacities, we have symmetric capacities.

Definition 2 *A capacity μ is said to be symmetric if for any $A, B \in \mathcal{P}(X)$ such that $|A| = |B|$, we have $\mu(A) = \mu(B)$.*

The Möbius transform is an invertible linear transform of set functions, and is a fundamental notion in capacity theory [Chateauneuf and Jaffray (1989)].

Definition 3 [Rota (1964)] *Let $\mu : \mathcal{P}(X) \rightarrow \mathbb{R}$ be a set function on X . The Möbius transform (or inverse) of μ is defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \forall A \subseteq X.$$

The Möbius transform being given, μ is recovered by the Zeta transform

$$\mu(A) = \sum_{B \subseteq A} m(B). \tag{1}$$

It is trivial to see that a capacity μ is symmetric if and only if for any $A, B \in \mathcal{P}(X)$ such that $|A| = |B|$, we have $m(A) = m(B)$.

The monotonicity constraints in terms of m are given by:

Proposition 1 [Chateauneuf and Jaffray (1989)] *A set of 2^n coefficients $m(A), A \subseteq X$ corresponds to the Möbius representation of a capacity if and only if*

$$(i) \quad m(\emptyset) = 0, \quad \sum_{A \subseteq X} m(A) = 1,$$

$$(ii) \quad \sum_{i \in B \subseteq A} m(B) \geq 0, \quad \text{for all } A \subseteq X, \quad \text{for all } i \in A.$$

Another well-known example of non-additive measures are belief functions:

Definition 4 [Dempster (1967), Shafer (1976)] *A belief function is a capacity $Bel : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying for any family $A_1, \dots, A_k \subseteq X$ the following property:*

$$Bel\left(\bigcup_{i=1}^k A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right),$$

for any $k \geq 2$.

Related to belief functions, the following result can be proved:

Proposition 2 [Shafer (1976)] *μ is a belief function if and only if its corresponding Möbius inverse is non-negative.*

Definition 5 [Grabisch (1996b)] *A capacity μ is said to be **k -order additive** or **k -additive** for short for some $k \in \{1, \dots, n\}$ if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$, and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$.*

We call any function $f : X \rightarrow \mathbb{R}$ an *act* or *alternative*, and we denote $\mathcal{F} := \mathbb{R}^X$ the set of all acts on X , while \mathcal{F}_M is the set of non decreasing acts. For convenience, we will often denote $f(i)$ by f_i , and identify f with the vector (f_1, \dots, f_n) of its values. Constant acts $f_i = \alpha$ for all $i \in X$ are denoted by α if no confusion occurs.

Definition 6 [Hardy et al. (1952)] *A pair of acts f, g on \mathcal{F} is said to be **comonotone** if and only if*

$$(f(i) - f(j))(g(i) - g(j)) \geq 0, \quad \forall i, j \in X.$$

Definition 7 [Choquet (1953)] *Let μ be a capacity over X and an act $f \in \mathcal{F}$. The **Choquet integral** of f with respect to μ is defined by*

$$\mathcal{C}_\mu(f) := \int_0^\infty \mu(f \geq \alpha) d\alpha + \int_{-\infty}^0 (\mu(f \geq \alpha) - 1) d\alpha,$$

which, since X is finite, reduces to

$$\mathcal{C}_\mu(f) = \sum_{i=1}^n (f_{(i)} - f_{(i-1)})\mu(B_i),$$

where (\cdot) stands for a permutation on X such that $0 =: f_{(0)} \leq f_{(1)} \leq \dots \leq f_{(n)}$, and $B_i := \{(i), \dots, (n)\}$.

Choquet integral in terms of Möbius transform is given by:

Proposition 3 [Walley (1991)] *Let μ be a capacity and m its Möbius transform. Then, the Choquet integral of $f \in \mathcal{F}$ with respect to μ in terms of m is expressed by:*

$$\mathcal{C}_\mu(f) = \sum_{A \subseteq X} m(A) \bigwedge_{i \in A} f_i.$$

Definition 8 [Yager (1988)] *An ordered weighted averaging operator (OWA) is an operator on \mathcal{F} defined by*

$$\text{OWA}_w(f) := \sum_{i=1}^n w_i f_{(i)},$$

where $w = (w_1, \dots, w_n) \in [0, 1]^n$ is such that $\sum_{i=1}^n w_i = 1$, (called a weight vector), and $f_{(i)}$ is defined the same way as for Choquet integral.

OWA operators and symmetric capacities are related through the following result:

Proposition 4 [Grabisch (1995), Grabisch (1996a), Murofushi and Sugeno (1993)] *Let μ be a capacity on X . Then, the following statements are equivalent:*

1. *There exists a weight vector w such that $\mathcal{C}_\mu(f) = \text{OWA}_w(f)$, for any $f \in \mathcal{F}$.*
2. *μ is a symmetric capacity.*

Finally, we consider a preference relation \succeq on $\mathcal{F} \times \mathcal{F}$, assumed to be reflexive, transitive and complete. As usual, the symmetric part of \succeq is denoted \sim , while \succ denotes the asymmetric part. We say that $V : \mathcal{F} \rightarrow \mathbb{R}$ is a **representation** of \succeq if for any pair of acts $f, g \in \mathcal{F}$, we have

$$f \succeq g \Leftrightarrow V(f) \geq V(g).$$

Our goal in next sections will be to find a set of axioms over \succeq such that the Choquet integral w.r.t. a k -additive capacity is a representation of \succeq .

3 Characterization of OWA operators

In this section we deal with the problem of characterizing the preference relation induced by an OWA operator, i.e. the Choquet integral with respect to a symmetric capacity (Proposition 4). Let us consider an OWA_w operator, with weight vector $w = (w_1, \dots, w_n)$, and the preference relation defined on \mathcal{F} by

$$f \succeq g \Leftrightarrow \sum_{i=1}^n w_i f_{(i)} \geq \sum_{i=1}^n w_i g_{(i)}.$$

We introduce the following axioms, defined and interpreted in [Weymark (1981)], in the context of social welfare:

- **A1.** Weak order: \succeq is complete, reflexive and transitive.
- **A2.** Continuity: For every $f \in \mathcal{F}$, if \succ denotes the strict preference, the sets $\{g \in \mathcal{F} | g \succ f\}$ and $\{g \in \mathcal{F} | g \prec f\}$ are open sets (in the topology of \mathcal{F} induced by the natural topology on \mathbb{R}^n).
- **A3.** Symmetry: For every $f, g \in \mathcal{F}$, if there is a permutation π on X such that $f = \pi g$, then $f \sim g$.
- **A4.** Weak independence of income source: For all acts $f, g, h \in \mathcal{F}_M$, $f \succeq g \Leftrightarrow f+h \succeq g+h$.

With these axioms, Weymark proved:

Theorem 1 [Weymark (1981)] *Let \succeq be a preference relation over $\mathcal{F} \times \mathcal{F}$. Then, \succeq satisfies **A1**, **A2**, **A3** and **A4** if and only if $\exists \mathcal{H} : \mathbb{R}^n \rightarrow \mathbb{R}$ which is a representation of \succeq , such that*

$$\mathcal{H}(f) = \sum_{i=1}^n w_i f_{(i)},$$

where $f_{(1)} \leq f_{(2)} \leq \dots \leq f_{(n)}$, and $w_i \in \mathbb{R}$.

Note that \mathcal{H} is not an OWA operator, since w is not necessarily a weight vector. Then, it suffices to add some axioms to Weymark's system in a way such that they guarantee that the operator obtained in Theorem 1 is an OWA operator. For the sake of simplicity, we will establish our result in several propositions.

First, as a consequence of Proposition 4, the following can be proved:

Lemma 1 *Let us suppose that we are given an operator defined for any $f \in \mathcal{F}$ by*

$$\mathcal{H}(f) = \sum_{j=1}^n w_j f_{(j)}.$$

Then, this operator is the Choquet integral with respect to a symmetric capacity if and only if the following conditions hold:

- **C1.** $w_i \geq 0, \forall i.$
- **C2.** $\sum_{i=1}^n w_i = 1.$

Proof: Remark that **C1** and **C2** are the conditions for w to be a weight vector. Then, applying Proposition 4, it is clear that if we have a Choquet integral with respect to a symmetric capacity, then **C1** and **C2** hold.

The reciprocal is also true. Just recall [Fodor *et al.* (1995)] that the capacity is defined as

$$\begin{array}{rcl} \mu(\{i\}) & = & w_n, \quad \forall i \\ \mu(\{i, j\}) & = & w_n + w_{n-1}, \quad \forall i, j \\ \vdots & \vdots & \vdots \end{array}$$

whence the result. ■

Let us consider the following axiom:

- **A5.** Monotonicity: Given $f, g \in \mathcal{F}$, if $f_i \geq g_i, \forall i \Rightarrow f \succeq g.$

Axiom **A5** states that if an act f is better or equal than another act g for all criteria, then this act must be considered better or equal in our preference relation.

It is clear that the Choquet integral satisfies **A5**. We have the following proposition:

Proposition 5 *Given a preference relation over $\mathcal{F} \times \mathcal{F}$, if **A1**, **A2**, **A3** and **A4** hold, then **A5** is equivalent to **C1**.*

Proof: As **A1**, **A2**, **A3**, **A4** hold, applying Theorem 1 our binary relation can be represented by

$$\mathcal{H}(f) = \sum_{i=1}^n w_i f_{(i)}.$$

\Rightarrow) Let us suppose that **A5** holds and that there exists i such that $w_i < 0$. Let f, g be defined by $g_j = 0, \forall j \leq i, g_j = 1, \forall j > i, f_j = 0, \forall j < i, f_j = 1, \forall j \geq i$. Then, by **A5**, $f \succeq g$.

On the other hand, if $w_i < 0$, we have

$$\mathcal{H}(f) = \sum_{j=1}^n w_j f_{(j)} = \sum_{j=1}^n w_j g_{(j)} + w_i < \sum_{j=1}^n w_j g_{(j)} = \mathcal{H}(g) \Rightarrow f \prec g,$$

a contradiction. Thus, $w_i \geq 0$.

\Leftarrow) Suppose on the other hand $w_i \geq 0$. If $f_i \geq g_i, \forall i \Rightarrow \sum_{j=1}^n f_{(j)} w_j \geq \sum_{j=1}^n g_{(j)} w_j \Rightarrow f \succeq g$, and

hence **A5** holds. ■

Let us now consider the following axiom:

- **A6.** Non-triviality: There exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Axiom **A6** is just needed in order to avoid the trivial relation, i.e. the preference relation in which all alternatives are considered equally good by the decision maker.

Then, the following proposition holds:

Proposition 6 *Consider a preference relation over $\mathcal{F} \times \mathcal{F}$. If **A1**, **A2**, **A3**, **A4**, **A5** hold, then **A6** is equivalent to $\sum_{i=1}^n w_i > 0$.*

Proof: By Proposition 5 we know that our binary relation can be represented by

$$\mathcal{H}(f) = \sum_{i=1}^n w_i f_{(i)}, \quad w_i \geq 0, \quad \forall i.$$

\Rightarrow) Now, if $w_i = 0, \forall i \Rightarrow \mathcal{H}(f) = 0, \forall f \in \mathcal{F}$, and then $f \sim g, \forall f, g \in \mathcal{F}$, contradicting our hypothesis (**A6**).

\Leftarrow) If $\sum_{i=1}^n w_i > 0 \Rightarrow \mathcal{H}(1) > 0 = \mathcal{H}(0) \Rightarrow 1 \succ 0$. ■

Now, we can normalize, and hence $\sum_{i=1}^n w_i = 1$. Then, applying Lemma 1 and Proposition 4 we obtain the following:

Theorem 2 [Miranda and Grabisch (2000)] *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following statements are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A3**, **A4**, **A5** and **A6**.
2. There is a unique symmetric capacity μ such that \succeq is represented by \mathcal{C}_μ .

4 Characterization of 2-additive OWA

In this section we deal with the problem of characterizing preferences represented by the Choquet integral with respect to a 2-additive symmetric capacity. In order to do this, we are going to use a result proved by Ben Porath and Gilboa in [Ben Porath and Gilboa (1994)]. Let us consider the following definition:

Definition 9 *Let f be an act. We say that i f -precedes j if $f_i < f_j$ and there is no $k \in X$ such that $f_i < f_k < f_j$.*

Ben Porath and Gilboa use the set of axioms of Weymark except **A4** which is changed into **A4'**; **A5** is changed into a stronger version **A5'**. Finally they also add some other axioms. Specifically:

- **A4'**. Order-preserving gift: For every f, f', g, g' in \mathcal{F}_M , for every $i \in X$, if $f_j = f'_j$ and $g_j = g'_j$ for every $j \neq i$ and $f'_i = f_i + t, g'_i = g_i + t$ for some $t \in \mathbb{R}$, then $f \succeq g$ if and only if $f' \succeq g'$.
- **A5'**. Strong monotonicity: For every $f, g \in \mathcal{F}$, if $f_i \geq g_i$ for all $i \in X$, and there exists a j such that $f_j > g_j$, then $f \succ g$.
- **A7**. Order-preserving transfer: For every f, f', g, g' in \mathcal{F} , and for all $i, j \in X$, if i f -, g -, f' - and g' -precedes j , if $f'_i = f_i + t, g'_i = g_i + t, f'_j = f_j - t, g'_j = g_j - t$ for some $t > 0$, and $f'_k = f_k, g'_k = g_k$ for all $k \neq i, j$, then $f \succeq g$ if and only if $f' \succeq g'$.
- **A8**. Inequality aversion: For every f, f' in \mathcal{F}_M , for all $i \in X$, if $f'_i = f_i + t, f'_{i+1} = f_{i+1} - t$ for some $t > 0$, if $f'_j = f_j$ for all $j \neq i, i + 1$, then $f' \succ f$.

Ben Porath and Gilboa work in the field of social welfare; thus, they consider X as a set of individuals, and acts are considered as the income profile distribution of a society. Then, they try to find a society where individuals are all equally rich, i.e where inequalities are reduced by sharing wealth. An explanation of the axioms can be found in [Ben Porath and Gilboa (1994)].

They proved the following:

Theorem 3 [Ben Porath and Gilboa (1994)] *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following statements are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A3**, **A4'**, **A5'**, **A7** and **A8**.
2. There is a unique number $\delta, 0 < \delta < 1/(n - 1)$, such that \succeq is represented by the following functional:

$$\mathcal{H}(f) = \sum_{i \in X} f_i - \delta \left[\sum_{1 \leq i < j \leq n} |f_i - f_j| \right]. \quad (2)$$

Grabisch [Grabisch (1998)] has shown that Equation (2) is the Choquet integral with respect to a 2-additive symmetric capacity. However, this result does not cover all of them. The problem comes from the fact that for this result, the coefficient multiplying $f_{(i)}$ is $1 + (n - 2i)\delta$, and thus the weights of the OWA operator are decreasing. We will prove below that **A8** is equivalent to have a strictly decreasing order in the weights of the OWA operator. As this is not always the case for capacities, we have to remove it. Indeed, the functional in Theorem 3 gives more importance to the smallest value (the poorest individual) than to the biggest value (the richest one) and thus, it tries to penalize the differences among incomes in the sense that an increment in the poorest individual from a gift of the richest one should lead to a better (preferred) income distribution. Anyway, we can conclude that 2-additivity is given by **A7**, **A8** or a mixture of these two axioms.

We will use this set of axioms in order to characterize 2-additive symmetric capacities. As in the previous section, we do it in several propositions.

Proposition 7 *Let us consider the Choquet integral w.r.t. a symmetric capacity μ , and its corresponding OWA operator with weight vector w . The following propositions are equivalent.*

(i) μ is at most 2-additive.

(ii) the weight vector has equidistant components

$$w_i - w_{i+1} = w_1 - w_2, \quad \forall i = 1, \dots, n-1.$$

Proof: Since μ is symmetric, the Choquet integral can be written under its OWA form (Proposition 4):

$$\mathcal{C}_\mu(f) = \sum_{i=1}^n w_i f_{(i)}. \quad (3)$$

(i) \Rightarrow (ii) since μ is at most 2-additive, the Choquet integral w.r.t. the Möbius transform of μ writes (Proposition 3)

$$\mathcal{C}_\mu(f) = \sum_{i=1}^n m((i))f_{(i)} + \sum_{i < j} m((i), (j))f_{(i)}.$$

Applying symmetry we have $m(A) = m(B)$ whenever $|A| = |B|$. Therefore,

$$\mathcal{C}_\mu(f) = \sum_{i=1}^n [k_1 + (n-i)k_2]f_{(i)}, \quad (4)$$

with $k_1 = m((i))$, $k_2 = m((i), (j))$. Identifying Equations (4) and (3), we conclude $w_i = k_1 + (n-i)k_2$ and thus the weights are equidistant.

(ii) \Rightarrow (i) Let us define $k_1 := w_n$, $k_2 := w_{n-1} - w_n$. Then, $w_n = k_1$, $w_{n-1} = k_1 + k_2$ and due to the hypothesis of equidistance, we have $w_i = k_1 + (n-i)k_2$. Thus,

$$\sum_{i=1}^n w_i f_{(i)} = \sum_{i=1}^n [k_1 + (n-i)k_2]f_{(i)},$$

and this is the Choquet integral of the 2-additive symmetric capacity defined by $m(i) = k_1$, $m(i, j) = k_2$ (a 1-additive capacity if $k_2 = 0$). As this is the only symmetric capacity whose corresponding Choquet integral has these weights, the result is proved. \blacksquare

In the case of 1-additive symmetric capacities, all coefficients are equal and thus $w_1 - w_2 = 0$.

Consequently, in order to characterize the preference relations over $\mathcal{F} \times \mathcal{F}$ that are represented through the Choquet integral with respect to a 2-additive (at most) symmetric capacity, it suffices to find an axiom leading to equidistance of the weights. This is solved in next proposition.

Proposition 8 *Let \succeq be a preference relation over $\mathcal{F} \times \mathcal{F}$. If **A1**, **A2**, **A3**, **A4**, **A5**, **A6** hold, then **A7** is equivalent to*

$$w_1 - w_2 = w_i - w_{i+1}, \quad \forall i = 1, \dots, n-1.$$

Proof: \Rightarrow) This is Lemma 3.2.1 in [Ben Porath and Gilboa (1994)].

\Leftarrow) Let us consider $f, g \in \mathcal{F}$ and suppose that $i f-, g-$ precedes j . Suppose that $f \succeq g$. Then, if f_i is the i_f -th smallest income in f , and g_i is the i_g -th in g , defining f' and g' as in **A7** and assuming that $i f'-, g'-$ precedes j , we have

$$\mathcal{H}(f') = \sum_{j=1}^n w_j f^{(j)} + t[w_{i_f} - w_{(i-1)_f}].$$

$$\mathcal{H}(g') = \sum_{j=1}^n w_j g^{(j)} + t[w_{i_g} - w_{(i-1)_g}].$$

But now, we know that $w_{i_f} - w_{(i-1)_f} = w_{i_g} - w_{(i-1)_g}$ by hypothesis of equidistance, and thus $f' \succeq g'$. Therefore, **A7** holds. \blacksquare

Then, we have proved the following:

Theorem 4 [Miranda and Grabisch (2000)] *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following statements are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A3**, **A4**, **A5**, **A6** and **A7**.
2. There is a unique 2-additive (or 1-additive) symmetric capacity μ such that \succeq is represented by \mathcal{C}_μ .

Remark that we have not used **A8**. If we add **A8**, it follows that the weights are strictly decreasing:

Lemma 2 *Let \succeq be a preference relation over $\mathcal{F} \times \mathcal{F}$. If **A1**, **A2**, **A3**, **A4**, **A5**, **A6** hold, then **A8** is equivalent to*

$$w_1 > w_2 > \dots > w_n.$$

Proof: We already know that the aggregation operator satisfying **A1**, **A2**, **A3**, **A4**, **A5**, **A6** is an OWA operator (Theorem 2). Then, it can be written as

$$\mathcal{H}(f) = \sum_{i=1}^n f_{(i)} w_i, \quad w_i \geq 0, \quad \sum_{i=1}^n w_i = 1.$$

Now, consider f and f' as in **A8**. It is easy to see that

$$\mathcal{H}(f) - \mathcal{H}(f') = w_{i+1}t - w_i t.$$

Suppose **A8** holds. Then, it follows that $w_i > w_{i+1}$.

Conversely, if $w_i > w_{i+1}$, **A8** holds. \blacksquare

Then, if we add **A8** in Theorem 4, we obtain a characterization of 2-additive symmetric capacities with strictly decreasing weights in the OWA operator. But this means $m(i, j) > 0$ (as $m(i, j) = w_i - w_{i+1}$). On the other hand, by monotonicity (Proposition 1) we have $m(i) \geq 0$. Hence, the 2-additive symmetric measure is also a belief function by Proposition 2. We write down this result in next corollary:

Corollary 1 *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A3**, **A4**, **A5**, **A6**, **A7** and **A8**.
2. There is a unique 2-additive symmetric capacity μ such that \succeq is represented by \mathcal{C}_μ and the weights of the OWA operator are strictly decreasing.
3. There is a unique 2-additive symmetric belief function μ such that \succeq is represented by \mathcal{C}_μ .

In [Ben Porath and Gilboa (1994)], Theorem B, it is proved that **A1**, **A2**, **A3**, **A6**, **A7** and **A8** characterize the Gini index. Consequently, a preference relation modelled through the Choquet integral w.r.t. a 2-additive symmetric measure is a special case of the Gini index, just imposing order preserving gift and monotonicity (**A4**, **A5**).

Remark that for this last result, we avoid the possibility of 1-additive belief functions (i.e. probabilities), as $m(i, j) \neq 0$. If we want to allow this possibility, **A8** should be changed into a weaker version:

- **A8'**. Weak inequality aversion: For every f, f' in \mathcal{F}_M , for all $i \in X$, if $f'_i = f_i + t$, $f'_{i+1} = f_{i+1} - t$ for some $t > 0$, if $f'_j = f_j$ for all $j \neq i, i + 1$, then $f' \succeq f$.

5 Characterization of k -additive OWA

Let us now turn to the general k -additive symmetric case. From the precedent analysis, we just have to replace **A7** by another axiom for the k -additive case, which we call **A7(k)**. We follow the same sequence as in the previous section.

Proposition 9 [Grabisch (1997a)] *Let μ be a symmetric capacity, m its Möbius transform, and w the weight function of the corresponding OWA operator. If μ is at most k -additive, then*

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} = c_k, \forall i = 1, \dots, n - k + 1,$$

and moreover $c_k = m(A)$, with $|A| = k$.

The reciprocal result is given by:

Proposition 10 *Let w be the weight vector of an OWA operator. If*

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{l+j}, \quad i, l = 1, \dots, n - k + 1,$$

then the corresponding capacity μ is at most k -additive.

Proof: We need a preliminary result:

Lemma 3 Let μ be a symmetric capacity, w the weight vector of the corresponding OWA, and $A \subseteq X$. If $|A| \geq k$, then $\forall \{i_1, \dots, i_k\} \subseteq A$, we have that

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(A \setminus B) (-1)^{|B|} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{n-|A|+j+1}.$$

Proof: We know that the capacity associated to an OWA is given by (see the proof of Lemma 1):

$$\mu(T) = \sum_{i=n-|T|+1}^n w_i, \quad T \subseteq X, \quad T \neq \emptyset.$$

Let us fix j and compute how many times each w_j appears. w_j appears in any $\mu(T)$ such that

$$n - |T| + 1 \leq j \Leftrightarrow |T| \geq n - j + 1.$$

In our case, $T = A \setminus B$, whence w_j appears in any B such that

$$|B| \leq |A| + j - n - 1. \tag{5}$$

We have three different cases:

- If $|A| + j - n - 1 < 0$, i.e. $j < n + 1 - |A|$, then no B satisfies the conditions of (5). It follows that w_j never appears in the definition of $\mu(A \setminus B)$, whatever B considered.
- Our second case stands when $|A| + j - n - 1 \geq k$, i.e. $j \in \{n - |A| + k + 1, \dots, n\}$ (if $|A| = k$, there is no j in these conditions). Then, any $|B|$ satisfies (5), whence it follows that w_j appears in the definition of any $\mu(A \setminus B)$. Thus, w_j appears $\sum_{i=0}^k \binom{k}{i} (-1)^i = (1 - 1)^k = 0$ times.
- Finally, assume $0 \leq |A| + j - n - 1 < k$, i.e. $j \in \{n + 1 - |A|, \dots, n - |A| + k\}$. In this case, w_j appears in the definition of all $A \setminus B$ satisfying such that $|B| \leq |A| + j - n - 1 := l$. For each $|B|$, we have $\binom{k}{|B|}$ subsets in these conditions, whence it suffices to prove

$$\sum_{p=0}^l (-1)^p \binom{k}{p} = (-1)^l \binom{k-1}{l}, \quad l \in \{0, \dots, k-1\},$$

but this is a well-known result in Combinatorics (see e.g. [Berge (1971)]). Then, the coefficient of w_j (with $j = n - |A| + 1 + l$) is $(-1)^l \binom{k-1}{l}$ and the result holds. \blacksquare

Let us now prove the proposition. Suppose μ is not k -additive. Then, it must exist A such that $|A| > k$ and $m(A) \neq 0$. Let us choose such an A of minimal cardinality.

Consider $\{i_1, \dots, i_k\}$. Then, by Lemma 3, we know that

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(i_1, \dots, i_k \setminus B) (-1)^{|B|} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{n-k+j+1}.$$

For a fixed $C \subseteq \{i_1, \dots, i_k\}$, let us find the coefficient of $m(C)$ in $\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(i_1, \dots, i_k \setminus B) (-1)^{|B|}$ expressed with the Möbius transform through Eq. (1). For this, we have to count the number of times each $m(C)$ appears. Given $C \subseteq \{i_1, \dots, i_k\}$, the corresponding $m(C)$ appears in the definition of any $\mu(i_1, \dots, i_k \setminus B)$ such that $C \subseteq \{i_1, \dots, i_k\} \setminus B$. For fixed $|B|$ we have $\binom{k-|C|}{|B|}$ subsets in these conditions. Consequently, if $|C| < k$, it follows that $m(C)$ appears

$$\sum_{|B|=0}^{k-|C|} \binom{k-|C|}{|B|} (-1)^{|B|} = 0$$

times. If $C = \{i_1, \dots, i_k\}$, then $m(C)$ appears once (when $B = \emptyset$), whence

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(i_1, \dots, i_k \setminus B) (-1)^{|B|} = m(i_1, \dots, i_k).$$

Let us now prove that $\forall \{i_1, \dots, i_k\} \subset A$, we have that

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(A \setminus B) (-1)^{|B|} = m(A) + m(i_1 \cdots i_k).$$

Let us consider $C \subseteq A$. Then, $m(C)$ appears in the expression of $\mu(A \setminus B)$ in terms of the Möbius transform for all B such that $C \subseteq A \setminus B$. Then, it follows that $B \subseteq \{i_1, \dots, i_k\} \setminus C$. We have the following cases:

- If $|\{i_1, \dots, i_k\} \setminus C| \neq 0$, the coefficient of $m(C)$ is

$$\sum_{i=0}^{|\{i_1, \dots, i_k\} \setminus C|} (-1)^i \binom{|\{i_1, \dots, i_k\} \setminus C|}{i} = (1-1)^{|\{i_1, \dots, i_k\} \setminus C|} = 0.$$

- When $|\{i_1, \dots, i_k\} \setminus C| = 0$, the only possible B is $B = \emptyset$ and in this case the coefficient is 1.

On the other hand, if $|C| > k$, $C \neq A$, then $m(C) = 0$ as A is of minimal cardinality satisfying $m(A) \neq 0$, $|A| > k$. Thus, the only possible non-null summands are $m(A)$ and $m(i_1, \dots, i_k)$, both of them multiplied by 1. As a conclusion, we obtain

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(A \setminus B) (-1)^{|B|} = m(A) + m(i_1 \cdots i_k).$$

Therefore, $\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{n-|A|+j+1}$ is not constant in A and this contradicts our hypothesis, whence the result holds. ■

Let us now consider the following axiom [Miranda and Grabisch (2000)]:

- **A7(k)** k -dimensional order preserving transfer: $\forall f, g, f', g' \in \mathcal{F}$ and $i \in X$ such that i precedes $i + 1$, which precedes $i + 2, \dots$, which precedes $i + k$, for f, f', g and g' , if

1. f' and g' are defined for some $t > 0$ by

$$f'_{i+j} = f_{i+j} + (-1)^j \binom{k-1}{j} t, j = 0, \dots, k-1, f'_k = f_k, k \neq i, i+1, \dots, i+k-1.$$

$$g'_{i+j} = g_{i+j} + (-1)^j \binom{k-1}{j} t, j = 0, \dots, k-1, g'_k = g_k, k \neq i, i+1, \dots, i+k-1.$$

2. f, f' and g, g' are comonotone,

then $f \succeq g \Leftrightarrow f' \succeq g'$.

Remark that this axiom, despite being a generalization of **A7**, is rather difficult to translate into natural language. We will deal with the problem of interpretation at the end of this section. The following proposition holds:

Proposition 11 *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. If \succeq satisfies **A1**, **A2**, **A3**, **A4**, **A5**, **A6**, then **A7(k)** is equivalent to*

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i'+j}, \forall i, i' = 1, \dots, n-k+1,$$

i.e. $\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j}$ does not depend on i .

Proof: \Rightarrow) The proof is an adaptation of the one of Lemma 3.2.1 in [Ben Porath and Gilboa (1994)]: First, it will prove useful to focus on the interior of \mathcal{F}_M , denoted $(\mathcal{F}_M)^o$ and given by

$$(\mathcal{F}_M)^o = \{f \in \mathcal{F}_M \mid f_1 < f_2 < \dots < f_n\}.$$

Let $f \in (\mathcal{F}_M)^o$ and take $\epsilon > 0$. Consider the auxiliary acts e^i defined by $e^i(j) = 1$ if $j = i$ and $e^i(j) = 0$ otherwise. We define

$$f' = f + \sum_{j=0}^{k-1} \epsilon e^{i+j} (-1)^{j-1} \binom{k-1}{j-1}.$$

Here $\epsilon > 0$ is small enough to maintain the order. Remark that we can take $\epsilon > 0$ because $f \in (\mathcal{F}_M)^o$, i.e. we are in the set of strictly increasing acts. Now, for fixed i , let π be the permutation on X defined by

1. If $i > k$, $\pi(1) = i, \pi(2) = i+1, \dots, \pi(k) = i+k-1, \pi(i) = 1, \pi(i+1) = 2, \dots,$

$$\pi(i+k-1) = k, \pi(l) = l, l \notin \{1, \dots, k, i, \dots, i+k-1\}.$$

2. If $i \leq k$, $\pi(1) = k + 1, \pi(2) = k + 2, \dots, \pi(i - 1) = i + k, \pi(i) = 1, \pi(i + 1) = 2, \dots,$

$$\pi(i + k - 1) = k, \pi(l) = l, l \notin \{1, \dots, i + k - 1\}.$$

Consider $g = \pi f$. Then, g' in the conditions of **A7(k)** is given by

$$g' = g + \sum_{j=1}^k \epsilon e^j (-1)^{j-1} \binom{k-1}{j-1}.$$

By symmetry **A3**, we have $g \sim f$. Then, by **A7(k)**, we get $f' \sim g'$ and thus

$$\mathcal{H}(f') - \mathcal{H}(f) = \mathcal{H}(g') - \mathcal{H}(g).$$

Consequently, as \mathcal{H} is indeed an OWA operator,

$$\mathcal{H}(f') - \mathcal{H}(f) = \sum_{j=1}^k \epsilon w_j (-1)^{j-1} \binom{k-1}{j-1} = \sum_{j=1}^k \epsilon w_{i+j-1} (-1)^{j-1} \binom{k-1}{j-1} = \mathcal{H}(g') - \mathcal{H}(g),$$

whence the result.

\Leftarrow) Consider $f, g, f', g' \in \mathcal{F}$ in the conditions of **A7(k)**, and let us suppose that $f \succeq g$. Then, we have

$$\begin{aligned} \mathcal{H}(f') &= \sum_{i=1}^n f_{(i)} w_i + t \left[\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} \right]. \\ \mathcal{H}(g') &= \sum_{i=1}^n g_{(i)} w_i + t \left[\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i'+j} \right]. \end{aligned}$$

By hypothesis, we know that

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i'+j}.$$

Then, we obtain that $f' \succeq g'$ and **A7(k)** holds. ■

Summarizing, we have proved the following:

Theorem 5 [Miranda and Grabisch (2000)] *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A3**, **A4**, **A5**, **A6** and **A7(k)**.
2. There is a unique at most k -additive symmetric capacity μ such that \succeq is represented by \mathcal{C}_μ .

Let us come back to the problem of interpreting **A7(k)**. The fact that this axiom is much more difficult to deal with than **A7** constitutes a weakness in this characterization. A possible interpretation has been given by T. Gajdos in [Gajdos (2002)], using an equivalent axiom which reads:

- **A7'(k)**. $\forall f \in \mathcal{F}_M, \forall i, j \in \{1, \dots, n - k + 1\}, \forall t > 0,$

1. Define f^i and f^j by

$$f_{i+r}^i = f_{i+r} + (-1)^r \binom{k-1}{r} t, r = 0, \dots, k-1, f_l^i = f_l, l \neq i, i+1, \dots, i+k-1.$$

$$f_{j+r}^j = f_{j+r} + (-1)^r \binom{k-1}{r} t, r = 0, \dots, k-1, f_l^j = f_l, l \neq j, j+1, \dots, j+k-1.$$

2. If f, f^i, f^j are comonotone,

then $f^i \sim f^j$.

Proposition 12 Consider a preference relation over $\mathcal{F} \times \mathcal{F}$. If **A1**, **A2**, **A3**, **A4**, **A5**, **A6** hold, then **A7'(k)** is equivalent to

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i'+j}, \forall i, i' = 1, \dots, n - k + 1.$$

Proof: If **A1**, **A2**, **A3**, **A4**, **A5**, **A6** hold, we already know (Theorem 2) that the functional \mathcal{H} representing \succeq is the Choquet integral with respect to a symmetric capacity μ . Let us denote by w the corresponding weight vector. Then,

$$\mathcal{H}(f) = \sum_{i=1}^n f_{(i)} w_i, \forall f \in \mathcal{F}.$$

Consider f, f^i, f^j fulfilling **A7'(k)**. It follows that

$$\mathcal{H}(f^i) = \mathcal{H}(f) + \sum_{r=0}^{k-1} (-1)^r t \binom{k-1}{r} w_{i+r}, \mathcal{H}(f^j) = \mathcal{H}(f) + \sum_{r=0}^{k-1} (-1)^r t \binom{k-1}{r} w_{j+r}.$$

Then, $f^i \sim f^j$ if and only if

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i+j} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} w_{i'+j}, \forall i, i' = 1, \dots, n - k + 1,$$

whence the result. ■

Then, by Proposition 11, we can interchange **A7(k)** and **A7'(k)** if **A1**, **A2**, **A3**, **A4**, **A5**, **A6** hold. In [Gajdos (2002)], Gajdos also showed the equivalence of these two axioms.

In **A7'(k)** the quantities added to individuals can be interpreted as a reward scheme. With this interpretation in mind, let us consider first a reward of ϵ to an individual. Now, we ask the following question: Does the decision maker care about which individual receives this reward? If not, then the decision maker does not care about equality as for him it is the same to give a reward to the richest individual as to give a reward to the poorest one. But if the answer is yes, the decision maker cares about inequality. Since f is strictly increasing, we may think that if the decision maker cares about inequality, then $f^i \succeq f^{i+1}$, or equivalently,

$$(f_1, \dots, f_i + \epsilon, f_{i+1}, \dots, f_n) \succeq (f_1, \dots, f_i, f_{i+1} + \epsilon, \dots, f_n)$$

which is in turn equivalent to

$$(f_1, \dots, f_i + \epsilon, f_{i+1} - \epsilon, \dots, f_n) \succeq (f_1, \dots, f_i, f_{i+1}, \dots, f_n),$$

provided the order in the incomes has not changed. But now, we can go a step further and ask the decision maker if he cares about which individual i is considered in this new reward scheme. If he does not care, then the decision maker satisfies **A7'** (or **A7'(2)**). In other case, he is quite sensitive to inequalities and we may think that

$$(f_1, \dots, f_i + \epsilon, f_{i+1} - \epsilon, f_{i+2}, \dots, f_n) \succeq (f_1, \dots, f_i, f_{i+1} + \epsilon, f_{i+2} - \epsilon, \dots, f_n),$$

or in other words, it is

$$(f_1, \dots, f_i + \epsilon, f_{i+1} - 2\epsilon, f_{i+2} + \epsilon, \dots, f_n) \succeq (f_1, \dots, f_i, f_{i+1}, f_{i+2}, \dots, f_n),$$

provided the order in the incomes has not changed. We can repeat the process for this scheme of rewards to sharpen the degree in which the decision maker cares about inequalities. Then, **A7'(k)** for different choices of k can be seen as a scale to measure the sensitivity to inequalities of the decision maker.

Another possibility of characterization for k -additive symmetric measures could be derived from the results of Calvo and De Baets [Calvo and De Baets (1998)] and Cao-Van and De Baets [Cao-Van and De Baets (2001)], in which they introduce the concept of binomial OWA operators.

Definition 10 *Let $k \in \{1, \dots, n\}$. The k -binomial OWA operator is the OWA operator with weight vector $w := (w_{k1}, \dots, w_{kn})$ defined by*

$$w_{ki} = \frac{\binom{n-i}{k-1}}{\binom{n}{k}}.$$

Then, they prove the following:

Theorem 6 *Let $k \in \{1, \dots, n\}$. Consider an aggregation operator \mathcal{H} ; then, the following equivalence holds:*

1. \mathcal{H} is the Choquet integral with respect to a symmetric k -additive capacity on X .
2. \mathcal{H} is a weighted sum of the first k -binomial OWA operators.

This result characterizes k -additive symmetric capacities from a mathematical point of view. We feel that this result can help to characterize k -additive symmetric measures. However, a wide study of the binomial OWA operators must be done. Besides, we need an axiom leading to a representation in terms of a weighted sum.

We have to remark as a conclusion that the properties that characterize the k -additive symmetric case are rather special. For the general k -additive case, it can be seen that these properties do not hold. In fact, the most important tool that we have used in the proofs of the results was the fact that $m(A) = m(B)$ whenever $|A| = |B|$, and this property comes from the symmetry. This implies that even if these results can give us some information about the general case, this one is much more difficult to handle.

This will not be the case for general measures. We will prove in the next section that removing the symmetry axiom, we can obtain a characterization of Choquet integral.

6 Characterization of Choquet integral for the general case

From an interpretational point of view, the difference between the symmetric case and the general case is given by the fact that for the symmetric case, the importance of an individual when computing Choquet integral comes from its relative position, while for the general case its importance depends also on its index.

To clarify this idea, suppose that we are in the symmetric case and we have an income distribution f . Then, if we give a reward of t to an individual i so that the order in the income distribution does not change, the amount of welfare is determined by the relative position of the income of i , but we do not really care about which individual is considered. This is not true in general for the non-symmetric situation. If we are in the general case, a gift of t to individual i determines an increment of welfare depending on the relative position of individual i and the concrete individual who has received the gift.

To characterize such preference relations, it is obvious that the symmetry axiom **A3** must be removed. In order to prove our result we need the following result due to Schmeidler in which we obtain the mathematical conditions that characterize Choquet integral:

Theorem 7 [Schmeidler (1986)] *Let X be a universal set (finite or infinite) and \mathcal{X} be a σ -algebra over X . Consider $\mathcal{B}(X, \mathcal{X})$ (or \mathcal{B} for short) the set of bounded, real valued, σ -measurable functions on X (i.e. the set of random variables). Let $\mathcal{H} : \mathcal{B} \rightarrow \mathbb{R}$ satisfying $\mathcal{H}(1_X) = 1$ be given. Suppose also that the functional \mathcal{H} satisfies*

1. *Comonotonic additivity: f and g comonotonic implies $\mathcal{H}(f + g) = \mathcal{H}(f) + \mathcal{H}(g)$.*
2. *Monotonicity: $f(x) \geq g(x), \forall x \in X$ implies $\mathcal{H}(f) \geq \mathcal{H}(g)$.*

Then, defining $\mu(A) = \mathcal{H}(1_A), \forall A \in \mathcal{X}$ we have

$$\mathcal{H}(f) = \int_0^\infty \mu(f \geq \alpha) d\alpha + \int_{-\infty}^0 (\mu(f \geq \alpha) - 1) d\alpha.$$

This expression is the Choquet integral for real functions (Definition 7). In our case, in which we are treating with finite universal sets, we can consider $\mathcal{X} = \mathcal{P}(X)$ and thus any act is measurable; and as X is finite, any act is also bounded.

Now, the following can be proved:

Theorem 8 [Miranda and Grabisch (2000)] *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following statements are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A4**, **A5** and **A6**.
2. There is a unique capacity μ such that \succeq is represented by \mathcal{C}_μ .

Proof: It is clear that Choquet integral satisfies all these axioms. Let us see that this is the only functional for which these conditions hold.

Thus, let \mathcal{H} be a functional satisfying **A1**, **A2**, **A4**, **A5**, **A6**. We will denote by \mathcal{H}_π the restriction of \mathcal{H} to a simplex H_π , which is defined as follows: for a given permutation π of the indices, $H_\pi = \{(x_1, \dots, x_n) \mid x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(n)}\}$. Now, we extend \mathcal{H}_π to \mathbb{R}_+^n by symmetry, i.e. given (x_1, \dots, x_n) , we take a permutation α such that $(x_{\alpha(1)}, \dots, x_{\alpha(n)}) \in H_\pi$ and we define $\bar{\mathcal{H}}(x_1, \dots, x_n) = \mathcal{H}_\pi(x_{\alpha(1)}, \dots, x_{\alpha(n)})$. Thus, $\bar{\mathcal{H}}$ is a symmetric functional. Symmetry axiom, together with **A1**, **A2** and **A4** is the set of axioms used by Weymark in [Weymark (1981)] (Theorem 1). Then, we know that $\bar{\mathcal{H}}$ is given by

$$\bar{\mathcal{H}}(f) = \sum_{i=1}^n p_i f_{(i)},$$

for some $p_i \in \mathbb{R}$, $i = 1, \dots, n$, and therefore $\bar{\mathcal{H}}$ satisfies comonotonic additivity.

Then, $\bar{\mathcal{H}}$ is linear in each simplex. In particular $(\bar{\mathcal{H}})_\pi$ is linear. But $(\bar{\mathcal{H}})_\pi = \mathcal{H}_\pi$. Since this is valid for any π , \mathcal{H} itself is linear in each simplex.

Now, we define $\mu(A) = \mathcal{H}(1_A)$.

By **A5** and **A6** we know that μ is a capacity. Then, we have the conditions of the result of Schmeidler (Theorem 7) characterizing the Choquet integral. So that \succeq is represented by a Choquet integral and thus the result holds. ■

Remark that this result is just a version of the conditions of Schmeidler while having a preference relation over the set of acts. In fact, **A4** is a version for acts of the comonotonic additivity. This result is very similar to the one proved by Chateauneuf in [Chateauneuf (1994)].

He uses the following set of axioms:

- **B1** \succeq is a non-trivial weak order.
- **B2** Continuity with respect to monotone uniform convergence
 1. $f_n, f, g \in \mathcal{F}, f_n \succeq g, f_n \downarrow^u f \Rightarrow f \succeq g$.
 2. $f_n, f, g \in \mathcal{F}, g \succeq f_n, f_n \uparrow^u f \Rightarrow g \succeq f$.
- **B3** Monotonicity $f_i \geq g_i + \epsilon, \forall i$ (where $\epsilon > 0$ is a constant) $\Rightarrow f \succ g$.

- **B4** Comonotonic independence $f, g, h \in \mathcal{F}$, f and h comonotonic, g and h comonotonic, then $f \sim g \Rightarrow f + h \sim g + h$.

and then he proves the following:

Theorem 9 *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following are equivalent:*

1. \succeq satisfies **B1**, **B2**, **B3** and **B4**.
2. there is a unique capacity μ such that \succeq is represented by \mathcal{C}_μ .

It is easy to see that **B1** is equivalent to **A1** and **A6**, **B2** is equivalent to **A2**, **B3** is similar to **A5** and finally **B4** is implied by **A4**. However, Theorem 9 is not restricted to finite sets and he only imposes f to be a bounded measurable function.

7 Characterization of 2-additive capacities

Now, we deal with the problem of characterizing preference relations on $\mathcal{F} \times \mathcal{F}$ that can be represented by the Choquet integral with respect to a 2-additive capacity. Of course, we only need to add an axiom to the set of axioms found in last section for the general case. However, **A7** does not suffice to characterize 2-additivity and therefore our new axiom must be a generalization of it.

Then, we have to find a new property characterizing 2-additive capacities. This property is obtained in next proposition.

Proposition 13 *A capacity μ is at most 2-additive if and only if $\forall A \subseteq X$, and $\forall i, j \in A$,*

$$\mu(A) - \mu(A \setminus i) - \mu(A \setminus j) + \mu(A \setminus i, j) = \mu(i, j) - \mu(i) - \mu(j).$$

Proof: Let us suppose that μ is not a 2-additive capacity (at most). Then, there exists A such that $|A| > 2$ and $m(A) \neq 0$.

Let us consider A of minimal cardinality in these conditions and let us take $i, j \in A$. This is always possible as $|A| > 2$. Then, we have the following:

$$\begin{aligned} \mu(A) - \mu(A \setminus i) - \mu(A \setminus j) + \mu(A \setminus i, j) &= \sum_{B \subseteq A} m(B) - \sum_{B \subseteq A \setminus i} m(B) - \sum_{B \subseteq A \setminus j} m(B) + \sum_{B \subseteq A \setminus i, j} m(B) \\ &= \sum_{i \in B \subseteq A} m(B) - \sum_{i \in B \subseteq A \setminus j} m(B) = m(A) + m(i, j), \end{aligned}$$

by hypothesis on A .

On the other hand, $\mu(i, j) - \mu(i) - \mu(j) = m(i, j)$ and thus the expression does not remain constant.

Reciprocally, proceeding the same way, we obtain that if $m(A) = 0$, $\forall A$ s.t. $|A| > 2$, then

$$\mu(A) - \mu(A \setminus i) - \mu(A \setminus j) + \mu(A \setminus i, j) = m(i, j) = \mu(i, j) - \mu(i) - \mu(j),$$

whence the result. ■

Let us introduce the following notation: Given an act $f \in \mathcal{F}$, for a fixed real value t , we define the act $f^B \in \mathcal{F}$ by

$$f_i^B = \begin{cases} f_i + t & \text{if } i \in B \\ 0 & \text{otherwise} \end{cases}$$

Consider now the following axiom:

- **A9.** Let $f, g \in \mathcal{F}$ be acts such that $f_i = f_j, g_i = g_j$. Consider the acts f^B, g^B for all $B \subseteq \{i, j\}$, with $t > 0$ so that f^B, f and g^B, g are comonotone for all B . Let $\mathcal{H} : \mathcal{F} \rightarrow \mathbb{R}$ be any functional representing the preference relation \succeq . Then,

$$\sum_{B \subseteq \{i, j\}} \mathcal{H}(f^B)(-1)^{|B|} = \sum_{B \subseteq \{i, j\}} \mathcal{H}(g^B)(-1)^{|B|}.$$

This axiom can be interpreted as follows: Suppose that we have an income distribution such that two individuals i, j are equally rich. Now, we give a gift $t > 0$ to both of them. Next step is to ask one of these individuals for this gift to be returned. This individual can be either i or j . Then, we obtain four income distributions and the global welfare with these operations could have changed. If the amount (or decrease) in the welfare just depend on individuals i, j but not on their relative position, then the decision maker acts following **A9**.

Let us introduce the following notation: For any $h \in \mathcal{F}$ we denote by A_h^i the set

$$A_h^i := \{l \in X \mid h_l > h_i\}.$$

The following can be proved:

Proposition 14 *Let \succeq be a preference relation on $\mathcal{F} \times \mathcal{F}$. If \succeq satisfies **A1**, **A2**, **A4**, **A5** and **A6**, then **A9** is equivalent to*

$$\mu(A) - \mu(A \setminus i) - \mu(A \setminus j) + \mu(A \setminus i, j) = \mu(i, j) - \mu(i) - \mu(j), \quad \forall A \subseteq X, \quad i, j \in A.$$

Proof: As **A1**, **A2**, **A4**, **A5** and **A6** hold, we know from Theorem 8 that our preference relation is modelled by the Choquet integral with respect to a capacity μ .

Consider $f, g \in \mathcal{F}$ fulfilling the conditions of **A9**. Then, it can be easily seen that

$$\mathcal{C}_\mu(f^{\{i, j\}}) = \mathcal{C}_\mu(f) + t[\mu(A_f^i \cup i, j) - \mu(A_f^i)].$$

$$\mathcal{C}_\mu(f^{\{i\}}) = \mathcal{C}_\mu(f) + t[\mu(A_f^i \cup i) - \mu(A_f^i)].$$

$$\mathcal{C}_\mu(f^{\{j\}}) = \mathcal{C}_\mu(f) + t[\mu(A_f^i \cup j) - \mu(A_f^i)].$$

The same can be done for g . Consequently,

$$\sum_{B \subseteq \{i, j\}} \mathcal{C}_\mu(f^B)(-1)^{|B|} = (\mu(A_f^i \cup i, j) - \mu(A_f^i \cup i) - \mu(A_f^i \cup j) + \mu(A_f^i))t,$$

$$\sum_{B \subseteq \{i, j\}} \mathcal{C}_\mu(g^B)(-1)^{|B|} = (\mu(A_g^i \cup i, j) - \mu(A_g^i \cup i) - \mu(A_g^i \cup j) + \mu(A_g^i))t,$$

whence **A9** holds for any f, g if and only if

$$\mu(A_f^i \cup i, j) - \mu(A_f^i \cup i) - \mu(A_f^i \cup j) + \mu(A_f^i) = \mu(A_g^i \cup i, j) - \mu(A_g^i \cup i) - \mu(A_g^i \cup j) + \mu(A_g^i),$$

whence the result. ■

Then, we have proved:

Theorem 10 *Let \succeq be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following are equivalent:*

1. \succeq satisfies **A1**, **A2**, **A4**, **A5**, **A6** and **A9**.
2. There is a unique 2-additive capacity μ such that \succeq is represented by \mathcal{C}_μ .

8 Characterization of k -additive capacities

In this section we deal with the general k -additive case. We will follow the same line as for the symmetric case. Then, we will use the results characterizing preference relations modelled through Choquet integral with respect to a capacity and change **A9** for another axiom **A9(k)** for the general k -additive case.

For the k -additive case, we have the following result:

Proposition 15 *A capacity μ is (at most) k -additive if and only if*

$$\sum_{\substack{B \subseteq \{i_1, \dots, i_k\} \\ i_1, \dots, i_k \in A}} \mu(A \setminus B)(-1)^{|B|} = \sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(i_1, \dots, i_k \setminus B)(-1)^{|B|}, \forall A \text{ such that } i_1, \dots, i_k \in A.$$

Proof: Suppose that μ is not k -additive. Then, there exist A such that $|A| > k$ and $m(A) \neq 0$. Let us consider A of minimal cardinality in these conditions. Let us prove that $\forall \{i_1, \dots, i_k\} \subseteq A$, we have

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(A \setminus B)(-1)^{|B|} = m(A) + m(\{i_1, \dots, i_k\}).$$

Consider $C \subseteq A$ such that $|C| = j$. We have the following cases:

- If $\{i_1, \dots, i_k\} \not\subseteq C$, then $m(C)$ appears $(1 - 1)^{k - |C \cap \{i_1, \dots, i_k\}|} = 0$ times.
- If $|C| > k$, $C \neq A$, then $m(C) = 0$ by hypothesis and the coefficient is not important.
- If $C = A$, then $m(A)$ appears once (when $B = \emptyset$).
- Finally, if $C = \{i_1, \dots, i_k\}$, then $m(C)$ appears once (only in $\mu(A)$).

On the other hand, repeating the process for $\{i_1, \dots, i_k\}$, we have

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(i_1, \dots, i_k \setminus B) (-1)^{|B|} = m(i_1, \dots, i_k).$$

Thus, the expression is constant if and only if μ is a k -additive measure. \blacksquare

We consider the axiom:

- **A9(k)** k -dimensional asymmetric order-preserving transfer: Consider $f, g \in \mathcal{F}$ such that $f_{i_1} = \dots = f_{i_k}, g_{i_1} = \dots = g_{i_k}$. We consider f^B, g^B for all $B \subseteq \{i_1, \dots, i_k\}$, with $t > 0$ so that f^B, f and g^B, g are comonotone for all B . Let $\mathcal{H} : \mathcal{F} \rightarrow \mathbb{R}$ be any functional representing the preference relation \succeq . Then,

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mathcal{H}(f^B) (-1)^{|B|} = \sum_{B \subseteq \{i_1, \dots, i_k\}} \mathcal{H}(g^B) (-1)^{|B|}.$$

Axiom **A9(k)** could be interpreted from **A9** the same way as **A7'(k)** is interpreted from **A7'**. Starting with the conditions of **A9**, if the decision maker does not care about the relative position of i, j , then he follows **A9**; otherwise he cares about the position, and then we turn to **A9(3)** and we restart again. The process finishes when we find the value of k . Then, **A9(k)** measures again the sensitivity of the decision maker to the relative positions of individuals.

The following can be proved:

Proposition 16 *Let \succeq be a preference relation on $\mathcal{F} \times \mathcal{F}$. If \succeq verifies **A1**, **A2**, **A4**, **A5** and **A6**, then **A9(k)** is equivalent to*

$$\sum_{\substack{B \subseteq \{i_1, \dots, i_k\} \\ i_1, \dots, i_k \in C}} \mu(C \setminus B) (-1)^{|B|} = \sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(B) (-1)^{k-|B|}, \forall C \text{ such that } i_1, \dots, i_k \in C.$$

Proof: As **A1**, **A2**, **A4**, **A5** and **A6** hold, we know from Theorem 8 that our preference relation is modelled by the Choquet integral with respect to a capacity μ .

Let us take $f, g \in \mathcal{F}$ in the conditions of **A9(k)**. Given i_1, \dots, i_k , consider $A_f^{i_k}$ (resp. $A_g^{i_k}$). Then,

$$\mathcal{C}_\mu(f_B) = \mathcal{C}_\mu(f) + t [\mu(B \cup A_f^{i_k}) - \mu(A_f^{i_k})], \mathcal{C}_\mu(g_B) = \mathcal{C}_\mu(g) + t [\mu(B \cup A_g^{i_k}) - \mu(A_g^{i_k})].$$

Now,

$$\begin{aligned} \sum_{B \subseteq \{i_1, \dots, i_k\}} \mathcal{H}(f^B) (-1)^{|B|} &= t \sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(B \cup A_f^{i_k}) (-1)^{|B|}, \\ \sum_{B \subseteq \{i_1, \dots, i_k\}} \mathcal{H}(g^B) (-1)^{|B|} &= t \sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(B \cup A_g^{i_k}) (-1)^{|B|}, \end{aligned}$$

and thus, **A9(k)** holds if and only if

$$\sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(B \cup A_f^{i_k}) (-1)^{|B|} = \sum_{B \subseteq \{i_1, \dots, i_k\}} \mu(B \cup A_g^{i_k}) (-1)^{|B|}, \forall f, g,$$

whence the result holds. ■

Then, we have proved:

Theorem 11 *Let \succsim be a binary relation on $\mathcal{F} \times \mathcal{F}$. The following statements are equivalent:*

1. \succsim satisfies **A1**, **A2**, **A4**, **A5**, **A6** and **A9(k)**.
2. There is a unique k -additive capacity μ such that \succsim is represented by \mathcal{C}_μ .

9 Conclusions

We have proposed a characterization of preference relations represented by the Choquet integral with respect to different types of capacities, from symmetric capacities to general k -additive measures, with an emphasis on 2-additive capacities.

Axioms for 2-additive measures (in the symmetric and in the general case) can be interpreted from an economical point of view and the theory of social welfare. Axioms for the k -additive case are just a generalization of those from the 2-additive case. For the symmetric case, an interesting interpretation of the axioms was given by Gajdos, but to our knowledge, no interpretation has been given for the general case. For this case an additional problem arises, as the relative strength of each individual is not the same. Then, we have to work with families of acts or income distributions, that make the resulting axioms rather difficult to handle and to translate into natural language (axioms **A9(k)**).

A drawback considering these last axioms is that the aggregation function \mathcal{H} intervenes into them. This could be overcome at the price of introducing *difference measurement* [Krantz *et al.* (1971)] (instead of relying on ordinal measurement), requiring from the decision maker to express a certain kind of intensity of preferences.

Another possible way to find a characterization would be to consider the Shapley interaction index [Grabisch (1996b)] instead of the Möbius transform; however, the Choquet integral in terms of Shapley interaction is rather complicated [Grabisch (1997c)], and thus a deep study should be done.

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