Order cones: A tool for deriving k-dimensional faces of cones of subfamilies of monotone games*

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Abstract

In this paper we introduce the concept of order cone. This concept is inspired by the concept of order polytopes, a well-known object coming from Combinatorics with which order cones share many properties. Similarly to order polytopes, order cones are a special type of polyhedral cones whose geometrical structure depends on the properties of a partially ordered set (brief poset). This allows to study the geometrical properties of these cones in terms of the subjacent poset, a problem that is usually simpler to solve. Besides, for a given poset, the corresponding order polytope and order cone are deeply related. From the point of view of applicability, it can be seen that many cones appearing in the literature of monotone TU-games are order cones. Especially, it can be seen that the cones of monotone games with restricted cooperation are order cones, no matter the structure of the set of feasible coalitions and thus, they can be studied in a general way applying order cones.

Keywords: Monotone games, restricted cooperation, order polytope, cone.

1 Introduction

Consider a finite set of n players $N = \{1, 2, ..., n\}$. We will denote subsets of N by capital letters $A, B, ...$ and by $\mathcal{P}(N)$ the set of parts of N. A **game** v is a function $v : \mathcal{P}(N) \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. The value $v(A)$ represents the minimal worth coalition A can obtain if all players in A agree to cooperate, no matter what players outside A might do.

In general, several additional conditions can be imposed on function v. One of the most natural conditions is monotonicity in v, i.e. $v(A) \leq v(B)$ if $A \subset B$. This means that if players add to a coalition, the corresponding worth increases. We will denote by $\mathcal{MG}(N)$ the set of all monotone games on N. Other popular conditions are additivity, supermodularity, and many others (see (Grabisch, 2016)).

On the other hand, it could be the case that some coalitions fail to form. Thus, v cannot be defined on some of the elements of $\mathcal{P}(N)$ and we have a subset $\mathcal{FC}(N)$ of $\mathcal{P}(N)$ containing all *feasible*

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coalitions. From now on, we will not include $\emptyset$ in $\mathcal{FC}(N)$. Usually, $\mathcal{FC}(N)$ has a concrete structure (for example, $\mathcal{FC}(N)$ could be a lattice if $\emptyset$ is added). Thus, depending on the structure of $\mathcal{FC}(N)$, the set of TU-games whose set of feasible coalitions is $\mathcal{FC}(N)$ has different properties.

With some abuse of notation, we will denote by v both the game and the corresponding point in $\mathbb{R}^{|\mathcal{FC}(N)|}$ given by $\boldsymbol{v} := (v(A))_{\{A:A\in\mathcal{FC}(N)\}}$. Then, the set of games on N satisfying a given condition (monotonicity, supermodluarity , ...) and/or such that the set of feasible coalitions is $\mathcal{FC}(N)$ can be seen as a set in $\mathbb{R}^{|\mathcal{FC}(N)|}$. In many cases, this set is usually a convex polyhedron. Hence, it can be given in terms of its vertices and extremal rays. Many papers have been devoted to solve the problem of obtaining different geometrical aspects of these polyhedra for particular cases (see e.g. (Grabisch and Kroupa, 2019; Shapley, 1971)).

Following this line, in this paper we introduce the concept of order cone. Order cones are defined in terms of a poset and its structure relays on the structure of the corresponding poset. Besides, we will show that order cones are deeply related to order polytopes. As many results are known for order polytopes, it is possible to translate such properties to order cones.

As it will become clear below, order cones are a class of cones including the cones of monotone games with restricted cooperation, no matter which the set $\mathcal{FC}(N)$ is. Thus, order cones allow to study this set of cones in a general way. As the properties of an order cone just depend on the structure of the corresponding poset, it suffices to study this poset, a problem that is simpler in general than studying the set $\mathcal{MG}(FC(N))$ as we will see in the examples of Section 4. Besides, it could help to identify which conditions need to be imposed on $\mathcal{FC}(N)$ so that $\mathcal{MG}(FC(N))$ satisfies a property. For example, we will see that the order cone is an (infinite) pyramid if $\mathcal{FC}(N)$ has a top element. Finally, using the relations between order cone and order polytope, many results known for order polytopes can be translated to order cones. For example, we will use this property to characterize the set of extremal rays of the cone $\mathcal{MG}(N)$, a problem that to our knowledge has not been solved yet (Grabisch, 2016).

Interestingly enough, order cones can be applied to situations where monotonicity is combined with other properties. As an example dealing with such a case, we study the cone of monotone k-symmetric games. This also adds more insight about the relationship between order cones and order polytopes.

Finally, order cones can be applied to other situations different to monotone games with restricted cooperation. It should be noted that order cones just rely on a poset structure and thus, they are very general. Thus, this concept can be applied to any family of games such that there are constraints $v(A) \leq v(B)$ whenever $A\mathcal{R}B$, where $\mathcal{R}$ is a relation on $\mathcal{FC}(N)$.

The rest of the paper goes as follows: In next section we introduce the basic concepts and results about cones and order polytopes. Next, we define order cones and study some of its geometrical properties. We then apply these results for some special cases of monotone games with restricted cooperation. We finish with the conclusions and open problems.

2 Basic results

In order to be self-contained and fix the notation, let us start introducing some concepts and results that will be needed throughout the paper.

A **cone** is a non-empty subset $\mathcal{C}$ of $\mathbb{R}^n$ such that if $\boldsymbol{x} \in \mathcal{C}$, then $\alpha\boldsymbol{x} \in \mathcal{C}$ for all $\alpha \geq 0$. Note that $\mathbf{0}$ is in any cone. Additionally, we say that the cone is **convex** if it is a convex set of $\mathbb{R}^n$; equivalently, a cone is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$, it follows

$$\boldsymbol{x} + \boldsymbol{y} \in \mathcal{C}.$$

Given a set $\mathcal{S}$, we define its **conic hull (or conic extension)** as the smallest cone containing $\mathcal{S}$.

A convex cone $\mathcal{C}$ is **polyhedral** if additionally it is a polyhedron. This means that it can be written as

$$\mathcal{C} := \{\boldsymbol{x} : A\boldsymbol{x} \leq \mathbf{0}\}, \tag{1}$$

for some matrix $A \in \mathcal{M}_{m\times n}$ of binding conditions. Two polyhedral cones are **affinely isomorphic** if there is a bijective affine map from one cone onto the other. Given a polyhedral cone $\mathcal{C}$ and $\boldsymbol{x} \in \mathcal{C}, \boldsymbol{x} \neq \mathbf{0}$, the set $\{\alpha\boldsymbol{x} : \alpha \geq 0\}$ is called a **ray**. In general we will identify a ray with the point $\boldsymbol{x}$. Notice also that for polyhedral cones, all rays pass through $\mathbf{0}$. Point $\boldsymbol{x}$ defines an **extremal ray** if $\boldsymbol{x} \in \mathcal{C}$ and there are $n-1$ binding conditions for $\boldsymbol{x}$ that are linearly independent. Equivalently, $\boldsymbol{x}$ cannot be written as a convex combination of two linearly independent points of $\mathcal{C}$.

It is well-known that a convex polyhedron only has a finite set of vertices and a finite set of extremal rays. The following result is well-known for convex polyhedra:

Theorem 1. *Let $\mathcal{P}$ be a convex polyhedron on $\mathbb{R}^n$. Let us denote by $\boldsymbol{x}_1, ..., \boldsymbol{x}_r$ the vertices of P and by $\boldsymbol{v}_1, ..., \boldsymbol{v}_s$ the vectors defining extremal rays. Then, for any $\boldsymbol{x} \in P$, there exists $\alpha_1, ..., \alpha_r$ such that $\alpha_1 + ... + \alpha_r = 1, \alpha_i \geq 0, i = 1, ..., r,$ and $\beta_1, ..., \beta_s$ such that $\beta_i \geq 0, i = 1, ..., s,$ satisfying that*

$$\boldsymbol{x} = \sum_{i=1}^{r} \alpha_i \boldsymbol{x}_i + \sum_{j=1}^{s} \beta_j \boldsymbol{v}_j.$$

Given a polyhedral cone, if $\boldsymbol{x} \in \mathcal{C}, \boldsymbol{x} \neq \mathbf{0}$, it follows that $\boldsymbol{x}$ cannot be a vertex of $\mathcal{C}$. Thus, for a polyhedral cone, the only possible vertex is $\mathbf{0}$. Thus, for the particular case of polyhedral cones, Theorem 1 writes as follows.

Corollary 1. *For a polyhedral cone $\mathcal{C}$ whose extremal rays are defined by $\boldsymbol{v}_1, ..., \boldsymbol{v}_s$, any $\boldsymbol{x} \in \mathcal{C}$ can be written as*

$$\boldsymbol{x} = \sum_{j=1}^{s} \beta_j \boldsymbol{v}_j, \quad \beta_j \geq 0, j = 1, ..., s.$$

Consequently, in order to determine the polyhedral cone it suffices to obtain the extremal rays.

We will say that a cone is **pointed** if $\mathbf{0}$ is a vertex. The following result characterizes pointed cones.

Theorem 2. *For a polyhedral cone $\mathcal{C}$ the following statements are equivalent:*

- *$\mathcal{C}$ is pointed.*
- *$\mathcal{C}$ contains no line.*
- *$\mathcal{C} \cap (-\mathcal{C}) = \mathbf{0}$.*

Finally, in this paper we will deal with the problem of obtaining the faces of order cones. Remember that given a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, a non-empty subset $\mathcal{F} \subseteq \mathcal{P}$ is a **face** if there exist $\boldsymbol{v} \in \mathbb{R}^n, c \in \mathbb{R}$ such that

$$\boldsymbol{v}^t\boldsymbol{x} \leq c, \forall \boldsymbol{x} \in \mathcal{P}, \quad \boldsymbol{v}^t\boldsymbol{x} = c, \forall \boldsymbol{x} \in \mathcal{F}.$$

We denote this face as $\mathcal{F}_{\boldsymbol{v},c}$. The **dimension** of a face is the dimension of the smallest affine space containing the face. A common way to obtain faces is turning into equalities some of the inequalities of (1) defining $\mathcal{P}$.

Theorem 3. *(Cook et al., 1988) Let $A \in \mathcal{M}_{m\times n}$. Then any non-empty face of $\mathcal{P} = \{\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}\}$ corresponds to the set of solutions to*

$$\begin{aligned} \sum_j a_{ij}x_j &= b_i \text{ for all } i \in I \\ \sum_j a_{ij}x_j &\leq b_i \text{ for all } i \notin I, \end{aligned}$$

for some set $I \subseteq \{1, \ldots, m\}$.

The set of faces with the inclusion relation determines a lattice known as the **face lattice** of the polyhedron.

Let us now recall the basic results about order polytopes. Consider a poset $(P, \preceq)$, or P for short, with p elements. Elements of P are denoted x, y and so on. We will say that x is **covered** by y, denoted $x \lessdot y$, if $x \preceq y$ and there is no $z \in P\backslash\{x, y\}$ such that $x \prec z \prec y$. A subset $F \subseteq P$ is an *upset* or *filter* if $x \in F$ and $x \prec y$ implies $y \in F$. We will denote by $\mathcal{F}(P)$ the set of upsets of P. It is well-known that $(\mathcal{F}(P), \subseteq)$ is a distributive lattice (Davey and Priestley, 2002). Posets are usually represented through *Hasse diagrams*. A poset is connected if the corresponding Hasse diagram is a connected graph.

For any poset, it is possible to define a polytope on $\mathbb{R}^p$, called the order polytope of P.

Definition 1. *(Stanley, 1986) Given a poset $(P, \preceq)$, we associate to P a polytope $\mathcal{O}(P)$ over $\mathbb{R}^p$, called the* **order polytope** *of P, formed by the p-uples f of real numbers satisfying*

- $0 \leq f(x) \leq 1$ *for every element x in P, and*
- $f(x) \leq f(y)$ *whenever $x \preceq y$ in P.*

Thus, the polytope $\mathcal{O}(P)$ consists in the order-preserving functions from P to $[0, 1]$. Note that we obtain an equivalent definition if the second condition turns into

$$f(x) \leq f(y) \text{ whenever } x \lessdot y.$$

The main advantage of order polytopes is that they allow to study the properties of the polytope in terms of the subjacent poset P. For example, if the poset is a chain, it can be shown that the corresponding order polytope is a **simplex**, i.e. a generalization of a triangle in the p-dimensional space.

Order polytopes has a tight relation with the set of capacities (see Definition 4 below). Indeed, it can be seen (Combarro and Miranda, 2010) that the set of capacities over a finite referential N of n elements, seen as a subset of $\mathbb{R}^{2^n-2}$, is the order polytope with respect to the set $\mathcal{P}(N)\backslash\{\emptyset, N\}$ with the inclusion order. Other families of normalized measures are order polytopes, too, as for example the set of k-symmetric measures when the partition of indifference is known (Combarro and Miranda, 2010).

The following facts related to order polytopes are well-known and are discussed in (Stanley, 1986).

Proposition 1. *Given a finite poset P, the vertices of $\mathcal{O}(P)$ are the characteristic functions v_F of upsets F of P, i.e.*

$$\boldsymbol{v}_F(x) := \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases}$$

Consequently, $\mathcal{O}(P)$ is a 0/1-polytope.

Next result characterizes whether two vertices are adjacent in $\mathcal{O}(P)$.

Theorem 4. *Let P be a finite poset and consider two upsets $F_1, F_2 \in \mathcal{F}(P)$. Then the vertices $\boldsymbol{v}_{F_1}$ and $\boldsymbol{v}_{F_2}$ are adjacent to each other if and only if $F_1 \subset F_2$ and $F_2 \backslash F_1$ is a connected subposet of P.*

For obtaining the k-dimensional faces of an order polytope, additionally to the general methods presented for general polyhedrons, we can derive another way using the order structure of P. For this, we need to consider the poset

$$\hat{P} := \perp \oplus P \oplus \top,$$

where we have added to P a minimum $\perp$ and a maximum $\top$. Then, $\mathcal{O}(P)$ is equivalent to the polytope given by

- $0 = f(\perp), f(\top) = 1$.
- $f(x) \leq f(y)$ whenever $x \preceq y$ in $\hat{P}$.

Now, note that turning an inequality of Definition 1 into an equality makes $f(x) = f(y)$ for some x, y such that $x \lessdot y$. Therefore, we can associate faces to partitions $\{B_1, ..., B_k\}$ of $\hat{P}$ in a way such that the face is the set of functions f such that $f(x) = f(y)$ for all x, y in the same block. However, not any partition defines a face. A partition $\{B_1, ..., B_k\}$ is *connected* if B_i is connected as a subposet of $\hat{P}$. Defining $B_i \prec B_j$ if there exists $x \in B_i, y \in B_j$ such that $x \preceq_P y$, we say that the partition is *compatible* if $\preceq$ is antisymmetric. Finally, the partition is *closed* if for $i \neq j$, there exists $g \in \mathcal{O}(P)$ constant in each block such that $g(B_i) \neq g(B_j)$. Now, the following holds.

Theorem 5. *A closed partition of $\hat{P}$ defines a face of $\mathcal{O}(P)$ if and only if it is compatible and connected.*

This result is especially useful for high-dimensional faces as for example facets, as it is easy to check if these conditions on the partition hold. For faces of small dimension, we can solve the problem in another way. Note that any face can be defined equivalently as the convex hull of the vertices in the face. Hence, a face can be associated to its vertices. However, not every set of vertices defines a face. Thus, it suffices to obtain a condition for a subset of vertices to define a face. On the other hand, in order polytopes vertices are related to upsets of P. If we focus on the set of upsets defining a face, the following characterization arises.

Theorem 6. *(Friedl, 2017) Let $L \subseteq \mathcal{F}(P)$. Then, L determines a face if and only if L is an* **embedded lattice** *of $\mathcal{F}(P)$, i.e. for any two upsets $F, F' \in \mathcal{F}(P)$*

$$J \cup J', J \cap J' \in L \Leftrightarrow J, J' \in L.$$

3 Order cones

Let us now turn to the concept of order cones. The idea is to remove the condition $f(a) \leq 1$ from Definition 1. Thus, the resulting set is no longer bounded. This is what we will call an order cone. Formally,

Definition 2. *Let P be a finite poset with p elements. The* **order cone** *$\mathcal{C}(P)$ is the intersection of the positive orthant and the set of points satisfying $f(x) \leq f(y)$ whenever $x \preceq y$ in P.*

In other words, the order cone of a poset is formed by the p-tuples satisfying

i) $0 \leq f(x)$ for every $x \in P$,

ii) $f(x) \leq f(y)$ whenever $x \preceq y$ in P.

For example, we will see in Section 4 that the set of monotone games $\mathcal{MG}(N)$ as a subset of $\mathbb{R}^{2^n-1}$ is an order cone with respect to the poset $P = \mathcal{P}(N)\backslash\{\emptyset\}$ with the partial order given by $A \prec B \Leftrightarrow A \subset B$. Another example is given at the end of the section.

The name order cone is consistent, as next lemma shows.

Lemma 1. *Given a finite poset P, then $\mathcal{C}(P)$ is a pointed polyhedral cone.*

Proof. It is a straightforward consequence of the definition that $\mathcal{C}(P)$ is a polyhedron. Let us then show that it is indeed a cone. For this, take $f \in \mathcal{C}(P)$ and consider $\alpha f, \alpha \geq 0$. For $x \preceq y$ in P, we have $f(x) \leq f(y)$ and thus, $\alpha f(x) \leq \alpha f(y)$. Hence $\alpha f \in \mathcal{C}(P)$ and the result holds.

Moreover, as $f(x) \geq 0, \forall x \in P, f \in \mathcal{C}(P)$, it follows that $\mathcal{C}(P) \cap -(\mathcal{C}(P)) = \{\mathbf{0}\}$, and by Theorem 2, $\mathcal{C}(P)$ is a pointed cone. □

Consequently, $\mathcal{C}(P)$ has just one vertex, $\mathbf{0}$.

Remark 1. *Consider the set $Q = \{1, ..., n\}$ and let us define on $Q \times Q$ a binary relation $\preceq$ that is reflexive and transitive. The pair $(Q, \preceq)$ is known as a* **preposet**. *When $\preceq$ also satisfies the antisymmetric property, then $(Q, \preceq)$ is a poset.*

For a preposet $(Q, \preceq)$, let us consider the semispaces on $\mathbb{R}^n$ given by $x_i \leq x_j$ iff $i \preceq j$. The intersection of these semispaces defines a polyhedral cone known as the **braid cone**[1] *(Postnikov et al., 2008). Braid cones has been used to define generalized permutahedra. Several properties and more insight about this concept can be found in (Postnikov et al., 2008).*

When Q is a poset, the corresponding braid cone is deeply related to the order polytope of Q. Indeed, it just suffices to include the condition $x_i \geq 0, i \in Q$ to obtain the order polytope.

Definition 2 suggests a strong relationship between order polytopes and order cones. The following results study some straightforward aspects of this relation.

Lemma 2. *Let P be a finite poset. Then, $\mathcal{C}(P)$ is the conical extension of $\mathcal{O}(P)$.*

Proof. If $f \in \mathcal{O}(P)$, it follows that for $x, y \in P, x \prec y$, it is $0 \leq f(x) \leq f(y)$. Thus, $f \in \mathcal{C}(P)$.

On the other hand, consider a cone $\mathcal{C}$ such that $\mathcal{O}(P) \subset \mathcal{C}$. For $f \in \mathcal{C}(P)$, and $\alpha > 0$ small enough, we have $\alpha f \in \mathcal{O}(P) \subset \mathcal{C}$. Then, $\frac{1}{\alpha}\alpha f = f \in \mathcal{C}$, and hence $\mathcal{C}(P) \subseteq \mathcal{C}$. □

Indeed, the following holds:

[1]We are very grateful to an anonymous reviewer for focusing our attention on braid cones.

Lemma 3. *Consider a finite poset P. Then,*

$$\mathcal{C}(P) \cap \{\boldsymbol{x} : \boldsymbol{x} \leq \boldsymbol{1}\} = \mathcal{O}(P).$$

Proof. $\subseteq$) Consider $f \in \mathcal{C}(P) \cap \{\boldsymbol{x} : \boldsymbol{x} \leq \boldsymbol{1}\}$. Hence, $f(x) \leq 1, \forall x \in P$, and if $x \preceq y$, then $0 \leq f(x) \leq f(y) \leq 1$. Thereofore, $f \in \mathcal{O}(P)$.

$\supseteq$) For $f \in \mathcal{O}(P)$, we have $f \in \mathcal{C}(P)$ by Lemma 2 and $f(x) \leq 1, \forall x \in P$. □

As $\mathcal{C}(P)$ is a polyhedral cone by Lemma 1 and according to Corollary 1, this cone can be given in terms of its corresponding extremal rays. Next theorem characterizes the set of extremal rays of $\mathcal{C}(P)$ in terms of upsets of P.

Theorem 7. *Let P be a finite poset and $\mathcal{C}(P)$ its associated order cone. Then, its extremal rays are given by*

$$\{\alpha \cdot \boldsymbol{v}_F : \alpha \in \mathbb{R}^+\},$$

where $\boldsymbol{v}_F$ is the characteristic function of a non-empty connected upset F of P.

Proof. We know that extremal rays of a pointed cone are rays passing through $\boldsymbol{0}$. Let us show that extremal rays of $\mathcal{C}(P)$ are related to vertices of $\mathcal{O}(P)$ adjacent to $\boldsymbol{0}$. Consider an extremal ray, that is given by a vector $\boldsymbol{v}$. We can assume that $\boldsymbol{v}$ is such that $\boldsymbol{v} \leq \boldsymbol{1}$ and there exists a coordinate i such that $v_i = 1$. Hence, by Lemma 3, $\boldsymbol{v} \in \mathcal{O}(P)$. Let us show that $\boldsymbol{v}$ is indeed a vertex of $\mathcal{O}(P)$. If not, there exist two different points $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{O}(P)$ such that

$$\boldsymbol{v} = \alpha \boldsymbol{w}_1 + (1-\alpha)\boldsymbol{w}_2, \quad \alpha \in (0,1).$$

Besides, $\alpha \boldsymbol{w}_1, (1-\alpha)\boldsymbol{w}_2 \in \mathcal{C}(P)$. Remark that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are linearly independent because there exists a coordinate i such that $v_i = 1$. Consequently, $\boldsymbol{v}$ does not define an extremal ray, a contradiction.

Next, let us now show that $\boldsymbol{v}$ is adjacent to $\boldsymbol{0}$. Otherwise, the segment $[\boldsymbol{0}, \boldsymbol{v}]$ is not an edge of $\mathcal{O}(P)$. Consequently, $\frac{1}{2}\boldsymbol{v}$ can be written as

$$\frac{1}{2}\boldsymbol{v} = \alpha \boldsymbol{y}_1 + (1-\alpha)\boldsymbol{y}_2,$$

where $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{O}(P)$ such that they are outside $[\boldsymbol{0}, \boldsymbol{v}]$. Thus,

$$\boldsymbol{v} = 2\alpha \boldsymbol{y}_1 + 2(1-\alpha)\boldsymbol{y}_2.$$

Finally, $2\alpha \boldsymbol{y}_1, 2(1-\alpha)\boldsymbol{y}_2 \in \mathcal{C}(P)$, so we conclude that $\boldsymbol{v}$ does not define an extremal ray, which is a contradiction.

Now, as $\boldsymbol{v}$ is a vertex, it is related to an upset $F \subseteq P$. On the other hand, $\boldsymbol{0}$ is related to the empty upset. As $\boldsymbol{v}$ is adjacent to $\boldsymbol{0}$, we can apply Theorem 4 to conclude that $F = F \setminus \emptyset$ is a connected upset of P.

Let us now prove the reverse. Consider $\boldsymbol{v}$ an adjacent vertex to $\boldsymbol{0}$ in $\mathcal{O}(P)$ and assume that $\boldsymbol{v}$ does not define an extremal ray. Then, there exists $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{C}(P)$ and not proportional to $\boldsymbol{v}$ such that

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2 = \frac{1}{2}2\boldsymbol{w}_1 + \frac{1}{2}2\boldsymbol{w}_2 = \frac{1}{2}\boldsymbol{w}'_1 + \frac{1}{2}\boldsymbol{w}'_2.$$

Now, for $\epsilon > 0$ small enough, we have

$$\epsilon \boldsymbol{v} = \frac{1}{2}\epsilon \boldsymbol{w}'_1 + \frac{1}{2}\epsilon \boldsymbol{w}'_2,$$

and $\epsilon \boldsymbol{w}'_1 \leq \mathbf{1}, \epsilon \boldsymbol{w}'_2 \leq \mathbf{1}$. Hence, $\epsilon \boldsymbol{w}'_1, \epsilon \boldsymbol{w}'_2 \in \mathcal{O}(P)$ by Lemma 3, and hence $[\mathbf{0}, \boldsymbol{v}]$ is not an edge of $\mathcal{O}(P)$, in contradiction with $\boldsymbol{v}$ adjacent to $\mathbf{0}$.

□

Let us now turn to the problem of obtaining the faces of $\mathcal{C}(P)$. As explained in Theorem 3, faces arise when inequalities turn into equalities. Let us consider the inequality $f(x) \leq f(y)$ for $x \lessdot y$ and assume this inequality is turned into an equality. This means that x and y identified to each other; let us call z this new element. In terms of posets, this translates into transforming P into another poset $(P', \preceq')$ defined as $P' := P \backslash \{x, y\} \cup \{z\}$ and $\preceq'$ given by:

$$\begin{cases} a \preceq' b \Leftrightarrow a \preceq b & \text{if } a, b \neq z \\ z \preceq' b \Leftrightarrow x \preceq b & \\ a \preceq' z \Leftrightarrow a \preceq y & \end{cases}$$

Similar conclusions arise when $0 \leq f(x)$ turns into an equality. Moreover, if $\mathcal{F}$ is the face obtained by turning an inequality into an equality, the projection

$$\begin{array}{cccc} \pi : & \mathcal{F} & \rightarrow & \mathcal{C}(P') \\ & (f(a), ..., f(x), f(y), ..., f(b)) & \hookrightarrow & (f(a), ..., f(z), ..., f(b)) \end{array}$$

is a bijective affine map. Consequently, the following holds.

Lemma 4. *The faces of an order cone are affinely isomorphic to order cones.*

Compare this result with the corresponding result for order polytopes (Stanley, 1986).

Lemma 5. *For an order cone $\mathcal{C}(P)$, the vertex $\mathbf{0}$ is in all non-empty faces. Consequently, all faces can be written as $\mathcal{F}_{\boldsymbol{v},0}$.*

Proof. It suffices to show that for a non-empty face $\mathcal{F}_{\boldsymbol{v},c}$, it is $c = 0$. First, $\boldsymbol{v}^t\mathbf{0} \leq c$, so that $c \geq 0$.

Suppose $c > 0$. As $\mathcal{F}_{\boldsymbol{v},c}$ is non-empty, there exist $\boldsymbol{x} \in \mathcal{C}(P)$ such that $\boldsymbol{v}^t\boldsymbol{x} = c$. But then, $\boldsymbol{v}^t 2\boldsymbol{x} = 2c > c$, a contradiction. Thus, $c = 0$ and $\mathbf{0} \in \mathcal{F}_{\boldsymbol{v},0}$. □

With this in mind, Theorem 7 can be extended to characterize all the faces of the order cone, not only the extremal rays.

Theorem 8. *Let P be a finite poset and $\mathcal{C}(P)$ and $\mathcal{O}(P)$ the corresponding order cone and order polytope, respectively. For a pair $(\boldsymbol{v}, 0)$, the set $\mathcal{F}'_{\boldsymbol{v},0} = \mathcal{C}(P) \cap \{\boldsymbol{x} : \boldsymbol{v}^t\boldsymbol{x} = 0\}$ is a face of $\mathcal{C}(P)$ if and only if $\mathcal{F}_{\boldsymbol{v},0} = \mathcal{O}(P) \cap \{\boldsymbol{x} : \boldsymbol{v}^t\boldsymbol{x} = 0\}$ is a face of $\mathcal{O}(P)$. Moreover, $dim(\mathcal{F}'_{\boldsymbol{v},0}) = dim(\mathcal{F}_{\boldsymbol{v},0})$.*

Proof. Let $\mathcal{F}_{\boldsymbol{v},0}$ be a face of $\mathcal{O}(P)$ containing $\mathbf{0}$ and let us show that it determines a face on $\mathcal{C}(P)$. First, let us show that $\boldsymbol{v}^t\boldsymbol{x} \leq 0, \forall \boldsymbol{x} \in \mathcal{C}(P)$. Otherwise, there exists $\boldsymbol{x}_0 \in \mathcal{C}(P)$ such that $\boldsymbol{v}^t\boldsymbol{x}_0 > 0$. But then $\boldsymbol{v}^t\epsilon\boldsymbol{x}_0 > 0, \forall \epsilon > 0$. As ϵ can be taken small enough so that $\epsilon\boldsymbol{x}_0 \leq \mathbf{1}$, it follows by Lemma 3 that $\epsilon\boldsymbol{x}_0 \in \mathcal{O}(P)$ and as $\boldsymbol{v}^t\epsilon\boldsymbol{x}_0 > 0$, and we get a contradiction. Hence, the pair $(\boldsymbol{v}, 0)$ determines a face $\mathcal{F}'_{\boldsymbol{v},0}$ of $\mathcal{C}(P)$.

Consider now a face $\mathcal{F}'_{\boldsymbol{v},0}$ of $\mathcal{C}(P)$. Hence, $\boldsymbol{v}^t\boldsymbol{x} \leq 0, \forall \boldsymbol{x} \in \mathcal{C}(P)$. But then, $\boldsymbol{v}^t\boldsymbol{x} \leq 0, \forall \boldsymbol{x} \in \mathcal{O}(P)$ and as $\mathbf{0} \in \mathcal{F}'_{\boldsymbol{v},0}$, this determines a face of $\mathcal{O}(P)$.

Let us now see that for each pair $(\boldsymbol{v}, 0)$, $dim(\mathcal{F}'_{\boldsymbol{v},0}) = dim(\mathcal{F}_{\boldsymbol{v},0})$. First, as $\mathcal{F}_{\boldsymbol{v},0} \subseteq \mathcal{F}'_{\boldsymbol{v},0}$, we have $dim(\mathcal{F}_{\boldsymbol{v},0}) \leq dim(\mathcal{F}'_{\boldsymbol{v},0})$.

On the other hand, let k be the dimension of $\mathcal{F}'_{\boldsymbol{v},0}$. This implies that there are k vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ linearly independent in $\mathcal{F}'_{\boldsymbol{v},0}$. But now, we can find $\epsilon > 0$ small enough such that $\epsilon\boldsymbol{v}_1 \leq \mathbf{1}, ..., \epsilon\boldsymbol{v}_k \leq \mathbf{1}$. Thus, $\epsilon\boldsymbol{v}_1, ..., \epsilon\boldsymbol{v}_k \in \mathcal{F}_{\boldsymbol{v},0}$ and hence, $dim(\mathcal{F}_{\boldsymbol{v},0}) \geq dim(\mathcal{F}'_{\boldsymbol{v},0})$. ☐

As a consequence, we can adapt Theorem 6 for order cones as follows.

Theorem 9. *Let $L \subseteq \mathcal{F}(P)$. Then, L determines a face of $\mathcal{C}(P)$ if and only if L is an embedded lattice of $\mathcal{F}(P)$ containing the empty upset.*

Remark 2. *From Theorem 8, in order to find faces of an order cone, we need to look for faces of the corresponding order polytope containing $\mathbf{0}$. As previously explained in Theorem 3, if we consider the expression of $\mathcal{O}(P)$ as a polyhedron, faces arise turning inequalities into equalities. Vertices in the face are the vertices of the polyhedron satisfying these equalities. If we consider $\hat{P}$, vertex $\mathbf{0}$ corresponds to function*

$$f(x) = \begin{cases} 0 & x \neq \top \\ 1 & x = \top \end{cases}$$

Consequently, $\mathbf{0}$ satisfies $f(x) = f(y)$ when $y \neq \top$. Thus, we look for the faces where the inequalities turned into equalities do not depend on $\top$.

In terms of Theorem 5, we have to look for partitions defining faces containing $\mathbf{0}$. Note that each block B_i defines a subset of P such that all elements in B_i attain the same value for all points in the face. Therefore, faces containing $\mathbf{0}$ mean that there is a block containing only $\top$.

Example 1. *Consider the polytope given in Figure 1 left.*

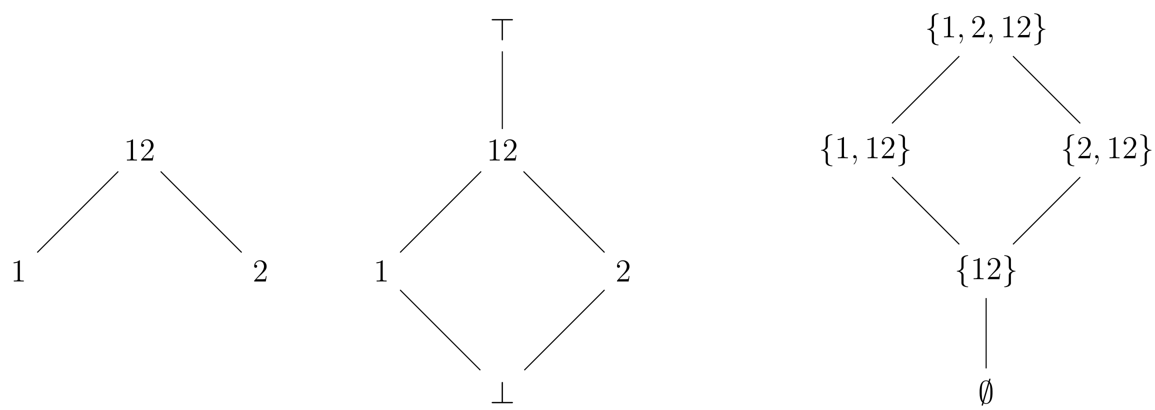


Figure 1: Example of poset P (left), his extension $\hat{P}$ (center) and his upset lattice (right).

In this case, we have three elements and both the order polytope and the cone order cone can be depicted in $\mathbb{R}^3$, with the first coordinate corresponding to 1, the second one to 2 and the third to 12, see Figure 2. The cone $\mathcal{C}(P)$ is given by 3-dimensional vectors f satisfying

$$0 \leq f(1),\ 0 \leq f(2),\ f(1) \leq f(12),\ f(2) \leq f(12).$$

Let us then explain the previous results for this poset. First, let us start obtaining the vectors defining extremal rays. According to Theorem 7, it suffices to obtain the non-empty upsets that are connected subposets of P. Non-empty upsets of P are:

$$\{\{12\}, \{1, 12\}, \{2, 12\}, \{1, 2, 12\}\}.$$

All of them are connected subposets of P. Hence, we have 4 extremal rays, whose respective vectors are

$$(0,0,1),(1,0,1),(0,1,1),(1,1,1).$$

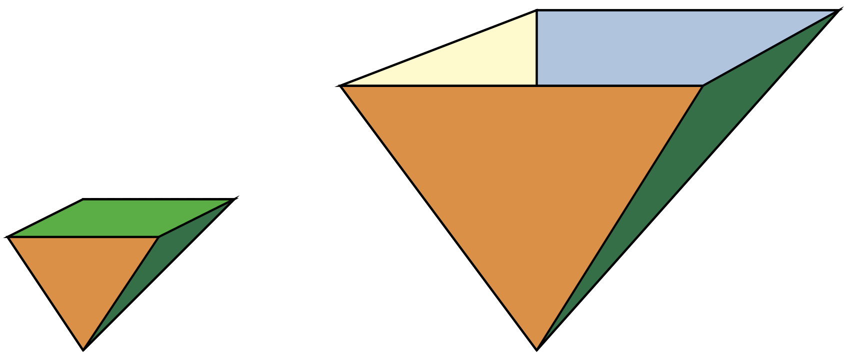


Figure 2: Order polytope $\mathcal{O}(P)$ (left) and order cone $\mathcal{C}(P)$ (right).

Let us now deal with the facets. For this, consider the poset $\hat{P} = \bot \oplus P \oplus \top$ (see Figure 1 center). According to Theorems 5 and 8, the facets are given by considering one of the following equalities:

$$f(\bot) = f(1),\ f(\bot) = f(2),\ f(1) = f(1,2),\ f(2) = f(1,2),\ f(1,2) = f(\top).$$

This translates into transforming poset $\hat{P}$ in a new poset where the elements in the equality identify to each other (see Lemma 4). The posets for the previous equalities are given in Figure 3.

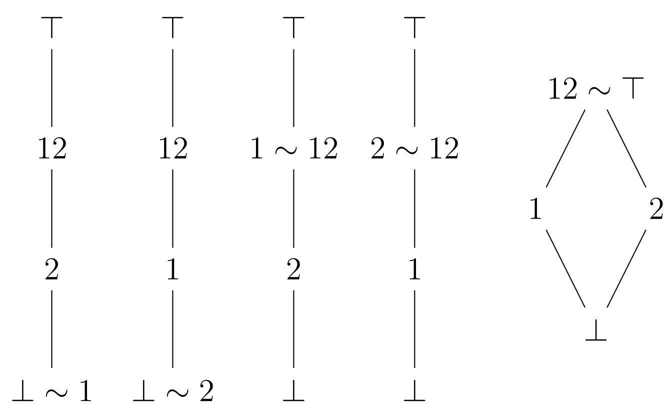


Figure 3: Subposets when turning an inequaity into an equality.

Note that the facets containing $\mathbf{0}$ are those whose defining equality does not involve $\top$, as $\mathbf{0}$ satisfies any other equality. In our case, they correspond to the first four cases. Thus, we have four facets containing $\mathbf{0}$ and all of them are simplices (indeed triangles) because the corresponding polytope is a chain.

For the 1-dimensional faces, we have to consider two equalities. However, we have to be careful with the selected equalities because they might imply other equalities. For example, if we consider $f(\bot) = f(1), f(1) = f(1,2)$, this also implies $f(\bot) = f(2)$, and hence we obtain a point instead of an edge. In our case, the edges containing $\mathbf{0}$ are given by the pairs of equalities defining an edge and not involving $\top$. There are four pairs in these conditions that are

$$\{f(\perp) = f(1), f(\perp) = f(2)\},\ \{f(\perp) = f(1), f(2) = f(1,2)\},$$

$$\{f(\perp) = f(2), f(1) = f(1,2)\},\ \{f(1) = f(1,2), f(2) = f(1,2)\}.$$

Alternatively, we could use the characterization given in Theorem 9. In this case, we have to consider the upset lattice (see Figure 1 right).

Hence, edges are given by pairs of upsets defining a sublattice and involving the empty upset. Thus, the possible choices are the following pairs:

$$\{\{\emptyset\}, \{12\}\},\ \{\{\emptyset\}, \{1, 12\}\},\ \{\{\emptyset\}, \{2, 12\}\},\ \{\{\emptyset\}, \{1, 2, 12\}\}.$$

Thus, the extremal rays of $\mathcal{C}(P)$ *are given by vectors* $(0,0,1), (1,0,1), (0,1,1), (1,1,1)$.

For 2-dimensional faces, we have to consider all possible sublattices of height 2 and involving the $\emptyset$ *upset. These sublattices are:*

$$\{\{\emptyset\}, \{1, 12\}, \{1, 2, 12\}\},\ \{\{\emptyset\}, \{2, 12\}, \{1, 2, 12\}\},\ \{\{\emptyset\}, \{12\}, \{1, 12\}\},\ \{\{\emptyset\}, \{12\}, \{2, 12\}\}.$$

Hence, the 2-dimensional faces for $\mathcal{C}(P)$ *are defined by vectors*

$$\{(1,0,1), (1,1,1)\}, \{(0,1,1), (1,1,1)\}, \{(0,0,1), (1,0,1)\}, \{(0,0,1), (0,1,1)\}.$$

Notice that we cannot consider

$$\{\{\emptyset\}, \{1, 12\}, \{2, 12\}, \{1, 2, 12\}\},\ \{\{\emptyset\}, \{12\}, \{1, 12\}, \{2, 12\}\},$$

because they are not embedded sublattices.

4 Application to Game Theory

In this section, we show that some well-known cones appearing in the field of monotone games can be seen as order cones. Hence, all the results developed in the previous section can be applied to these cones. The first example deals with the general case of monotone games when all coalitions are feasible. We next extend this to the case where $\mathcal{FC}(N) \subset \mathcal{P}(N)\backslash\{\emptyset\}$. As an example of applicability for subfamilies of monotone games satisfying a property on v but not on the set of feasible coalitions, we also treat the case of k-symmetric monotone games.

4.1 The cone of general monotone games

Consider monotone games when all coalitions are feasible, i.e. the set $\mathcal{MG}(N)$. We consider $\mathcal{MG}(N)$ as a subset of $\mathbb{R}^{2^n-1}$ (we have removed the coordinate for $\emptyset$ because its value is fixed).This set is given by all games satisfying $v(A) \leq v(B)$ whenever $A \subset B$. Thus, a game $v \in \mathcal{MG}(N)$ is characterized by the following conditions:

- $0 \leq v(A)$.
- $v(A) \leq v(B)$ if $A \subseteq B$.

Then, $\mathcal{MG}(N) = \mathcal{C}(\mathcal{P}(N)\backslash\{\emptyset\})$, where the order relation $\prec$ on $\mathcal{P}(N)\backslash\{\emptyset\}$ is given by $A \prec B$ if and only if $A \subset B$. For example, for $|N| = 3$, this poset is given in Figure 4. However, little else is known about $\mathcal{MG}(N)$; for example, the set of extremal rays is not known and this question appears in (Grabisch, 2016) as an open problem. We will study this set at the light of the results of the previous section. Let us first deal with the extremal rays.

Corollary 2. *The vectors defining an extremal ray of $\mathcal{MG}(N)$ are defined by non-empty upsets of $\mathcal{P}(N)\backslash\{\emptyset\}$.*

Proof. Following Theorem 7, we need to find the upsets of $\mathcal{P}(N)\backslash\{\emptyset\}$ that are connected. But in this case, all upsets except the empty upset, corresponding to vertex $\mathbf{0}$, contain N. Hence, all of them are connected. □

For obtaining the number of extremal rays, note that any upset in a poset is characterized in terms of its minimal elements and that these minimal elements are an antichain of the poset. For the boolean poset $\mathcal{P}(N)$, the number of antichains is known as the Dedekind numbers, D_n. The first values of D_n are given in Table 1.

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7 581
6	7 828 354
7	2 414 682 040 998
8	56 130 437 228 687 557 907 788

Table 1: First Dedekind numbers.

For $\mathcal{MG}(N)$, we have to remove the antichain $\{\emptyset\}$ because the poset defining the order cone is $\mathcal{P}(N)\backslash\{\emptyset\}$. Besides, the empty antichain corresponds to $\mathbf{0}$ and thus, it should be removed, too. Hence, the number of extremal rays of $\mathcal{MG}(N)$ is $D_n - 2$.

Example 2. *Let us compute the extremal rays of the order cone $\mathcal{MG}(N)$ where $N = \{1, 2, 3\}$. Note that $\mathcal{C}(P)$ is a cone in $\mathbb{R}^7$. Then, considering $P = B_3 \setminus \{\emptyset\}$, it suffices to compute the upsets of P.*

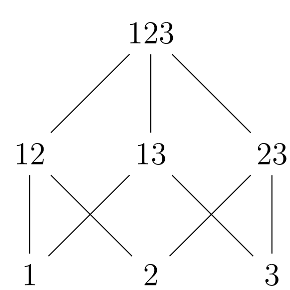


Figure 4: Boolean poset $P = B_3 \setminus \{\emptyset\}$.

A list with these upsets is:

$$\mathfrak{CF}(P) = \{\emptyset, \{123\}, \{12, 123\}, \{13, 123\}, \{23, 123\}, \{12, 13, 123\}, \{12, 23, 123\}, \{13, 23, 123\},$$

$$\{12, 13, 23, 123\}, \{1, 12, 13, 123\}, \{1, 12, 13, 23, 123\}, \{2, 12, 23, 123\}, \{2, 12, 13, 23, 123\}, \{3, 13, 23, 123\},$$

$$\{3, 12, 13, 23, 123\}, \{1, 2, 12, 13, 23, 123\}, \{1, 3, 12, 13, 23, 123\}, \{2, 3, 12, 13, 23, 123\}, \{1, 2, 3, 12, 13, 23, 123\}\}.$$

Removing $\emptyset$, we have a total of 18 extremal rays. Note that $D_3 = 20$.

Similarly, we can apply Theorem 8 to obtain all k-dimensional faces of the cone $\mathcal{MG}(N)$.

Corollary 3. *The non-empty faces of $\mathcal{MG}(N)$ are given by the non-empty faces of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset\})$ containing vertex $\mathbf{0}$.*

However, we will see that in this case we can do better.

Definition 3. *Let $\mathcal{P}$ be a convex polytope and $\boldsymbol{x}$ be a point outside the affine space generated by $\mathcal{P}$, denoted aff($\mathcal{P}$). Point $\boldsymbol{x}$ is called* **apex**. *We define a* **pyramid** *with base $\mathcal{P}$ and apex $\boldsymbol{x}$, denoted by pyr($\mathcal{P}, \boldsymbol{x}$), as the polytope whose vertices are the ones of $\mathcal{P}$ and $\boldsymbol{x}$.*

Note that for a pyramid $\text{pyr}(\mathcal{P}, \boldsymbol{x})$, any vertex of $\mathcal{P}$ is adjacent to $\boldsymbol{x}$. Moreover, there is a simple way to find faces containing $\boldsymbol{x}$ for a pyramid that we write below.

Proposition 2. *For a pyramid of apex $\boldsymbol{x}$ and base $\mathcal{P}$, the k-dimensional faces containing $\boldsymbol{x}$ are given by the $(k-1)$-dimensional faces of $\mathcal{P}$.*

From now on, in order to simplify the notation, we will assume that the last coordinate in vector $\boldsymbol{v}$ corresponds to the value $v(N)$.

Proposition 3. *Consider the poset $\mathcal{P}(N)\backslash\{\emptyset\}$ with the relation order $A \prec B \Leftrightarrow A \subset B$. Then, the order polytope $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset\})$ is a pyramid with apex $\mathbf{0}$ and base $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\}\}$.*

Proof. Note that for any non-empty upset F, it follows that $N \in F$. Then, the characteristic function of any non-empty upset v_F satisfies $v_F(N) = 1$. Hence, any vertex of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset\})$ except $\mathbf{0}$ is in the hyperplane $v(N) = 1$. Consequently, $\mathcal{O}(\mathcal{P}\backslash\{\emptyset\})$ is a pyramid with apex $\mathbf{0}$. Finally, the points $\boldsymbol{v}$ of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset\})$ in the hyperplane $v(N) = 1$ satisfy $v(A) \leq v(B)$ if $A \subseteq B$. Thus, these points can be associated to the order polytope $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$, where the order relation $\preceq$ is given by $A \preceq B \Leftrightarrow A \subseteq B$. □

This allows us to study the k-dimensional faces of $\mathcal{MG}(X)$ from a different point of view that the one of Theorems 9 and 8. In particular, as apex $\boldsymbol{x}$ is adjacent to every vertex in the base $\mathcal{P}$, edges are given by segments $[\boldsymbol{x}, \boldsymbol{y}]$ with $\boldsymbol{y}$ a vertex of $\mathcal{P}$, thus recovering the result of Corollary 2. In general, applying Proposition 2, the following holds.

Corollary 4. *The k-dimensional faces of $\mathcal{MG}(N)$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$.*

The order polytope $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$ is a well-known polytope corresponding to the set of capacities or fuzzy measures.

Definition 4. *A* **capacity** *on X is a map $\mu : \mathcal{P}(N) \to \mathbb{R}$, satisfying* *i)* $\mu(\emptyset) = 0, \mu(N) = 1$ *(normalization).*

ii) $\mu(A) \leq \mu(B), \quad \forall A \subseteq B$ *(monotonicity).*

This notion was proposed by Choquet (Choquet, 1953) and independently by Sugeno under the name of fuzzy measure (Sugeno, 1974). These measures are also called "non-additive mesaures" (Denneberg, 1994). From the point of view of Game Theory, capacities are just normalized monotone games. Capacities constitute an extension of a probability distribution, where additivity is turned into monotonicity and they have been applied in many different fields, as for example Decision Making (see (Grabisch, 2016) and references therein). The set of capacities on a referential set N is denoted by $\mathcal{FM}(N)$ and it can be seen (Combarro and Miranda, 2010) that

$$\mathcal{FM}(N) = \mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\}).$$

It is worth-noting that the geometrical structure (apart the dimension) of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$ is quite different from the geometrical structure of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset\})$. For example, in $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset\})$ all vertices are adjacent to $\mathbf{0}$, while this is not the case for $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$ (see (Combarro and Miranda, 2010)).

For this order polytope, many results are known, as for example wether two vertices are adjacent or the centroid (Combarro and Miranda, 2010, 2008). Applying Corollary 4, we conclude that 2-dimensional faces of $\mathcal{MG}(N)$ are given by an edge of $\mathcal{FM}(N) = \mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$. On the other hand, an edge in $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$ is given by two adjacent vertices $\boldsymbol{v}_{F_1}, \boldsymbol{v}_{F_2}$. Another characterization specific for $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$ is given in (Combarro and Miranda, 2008). Moreover, as both F_1, F_2 are adjacent to $\mathbf{0}$, the following holds.

Corollary 5. *Any 2-dimensional face of $\mathcal{MG}(N)$ are defined in terms of 2-dimensional simplices given by $\{\mathbf{0}, \boldsymbol{v}_{F_1}, \boldsymbol{v}_{F_2}\}$ where $F_2\backslash F_1$ is a connected subposet of $\mathcal{P}(N)\backslash\{\emptyset, N\}$.*

Example 3. *Continuing with the previous example, the previous discussion allows to derive the 2-dimensional faces of $\mathcal{MG}(N)$, as by Corollary 4 they can be given in terms of edges of $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$. The upsets of $\mathcal{P}(N)\backslash\{\emptyset, N\}$ were given before. Now, we have to search for pairs of adjacent vertices in $\mathcal{O}(\mathcal{P}(N)\backslash\{\emptyset, N\})$, for example using Theorem 4. It is easy but tedious to show that there are 76 pairs in these conditions.*

4.2 The cone of games with restricted cooperation

Let us now treat the problem when we face a situation of restricted cooperation. Then, several coalitions are not allowed and we have a set $\mathcal{FC}(N) \subset \mathcal{P}(N)\backslash\{\emptyset\}$ of feasible coalitions. Many papers have been devoted to this subject, usually imposing an algebraic structure on $\mathcal{FC}(N)$ (see e.g. (Faigle, 1989; Pulido and Sánchez-Soriano, 2006; Katsev and Yanovskaya, 2013; Grabisch, 2011)). From the point of view of polyhedra, if a coalition is not feasible, this implies that this subset is removed from $\mathcal{FC}(N)$. We will denote by $\mathcal{MG}_{\mathcal{FC}(N)}(N)$ the set of all monotone games whose feasible coalitions are $\mathcal{FC}(N)$. Thus, a game $v \in \mathcal{MG}_{\mathcal{FC}(N)}(N)$ is characterized by the following conditions:

- $0 \leq v(A), A \in \mathcal{FC}(N)$.
- $v(A) \leq v(B)$ if $A \subseteq B, A, B \in \mathcal{FC}(N)$.

Then, $\mathcal{MG}_{\mathcal{FC}(N)}(N) = \mathcal{C}(\mathcal{FC}(N))$, where the order relation $\prec$ on $\mathcal{FC}(N)$ is given by $A \prec B$ if and only if $A \subset B$.

Assume first that $N \in \mathcal{FC}(N)$. This is the usual situation, as most of the solution concepts on Game Theory assume that all players agree to form the grand coalition (see e.g. (Grabisch, 2013)). In this case, the following holds.

Corollary 6. *If $N \in \mathcal{FC}(N)$, then the set of extremal rays of $\mathcal{MG}_{\mathcal{FC}(N)}(N)$ is given by*

$$\{\boldsymbol{v}_F : \emptyset \neq F, F \text{upset of } \mathcal{FC}(N)\}.$$

Proof. Applying Theorem 7, the set of extremal rays is given by the set of vertices $\boldsymbol{v}_F$ of $\mathcal{O}(\mathcal{FC}(N))$ such that F is a connected upset in $\mathcal{FC}(N)$. As $N \in \mathcal{FC}(N)$, it follows that all upsets are connected subposets of $\mathcal{FC}(N)$, so that we have as many extremal rays as vertices in $\mathcal{O}(\mathcal{FC}(N))$ different from $\mathbf{0}$. And this value is given by the number of upsets minus one (for the empty upset corresponding to vertex $\mathbf{0}$). □

Indeed, we can translate in this case the results obtained for $\mathcal{MG}(N)$. Assuming the last coordinate corresponds to subset N, the following holds.

Proposition 4. *Assume $N \in \mathcal{FC}(N)$ and consider the poset $\mathcal{FC}(N)$ with the relation order $A \prec B \Leftrightarrow A \subset B$. Then, the order polytope $\mathcal{O}(\mathcal{FC}(N))$ is a pyramid with apex $\mathbf{0}$ and base $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(\mathcal{FC}(N) \backslash \{N\}\}$.*

Proof. It is a straightforward translation of the proof of Proposition 3. □

This implies that we have two possibilities for studying $\mathcal{MG}_{\mathcal{FC}(N)}(N)$. First, we can apply the general results for any order cones developed in Section 3. Alternatively, we can apply Proposition 2 and derive the results from the structure of the order polytope $\mathcal{O}(\mathcal{FC}(N)\backslash\{N\})$ just as it has been done for $\mathcal{MG}(N)$. In this last case, the following holds.

Corollary 7. *The k-dimensional faces of $\mathcal{MG}(\mathcal{FC}(N))$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(\mathcal{FC}(N)\backslash\{N\})$.*

Example 4. *Suppose a situation with four players, and assume that the only feasible coalitions are $\mathcal{FC}(N) = \{12, 23, 34, 1234\}$. The corresponding Hasse diagram is given in Figure 5.*

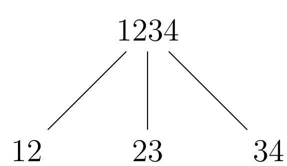


Figure 5: Hasse diagram of the poset of a game with restricted cooperation.

For this example, the non-empty upsets of $\mathcal{FC}(N)$ are:

$$F_1 = \{1234\},\ F_2 = \{12, 1234\},\ F_3 = \{23, 1234\},\ F_4 = \{34, 1234\},\ F_5 = \{12, 23, 1234\},$$

$$F_6 = \{12, 34, 1234\},\ F_7 = \{23, 34, 1234\},\ F_8 = \{12, 23, 34, 1234\}.$$

Thus, we have 8 extremal rays. For example, the extremal ray corresponding to F_5 is given by vector $\boldsymbol{v} = (1, 1, 0, 1)$, where the third coordinate corresponds to subset $\{34\}$.

For k-dimensional faces, it just suffice to note that $\mathcal{FC}(N)\backslash\{1234\}$ *is an antichain. Then,* $\mathcal{O}(\mathcal{FC}(N)\backslash\{N\})$ *is a cube. For example, for finding 2-dimensional faces, we have to consider pairs of adjacent vertices of the cube* $\mathcal{O}(\mathcal{FC}(N)\backslash\{N\})$ *(there are 12 pairs). Similarly, for 3-dimensional faces we have to consider 2-dimensional faces of the cube (six cases), and there is just one 4-dimensional face.*

Now, assume $N \notin \mathcal{FC}(N)$. This situation is more tricky and needs to study each case applying Theorems 7 and 8. For example, in this situation it could happen that some vertices are not adjacent to $\mathbf{0}$ and thus, they do not define an extremal ray. Moreover, the 2-dimensional faces are not defined necessarily via 2-dimensional simplices.

As examples for this case, we study two situations. First, assume $\mathcal{FC}(N)$ is a poset with a top element $\top$. Thus, we can extend all the results that we have obtained when $N \in \mathcal{FC}(N)$.

Proposition 5. *Consider the poset with top element* $\mathcal{FC}(N)$ *with the relation order* $A \prec B \Leftrightarrow A \subset B$ *and top element* $\top$*. Then, the order polytope* $\mathcal{O}(\mathcal{FC}(N)\backslash\{\emptyset\})$ *is a pyramid with apex* $\mathbf{0}$ *and base* $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(\mathcal{FC}(N)\backslash\{\top\}\}$.

Corollary 8. *The k-dimensional faces of* $\mathcal{MG}(\mathcal{FC}(N))$ *are given by the* $(k-1)$*-dimensional faces of* $\mathcal{O}(\mathcal{FC}(N)\backslash\{\top\})$.

Suppose as a second example that $\mathcal{FC}(N)$ is a union of connected posets

$$\mathcal{FC}(N) = P_1 \cup ... \cup P_r, \quad P_i \text{ connected.}$$

In this case, the only connected upsets are the connected upsets $F_i \subseteq P_i$. Then, we have:

Proposition 6. *If* $\mathcal{FC}(N) = P_1 \cup ... \cup P_r$*, where* P_i *is a connected poset,* $i = 1, ..., r$*, then the extremal rays of* $\mathcal{MG}(\mathcal{FC}(N))$ *are given by* $\boldsymbol{v}_{F_i}$ *where* F_i *is a non-empty connected upset of* P_i.

For example, if $|P_i| = 1\ \forall i$, then $\mathcal{FC}(N)$ is an antichain and the only connected upsets are the singletons. Thus, there are just r extremal rays for $\mathcal{MG}(\mathcal{FC}(N))$. Indeed, note that the corresponding order polytope is the r-dimensional cube and thus the vertices adjacent to $\mathbf{0}$ are $\boldsymbol{e}_i, i = 1, ..., r$.

In general, we have to study the properties of the corresponding poset.

Example 5. *Assume again a 4-players game and let us consider the coalitions given in Figure 6 left. We have in this case a 4-dimensional cone order.*

Fixing the order for coordinates 12, 13, 34, 123*, the vertices of the corresponding order polytope are given in Table 2.*

Upset	$\emptyset$	**123**	**34**	34, 123	**12, 123**
Vertex	(0,0,0,0)	**(0,0,0,1)**	**(0,0,1,0)**	(0,0,1,1)	**(1,0,0,1)**
Upset	**13, 123**	**12, 12, 123**	12, 34, 123	13, 34, 123	12, 13, 34, 123
Vertex	**(0,1,0,1)**	**(1,1,0,1)**	(1,0,1,1)	(0,1,1,1)	(1,1,1,1)

Table 2: Upsets and vertices of poset of Figure 6.

Vertices defining an extremal ray are those whose corresponding upset is connected. The five vertices in these conditions are written in boldface.

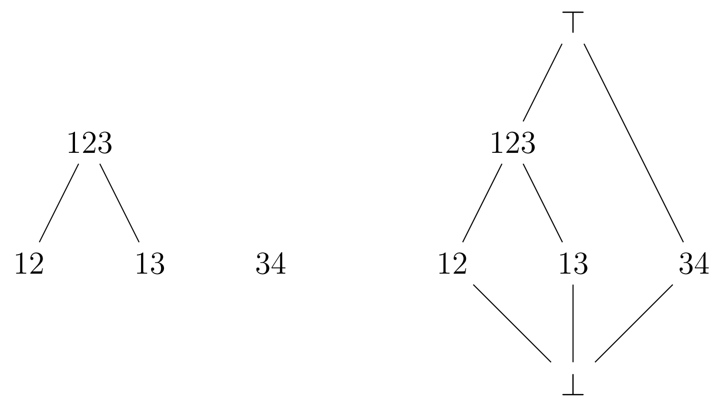


Figure 6: Hasse diagram of the poset P of a game with restricted cooperation (left) and his extension $\hat{P}$.

In order to obtain the facets of this order cone, we look for facets of the corresponding order polytope containing $\mathbf{0}$ *(Theorem 8). For this, we consider* $\perp \oplus P \oplus \top$ *(see Figure 6 right). As we are looking for facets, we just turn an inequality not involving* $\top$ *into an equality. Then, the facets are given in Table 3.*

Restriction	$f(\perp) = f(12)$	$f(\perp) = f(13)$	$f(\perp) = f(34)$	$f(12) = f(123)$	$f(13) = f(123)$
Vertices	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
	(0,0,0,1)	(0,0,0,1)	(0,0,0,1)	(1,0,0,1)	(0,1,0,0)
	(0,0,1,0)	(0,0,1,0)	(1,0,0,1)	(1,0,1,1)	(0,1,1,1)
	(0,0,1,1)	(0,0,1,1)	(0,1,0,1)	(1,1,0,1)	(1,1,0,1)
	(0,1,0,1)	(0,1,0,1)	(1,1,0,1)	(1,1,1,1)	(1,1,1,1)
	(0,1,1,1)	(1,0,1,1)			

Table 3: Facets of the order cone of poset of Figure 6.

Another way to look for extremal rays is Theorem 6. For this, we need to build the lattice of upsets, that is given in Figure 7.

Then, the extremal rays are given by upsets that together with $\emptyset$ *form an embedded sublattice. These upsets are*

$$\{123\}, \{34\}, \{12, 123\}, \{13, 123\}, \{12, 13, 123\}.$$

4.3 The cone of k-symmetric measures

As explained before, order cones can be applied to more general situations than games with restricted cooperation. In this subsection we will apply it to k-symmetric monotone games. We have chosen this case because the set of k-symmetric capacities with respect to a fixed partition is an order polytope (Combarro and Miranda, 2010).

The concept of k-symmetry appears in the theory of capacities as an attempt to reduce the complexity (Miranda et al., 2002). The subjacent idea is that several players could act exactly in the same way, so that we do not need to care about which players in these conditions are in a coalition and we just need to know how many players are inside it. The key concept of k-symmetric monotone

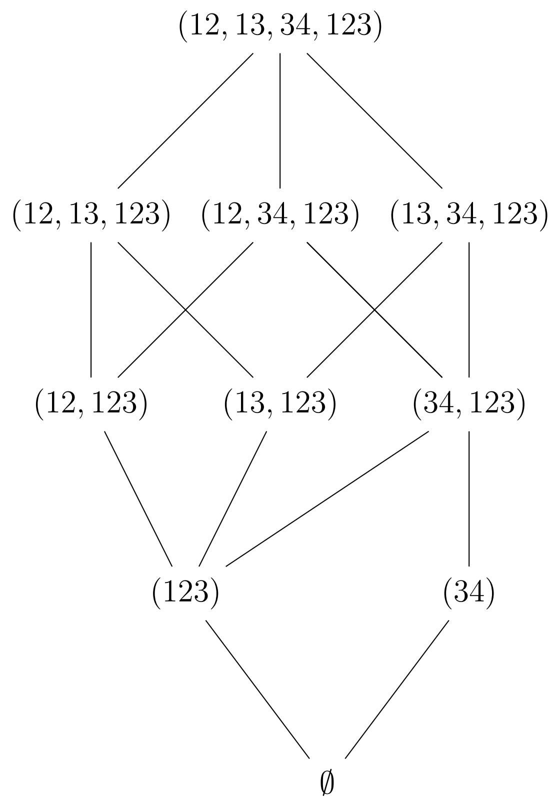


Figure 7: Lattice of upsets.

game is *subset of indifference*. Basically, a subset of indifference is a group of indistinguishable elements in terms of game v. Mathematically, this translates into

$$v(B_1 \cup C) = v(B_2 \cup C), \forall C \subseteq X \backslash A, \quad B_1, B_2 \subset A, |B_1| = |B_2|.$$

This allows us to write a coalition in terms of the number of players inside each subset of indifference.

Lemma 6. *(Miranda et al., 2002) If $\{A_1, ..., A_k\}$ is a partition of indifference for N, then any $C \subseteq N$ can be identified with a k-dimensional vector $(c_1, ..., c_k)$ with $c_i := |C \cap A_i|$.*

Then, each coalition writes $(c_1, ..., c_k)$ with $c_i = 0, ..., |A_i|$. For a given game v, it can be seen that it is always possible to partitionate N in several subsets of indifference. Several partitions are possible, but it can be proved (Miranda et al., 2002) that there is an only one being the coarsest.

Definition 5. *We say that a game is k-symmetric with respect to the partition $A_1, ..., A_k$ if this is the coarsest partition of N in subsets of indifference.*

We denote by $\mathcal{MG}^k(A_1, ..., A_k)$ the set of monotone games v such that $A_1, ..., A_k$ are subsets of indifference for v (but not necessarily k-symmetric; for example, any symmetric monotone game, in which all players are indifferent, belongs to $\mathcal{MG}^k(A_1, ..., A_k)$). Then, $v \in \mathcal{MG}^k(A_1, ..., A_k)$ is characterized as follows:

- $v(0, ..., 0) = 0$.
- $v(a_1, ..., a_k) \leq v(b_1, ..., b_k)$ if $a_i \leq b_i, i = 1, ..., k$.

Consider then the poset

$$P = \{(c_1, ..., c_k) : c_i = 0, ..., |A_i|, i = 1, ..., k\}$$

with the order relation $(c_1, ..., c_k) \preceq (b_1, ..., b_k)$ if and only if $c_i \leq b_i, i = 1, ..., k$.

Then, it follows that $\mathcal{MG}^k(A_1, ..., A_k) = \mathcal{C}(P \backslash \{(0, ..., 0)\})$ and the results of Section 3 can be applied to obtain the geometrical aspects of this cone. Moreover, as $(|A_1|, ..., |A_k|)$ is a top element in the poset, we can apply the results obtained for $\mathcal{MG}(N)$.

Corollary 9. *The vectors defining an extremal ray of $\mathcal{MG}^k(A_1, ..., A_k)$ are defined by non-empty upsets of $P \backslash \{(0, ..., 0)\}$.*

Proposition 7. *Consider the poset $P = \{(c_1, ..., c_k) : c_i = 0, ..., |A_i|, i = 1, ..., k\}$. Then, the order polytope $\mathcal{O}(P \backslash \{(0, ..., 0)\})$ is a pyramid with base $\{(\boldsymbol{x}, 1) : \boldsymbol{x} \in \mathcal{O}(P \backslash \{(0, ..., 0), (|A_1|, ..., |A_k|)\})\}$ and apex $\mathbf{0}$.*

Corollary 10. *The k-dimensional faces of $\mathcal{MG}^k(A_1, ..., A_k)$ are given by the $(k-1)$-dimensional faces of $\mathcal{O}(P \backslash \{(0, ..., 0), (|A_1|, ..., |A_k|)\}$.*

Let us study two particular cases.

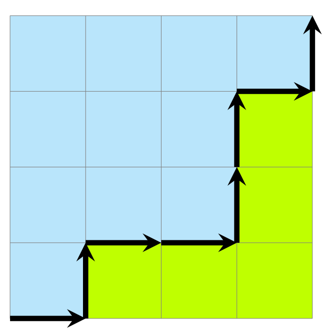
Example 6. *For $\mathcal{MG}^1(N)$, the set of monotone symmetric games, the corresponding order polytope is a chain of n elements. Thus, we have n non-empty upsets $F_1, ..., F_n$, given by $F_i := \{i, ..., n\}$ and $\boldsymbol{v}_{F_i} = (0, ..., 1, ..., 1)$. Therefore, we have n extremal rays.*

Besides, by Theorem 4, we conclude that all vertices are adjacent to each other. Hence, we have $\binom{n}{2}$ 2-dimensional faces and in general, the number of k-dimensional faces is $\binom{n}{k}$, for $k \geq 2$.

Example 7. *For the 2-symmetric case $\mathcal{MG}^2(A_1, A_2)$, it has been proved in (García-Segador and Miranda, 2020) that the order polytope $\mathcal{FM}^2(A_1, A_2)$ can be associated to a Young diagram (Bandlow, 2008) of shape $\boldsymbol{\lambda} = (|A_2|, ..., |A_2|)$.*

Moreover, there is a correspondence between upsets and staircase walks from $(0, 0)$ to (a_1, a_2) in a $(|A_1| + 1) \times (|A_2| + 1)$ grid (see Figure 8). Cell (i, j) represents the subset (i, j). In this sense, the walk separates subsets with value 0 from subsets with value 1 (see (García-Segador and Miranda, 2020)). For example, the empty upset corresponds to the staircase walk going from $(0, 0)$ to $(a_1, 0)$ and then to (a_1, a_2).

Figure 8: Staircase walk in a 4×4 grid and a staircase walk.



Then, the number of vertices of $\mathcal{FM}^2(A_1, A_2)$ is the number of possible staircase walks, that is given by

$$\binom{a_1 + a_2 + 2}{a_1 + 1},$$

and by Corollary 9 the number of vertices determining an extremal ray is $\binom{a_1+a_2+2}{a_1+1} - 1$.

5 Conclusions

In this paper we have introduced the concept of order cones. This concept is a natural extension of order polytopes, a well-known object in Combinatorics with which order cones share many properties. We have shown that all order cones are pointed, and we have derived some of their geometrical properties. Namely, we have characterized its k-dimensional faces. In particular, we have obtained a characterization of extremal ray in terms of the corresponding subjacent poset. The results in the paper show that the geometrical structure of order cones can be derived from the order structure of the subjacent poset, thus simplifying many results.

We feel that order cones could be a powerful tool to study different cones appearing in Game Theory in a general way. As examples of applicability, in the second part of the paper, we have applied these results to some special subfamilies of monotone games that satisfy the conditions of order cone. We have shown that the results derived in the first part can be applied to the set of monotone games with restricted cooperation, no matter the structure of the set of feasible coalitions. Then, we have studied in the first place the set of monotone games when all coalitions are allowed. For this case, we have shown that it is closely related to the order polytope of capacities, although there are differences. In a second step, we have studied this set when a set of feasible coalitions arises. We have shown that the set of monotone games with restricted cooperation always leads to an order cone whose structure relays on the poset of feasible coalitions. And we have seen that roughly speaking, there are two possible cases: the one with a top element (usually N) as a feasible coalition, that is very similar to the general case, and the case where there are several maxima, that leads to a more complicated problem.

Finally, we have studied an example where an order cone arises if constraints are added to the values of the game. This shows that order cones can be applied to situations different of monotone games with restricted cooperation. More concretely, we have studied the set of k-symmetric monotone games.

It should be noted that order cones just rely on a poset structure and thus, they are very general. In this paper, we have dealt with monotone games, as they are defined in terms of a natural order structure. However, this concept is more general and can be applied to any other order. The only condition is that there are constraints such as $v(A) \leq v(B)$ whenever $A\mathcal{R}B$, where $\mathcal{R}$ is a relation on $\mathcal{FC}(N)$. Indeed, general games can be seen as an order cone where the corresponding poset is an antichain, and the same can be said for general games with restricted cooperation. From this trivial situation, we can add other order conditions. For example, imagine that a player, say 1, always enhances coalitions, i.e. $v(B) \leq v(B \cup 1), \forall B \subseteq N \backslash \{1\}$. The geometry of the set of games satisfying this condition can be treated as an order cone.

We have introduced order cones as a tool for dealing with the geometry of a special type of convex cones. However, other applications may arise, as it can be seen in (Postnikov et al., 2008) for braid cones.

In this paper, we have removed $\emptyset$ from the set of feasible coalitions as $v(\emptyset)$ is fixed. Note that indeed this condition appears in an indirect way in the definition of order cone, as $\emptyset \subseteq A$ and $f(A) \geq 0$. Thus, this concept can be extended when the values for some coalitions are fixed and these coalitions appear in some monotonicity conditions. For example, if monotonicity is kept and we add, say $v(A) = 0.5$, the corresponding set of monotone games satisfying $v(A) = 0.5$ can be seen as an extended order cone. We intend to study this situation in the future.

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