Pointed order polytopes: Studying geometrical aspects of the polytope of bi-capacities

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Abstract

In this paper we study some geometrical questions about the polytope of bi-capacities. For this, we introduce the concept of pointed order polytope, a natural generalization of order polytopes. Basically, a pointed order polytope is a polytope that takes advantage of the order relation of a partially ordered set and such that there is a relevant element in the structure. We study which are the set of vertices of pointed order polytopes and sort out a simple way to determine whether two vertices are adjacent. We also study the general form of its faces. Next, we show that the set of bi-capacities is a special case of pointed order polytope. Then, we apply the results obtained for general pointed order polytopes for bi-capacities, allowing to characterize vertices and adjacency, and obtaining a bound for the diameter of this important polytope arising in Multicriteria Decision Making.

Keywords: Bi-capacities, poset, polytope, vertices, adjacency, diameter.

1 Introduction

Capacities [7], also called fuzzy measures [26] or non-additive measures [11], have proved themselves to be a very important tool in the field of Decision Making. The reason for this success is that capacities provide a wide flexibility that allows to model very different situations. For example, they can model Ellsberg and Allais paradoxes in Decision Under Uncertainty and Risk (see e.g. [14]). In Multicriteria Decision Making (MCDM), capacities can model interactions among criteria [12], as well as situations of veto and favor [13]. This has led to a huge number of works dealing with capacities, both from a theoretical and practical point of view [17, 23, 20, 4, 2], being a popular theory in Decision Making.

Basically, for each subset of criteria A in MCDM, a capacity μ assigns a value $\mu(A)$ representing the degree of satisfaction of an object being completely satisfactory for criteria in A and completely unsatisfactory outside A. This means that capacities assume that there is a polar scale (good/bad) modeling the problem. However, in many practical situations, decision makers do not behave the same for good valuations and bad valuations with respect to a criteria (see e.g. [24]). Thus, in these situations it is necessary to consider a bi-polar scale, in which there is a *neutral* value separating good and bad scores.

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To adapt capacities to a bipolar context, Grabisch and Labreuche have recently defined the concept of bi-capacity [16, 15]. In the definition of bi-capacities, it is taken into account that for an object some criteria might be completely satisfactory, some completely unsatisfactory and some have a neutral score. Hence, they are a suitable option for dealing with these situations and several papers based on this concept have appeared since then [22, 18, 27, 19, 1].

From a geometrical point of view, it can be seen that for a given set of criteria, the set of bi-capacities is a convex polytope. However, to our knowledge, there are no results about the vertices of this polytope, neither results to determine whether two vertices are adjacent. Similarly, the general form of its faces is unknown. This is the problem we tackle in this paper. In this sense, it should be noted that although the set of vertices, adjacency and similar problems have a very simple and intuitive formulation, the practical and mathematical aspects of these problems usually lead to very complex problems. Indeed, solving some of them for particular families of polytopes is relevant and it is hot topic in Combinatorics [3, 5].

Appart the mathematical interest, this problem arises in the practical use of bi-capacities. For example, let us suppose that we address the problem of identifying a bi-capacity from a sample data. Proceeding as in [8] for capacities, it is possible to develop a procedure based on genetic algorithms and such that the cross-over operation is the convex combination. In this case, the search region reduces at each iteration and then the initial population should be the set of vertices. Similarly, an appealing mutation operation is the convex combination with a randomly chosen vertex.

To deal with the geometrical structure of the polytope of bicapacities, in this paper we introduce the concept of pointed order polytope. Roughly speaking, a pointed order polytope is a polytope coming from a partially ordered set (brief poset) P such that there is an element in the poset that plays a special role. As we will see in the paper, this situation can be applied to bi-capacities and the special element is $(\emptyset, \emptyset)$. Hence, bi-capacities can be seen as a special case of pointed order polytope.

Pointed order polytopes are a natural generalization of order polytopes, a well-known object in Combinatorics [25]. However, we will see below that the existence of a relevant element in the poset makes the structure of the corresponding pointed order polytope much more difficult to handle. For example, we will see below that there are 49 vertices in the polytope of bi-capacities for a set of two criteria, a number far away the 4 vertices arising in the set of capacities for the same referential set. In the same line, the characterization of vertices is more tricky than the one for order polytopes.

The rest of the paper goes as follows. First, we review the basic concepts about bi-capacities and order polytopes in next section. Then, we introduce pointed order polytopes and study some of their properties. Specifically, we characterize the vertices of a pointed order polytope, as well as the adjacency and the general form of k-dimensional faces. Next, in Section 4 we apply these results to the set of bi-capacities for a fixed referential set. Besides, we obtain a bound for the diameter of this polytope. We finish with the conclusions and open problems.

2 Basic concepts

Consider $X = \{1, ..., n\}$ a set of criteria. Subsets of X are denoted by $A, B, ...$ and so on. The set of subsets of X is denoted $\mathcal{P}(X)$.

Definition 1. *[7, 26, 11] A* **capacity** *is a set function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying*

- *Monotonicity: $\mu(A) \leq \mu(B)$ if $A \subseteq B$.*
- *Boundary conditions: $\mu(X) = 1, \mu(\emptyset) = 0$.*

The set of capacities on a fixed referential set X is denoted by $\mathcal{FM}(X)$. If we identify μ with the vector $(\mu(A))_{(A)\in\mathcal{P}(X)}$, it follows that $\mathcal{FM}(X)$ is a convex polytope.

In order to extend capacities to the framework of bipolar scales, Grabisch and Labreuche have introduced the concept of bi-capacity. Let us define

$$\mathcal{Q}(X) := \{(A, B) : A \cap B = \emptyset, A, B \subseteq X\},$$

where A denotes the set of criteria that are completely satisfactory and B the set of criteria being completely unsatisfactory.

Definition 2. *[16] A* **bi-capacity** *is a function* $\nu : \mathcal{Q}(X) \to [-1, 1]$ *satisfying*

- *Monotonicity:* $\nu(A, B) \leq \nu(C, D)$ *if* $A \subseteq C, B \supseteq D$.
- *Boundary conditions:* $\nu(X, \emptyset) = 1, \nu(\emptyset, X) = -1, \nu(\emptyset, \emptyset) = 0$.

Let us denote by $\mathcal{BCAP}(X)$ the set of all bi-capacities on X. From Definition 2, if we identify ν with the vector $(\nu((A, B))_{(A,B)\in\mathcal{Q}(X)}$, it follows that $\mathcal{BCAP}(X)$ is a convex polytope. Although it is included in $\mathbb{R}^{|\mathcal{Q}(X)|}$, as the values for $(X, \emptyset), (\emptyset, X)$ and $(\emptyset, \emptyset)$ are fixed, this polytope can be projected into $\mathbb{R}^{|\mathcal{Q}(X)|-3}$ removing these coordinates.

Proposition 1. *[16] Let* $n = |X| > 1$. *The dimension of* $\mathcal{BCAP}(X)$ *is* $3^n - 3$.

Proof. It suffices to compute $|\mathcal{Q}(X)|$. For this, remark that $\mathcal{Q}(X)$ can be identified to the set of functions $f : X \to \{-1, 0, 1\}$. Hence, $|\mathcal{Q}(X)| = 3^n$. □

Let us now introduce the basic facts about order polytopes. This will help understand and compare many results about pointed order polytopes that we will state below. Let $(P, \preceq)$ (or P for short) be a finite partially ordered set (brief **poset**) of p elements, i.e. a set P endowed with a partial relation $\preceq$ that is reflexive, antisymmetric and transitive. Elements of P are denoted $x, y, ...$ Subsets of P are denoted by capital letters $A, B, ...$ or $A_1, A_2, ...$ Posets can be represented as graphs via *Hasse diagrams* (see e.g. Figure 1 left). We will write $x \lessdot y$ to mean that $x \prec y$ and there is no $z \in P\backslash\{x, y\}$ such that $x \prec z \prec y$. In the Hasse diagram, this translates into there is a line joining x and y. For a poset, an **upset** or filter F is a subset of P such that $x \in F, x \preceq y$ implies $y \in F$. Dually, a **downset** or ideal I of P is a subset such that $x \in I, y \preceq x$ implies $y \in I$. When no pair of elements can be compared, the poset is called an **antichain**. The **width** of P, denoted $w(P)$, is the size of the largest subset of P forming an antichain.

Given two posets, $(P, \preceq_P), (Q, \preceq_Q)$, their **disjoint union**, denoted $P \biguplus Q$, is a poset over the referential $P \cup Q$ (disjoint union) and whose partial order $\preceq_{P\biguplus Q}$ is defined as follows: $x \preceq_{P\biguplus Q} y$ whenever $x, y \in P$ and $x \preceq_P y$ or $x, y \in Q$ and $x \preceq_Q y$. Their **direct sum**, denoted $P \oplus Q$, is a poset over the referential $P \cup Q$ (disjoint union) and whose partial order $\preceq_{P\oplus Q}$ is defined as follows: if $x, y \in P$ then $x \preceq_{P\oplus Q} y$ if and only if $x \preceq_P y$; if $x, y \in Q$ then $x \preceq_{P\oplus Q} y$ if and only if $x \preceq_Q y$; and if $x \in P, y \in Q$ then $x \preceq_{P\oplus Q} y$.

Definition 3. *[25] Let* P *be a poset. We define the* **order polytope** *associated to* P, *denoted* $\mathcal{O}(P)$, *as the set of set of points* $f \in \mathbb{R}^p$ *ordered by the elements of* P *satisfying*

- $0 \leq f(x) \leq 1, \forall x \in P$.
- $f(x) \leq f(y)$ *if* $x \preceq y$.

There are many polytopes appearing in the Theory of Capacities that are order polytopes. For example, it has been shown in [9] that $\mathcal{FM}(X)$ is the order polytope $\mathcal{O}(\mathcal{P}(X)\backslash\{X,\emptyset\})$, where $A \preceq B$ if and only if $A \subseteq B$. Similarly, the set of normalized monotone games with restricted cooperation is an order polytope, no matter the set of feasible coalitions [21].

Order polytopes have the advantage that the combinatorial structure of the polytope can be studied in terms of the subjacent poset, and this is usually a simpler problem. For example, it is easy to characterize the vertices of an order polytope in terms of P.

Theorem 1. *[25] The vertices of $\mathcal{O}(P)$ are the characteristic functions of upsets of P.*

Similarly, it is possible to find an easy condition to determine if two vertices are adjacent.

Theorem 2. *Given two vertices of $\mathcal{O}(P)$ whose corresponding upsets are F_1 and F_2, they are adjacent if and only if $F_1 \subseteq F_2$ and $F_2\backslash F_1$ is a connected subposet of P.*

In order to determine faces of the order polytope, it is useful to consider the poset

$$\hat{P} := \bot \oplus P \oplus \top,$$

where we add a maximum $\top$ and a minimum $\bot$. Hence, $\mathcal{O}(P)$ can be defined in terms of $\hat{P}$ as

- $f(\top) = 1, f(\bot) = 0.$
- $f(x) \leq f(y)$ if $x \lessdot y, x, y \in \hat{P}$.

Now, for determining a face of a polytope we need to turn some inequalities of the definition into equalities. Hence, we obtain a partition of $\hat{P}$ into several blocks $A_\top, A_\bot, A_1, ..., A_r$, where $f(x) = f(y)$ whenever x, y are in the same block and $A_\top, A_\bot$, represent the blocks containing $\top$ and $\bot$, respectively. Therefore, a face can be given in terms of a partition of $\hat{P}$. Note however that it is not true that any partition determines a face and it is necessary to impose additional conditions.

A partition $\mathfrak{P} = \{A_\top, A_\bot, A_1, ..., A_r\}$ of P is **connected** if all A_i are connected suposets of $\hat{P}$.

Let us define the relation $\preceq_\mathfrak{P}$ on $\{A_\top, A_\bot, A_1, ..., A_r\}$ by

$$A_i \preceq_\mathfrak{P} A_j \Leftrightarrow \exists x \in A_i, y \in A_j, x \preceq y.$$

A partition $\mathfrak{P}$ is **compatible** if $\preceq_\mathfrak{P}$ antisymmetric. Finally, a partition $\mathfrak{P}$ is **closed** if for any $A_i, A_j, i \neq j$, there exists $f \in \mathcal{O}(P)$ constant on each block such that $f(A_i) \neq f(A_j)$. Note that compatibility and connectivity are defined in terms of the poset, while the notion of closedness depends on the (order) polytope. Now, the following holds.

Theorem 3. *[25] A partition $\{A_\top, A_\bot, A_1, \ldots, A_r\}$ of $\widehat{P}$ is closed and determines a r-dimensional face of $\mathcal{O}(P)$ if and only if it is compatible and connected.*

For a polytope $\mathcal{P}$, we define its **skeleton** as the graph whose vertices are the vertices of $\mathcal{P}$ and two vertices are joined by an edge if they are adjacent vertices in $\mathcal{P}$. Given two vertices of $\mathcal{P}$, we define the *distance* between them as the number of edges of the minimal path connecting them in the skeleton of $\mathcal{P}$. The **diameter** of $\mathcal{P}$ is the maximal distance between two vertices. The diameter of a polytope gives information about its geometric complexity.

Theorem 4. *[9] Let P be a finite poset and $\mathcal{O}(P)$ its associated order polytope. Then:*

i) *The diameter of $\mathcal{O}(P)$ is at most $w(P)$.*

ii) *If P has r connected components $P_1, \dots, P_r$, i.e. $P = P_1 \biguplus \dots \biguplus P_r$, and the diameter of $\mathcal{O}(P_i)$ is d_i, then the diameter of $\mathcal{O}(P)$ is $\sum_{i=1}^{r} d_i$.*

iii) *If P has a maximum or a minimum element, then the diameter of $\mathcal{O}(P)$ is at most 2. In addition, if there are two incomparable elements in $\mathcal{O}(P)$, then the diameter is exactly 2.*

3 Pointed order polytopes

3.1 Definition and vertices

We are now in position to define pointed order polytopes.

Definition 4. *Let P be a poset and take $a \in P$. We define the* **pointed order polytope** *associated to P and a as the set of points $f \in \mathbb{R}^p$ ordered by the elements of P satisfying*

- $-1 \leq f(x) \leq 1, \forall x \in P.$
- $f(x) \leq f(y)$, *if* $x \preceq y$.
- $f(a) = 0.$

We will denote the pointed order polytope on P and a as $\mathcal{O}(P, a)$. Note that it is a polytope of dimension $|P| - 1$, because the value of $f(a)$ is fixed.

Example 1. *Consider the poset* $\mathbf{1} \equiv (P, \preceq)$ *where* $P := \{x, y, z, a\}$ *and whose Hasse diagram is given in Figure 1 left. Then, the pointed order polytope $\mathcal{O}(P, a)$ is defined by the equations*

$$0 \leq f(y) \leq 1, \quad -1 \leq f(x) \leq f(y), \quad f(a) = 0, \quad -1 \leq f(z) \leq 0.$$

As $f(a)$ is fixed, we can draw this polytope in $\mathbb{R}^3$. A graph is given in Figure 1 right.

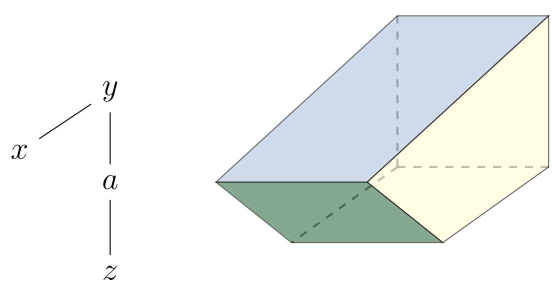

Figure 1: The poset $\mathbf{1}$ and its corresponding pointed order polytope $\mathcal{O}(P, a)$.

Remark 1. *As in the case for order polytopes, the concept of pointed order polytope can be established equivalently if we consider the poset*

$$\hat{P} := \bot \oplus P \oplus \top,$$

where we add a maximum $\top$ and a minimum $\bot$. Hence, $\mathcal{O}(P, a)$ can be written in terms of $\hat{P}$ as

- $f(\top) = 1, f(\bot) = -1, f(a) = 0.$
- $f(x) \leq f(y)$ *if* $x \lessdot y, x, y \in \hat{P}$.

Remark 2. *Bounds -1 and 1 are arbitrary and need not be symmetric. We have chosen these values because they are the typical values when bipolar scales are considered and these are the values appearing in bi-capacities. Similarly, the values 1,0 for order polytopes are arbitrary. Hence, we can define an order polytope equivalently fixing the lower and upper bounds on v_1 and v_2, respectively. We will denote this polytope by $\mathcal{O}_{[v_1,v_2]}(P)$.*

Remark 3. *Let (P, a) be a pointed order polytope such that there are posets P_1 and P_2 satisfying*

$$P = P_1 \oplus a \oplus P_2.$$

Then, $\mathcal{O}(P, a)$ is isometric to $\mathcal{O}(P_1) \times \mathcal{O}(P_2)$. This isometry $\phi : \mathcal{O}(P_1) \times \mathcal{O}(P_2) \longrightarrow \mathcal{O}(P, a)$ is given by $\phi(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x} - 1, 0, \boldsymbol{y})$. When P_1 (resp. P_2) is the empty set, a is the minimum (resp. maximum) of P and $\mathcal{O}(P, a)$ is just the order polytope $\mathcal{O}(P_2)$ (resp. $\mathcal{O}_{[-1,0]}(P_1)$). Thus considered, order polytopes are a special case of pointed order polytopes.

As pointed order polytopes are polytopes, they can be defined in terms of their vertices. Let us then deal with the problem of characterizing the vertices of $\mathcal{O}(P, a)$ in terms of P.

Proposition 2. *Let P be a poset and $a \in P$. Consider $f \in \mathcal{O}(P, a)$. If f is a vertex of $\mathcal{O}(P, a)$, then*

$$f(x) \in \{-1, 0, 1\}, \forall x \in P.$$

Proof. Let f be a vertex of $\mathcal{O}(P, a)$ and assume there exists $x \in P$ such that $f(x) \notin \{-1, 0, 1\}$. Suppose that $f(x) > 0$ (the case $f(x) < 0$ is completely symmetric) and let us define

$$A^+ := \{y \in P : f(y) \in (0, 1)\}.$$

For A^+ we define

$$\epsilon := \min\{f(y), 1 - f(y) : y \in A^+\} > 0.$$

We build g_1, g_2 as follows:

$$g_1(y) := \begin{cases} f(y) + \epsilon & \text{if } y \in A^+ \\ f(y) & \text{otherwise} \end{cases}, \qquad g_2(y) := \begin{cases} f(y) - \epsilon & \text{if } y \in A^+ \\ f(y) & \text{otherwise} \end{cases}.$$

Consequently, $f = \frac{1}{2}g_1 + \frac{1}{2}g_2$. It just suffices to show that $g_1, g_2 \in \mathcal{O}(P, a)$. We prove it for g_1, as the case for g_2 is symmetric. First, note that $g_1(y) \in [-1, 1], \forall y \in P$ by definition of ϵ. Now, take $y, z \in P, y \preceq z$. We have the following cases:

- If $f(z) = 1$, then $g_1(z) = 1$, and hence $g_1(y) \leq g_1(z)$.
- If $f(z) \leq 0$, then $f(y) \leq 0$ by monotonicity and $g_1(y) = f(y) \leq f(z) = g_1(z)$.
- If $f(z) \in (0, 1)$, then $g_1(y) \leq f(y) + \epsilon \leq f(z) + \epsilon = g_1(z)$.

Hence, the result holds. □

Note however that, contrary to the case of order polytopes, it could be the case that $f \in \mathcal{O}(P, a)$, and $f(x) \in \{-1, 0, 1\}$ but f is not a vertex of $\mathcal{O}(P, a)$.

Example 2. *(Continued Example 1) In this case, note that point* $(f(x) = 0, f(y) = 1, f(z) = 0, f(a) = 0)$ *is not a vertex of the polytope, as*

$$(0, 1, 0, 0) = \frac{1}{2}(1, 1, 0, 0) + \frac{1}{2}(-1, 1, 0, 0),$$

and both $(1, 1, 0, 0)$ *and* $(-1, 1, 0, 0)$ *are in* $\mathcal{O}(P, a)$.

Let us then study which are the conditions for $f \in \mathcal{O}(P, a)$ to be a vertex. From Proposition 2, we have to look for conditions on partitions of P consisting on three subsets attaining values 1, 0, -1 such that they lead to a vertex. Consider a partition $\{A_1, A_0, A_{-1}\}$ of P and define $f_{A_1,A_0,A_{-1}}$ by

$$f_{A_1,A_0,A_{-1}}(x) := \begin{cases} -1 & \text{if } x \in A_{-1} \\ 0 & \text{if } x \in A_0 \\ 1 & \text{if } x \in A_1 \end{cases}$$

Remark that A_1 (resp. A_{-1}) could be empty. However, $a \in A_0$.

Definition 5. *Let P be a poset and $a \in P$. We say that a partition $\{A_{-1}, A_0, A_1\}$ is a* **vertex partition** *of P if it satisfies the following conditions*

1. *$A_1 \subseteq P \backslash \{x : x \preceq a\}$ and A_1 is an upset of P.*
2. *$A_{-1} \subseteq P \backslash \{x : a \preceq x\}$ and A_{-1} is a downset of P.*
3. *A_0 is a connected subset of P.*

Note that in a vertex partition, $a \in A_0$.

Proposition 3. *Let $\mathcal{O}(P, a)$ be a pointed order polytope and consider $f_{A_1,A_0,A_{-1}}$ where $\{A_1, A_0, A_{-1}\}$ is a vertex partition. Then, $f_{A_1,A_0,A_{-1}}$ is a vertex of $\mathcal{O}(P, a)$.*

Proof. Suppose $f_{A_1,A_0,A_{-1}}$ is not a vertex of $\mathcal{O}(P, a)$. Then, there exist $g_1, g_2 \in \mathcal{O}(P, a), g_1 \neq g_2$, and $\alpha \in (0, 1)$ such that

$$f_{A_1,A_0,A_{-1}} = \alpha g_1 + (1 - \alpha) g_2.$$

We will show that $f_{A_1,A_0,A_{-1}}(x) = g_1(x) = g_2(x)$. For $x \in A_1$, we have $f_{A_1,A_0,A_{-1}}(x) = 1$ and hence $g_1(x) = g_2(x) = 1$. Similarly, for $x \in A_{-1}$, we have $g_1(x) = g_2(x) = -1$.

Let us then consider $x \in A_0, x \neq a$. Assume $x \preceq a$ or $a \preceq x$. We will denote the set of elements of A_0 in these conditions as A_0^1. W.l.g. let us study the case $a \preceq x$. Therefore,

$$\left. \begin{array}{l} g_1(x) \geq g_1(a) = 0 \\ g_2(x) \geq g_2(a) = 0 \end{array} \right\} \Rightarrow g_1(x) = 0 = g_2(x),$$

as $f_{A_1,A_0,A_{-1}}(x) = 0$. Now, suppose $x \notin A_0^1$ but there exists $y \in A_0^1$ such that $x \preceq y$ or $y \preceq x$. W.l.g. let us study the case $y \preceq x$. As before,

$$\left. \begin{array}{l} g_1(x) \geq g_1(y) = 0 \\ g_2(x) \geq g_2(y) = 0 \end{array} \right\} \Rightarrow g_1(x) = 0 = g_2(x),$$

as $f_{A_1,A_0,A_{-1}}(x) = 0$. As A_0 is connected, for $x \in A_0$ there is a chain

$$x - y_r - y_{r-1} - ... - y_1 - a.$$

Then, we can repeat the previous process for $y_1, ..., y_r, x$ so that

$$g_1(y_1) = 0 = g_2(y_1), ..., g_1(y_{r-1}) = 0 = g_2(y_{r-1}), g_1(y_r) = 0 = g_2(y_r), g_1(x) = g_2(x) = 0,$$

and we conclude that $f_{A_1,A_0,A_{-1}}$ is a vertex of $\mathcal{O}(P, a)$. □

Proposition 4. *Let $\mathcal{O}(P, a)$ be a pointed order polytope and consider f a vertex. Then, f can be written as $f_{A_1,A_0,A_{-1}}$ with $\{A_1, A_0, A_{-1}\}$ a vertex partition.*

Proof. If f is a vertex, we know from Prop. 2 that f can be written as $f_{A_1,A_0,A_{-1}}$, where $A_i := \{x : f(x) = i\}, i = -1, 0, 1$. Hence, it just suffices to show that these A_i are in the conditions of Def. 5.

- A_1 is an upset by monotonicity. Besides, if $x \preceq a$, monotonicity implies $f(x) \leq f(a) = 0$. Thus,

$$A_1 \subseteq P \backslash \{x : x \preceq a\}.$$

- Similarly, A_{-1} is a downset and $A_{-1} \subseteq P \backslash \{x : a \preceq x\}$.

- Let us show that A_0 is connected. Suppose that A_0 has at least two connected components. As $a \in A_0$, let us denote by C_1 a connected component of A_0 such that $a \notin C_1$. Hence, if $x \in C_1$, this element cannot be compared to any other element of $A_0 \backslash C_1$. For a fixed $\epsilon \in (0, 1)$, define

$$g_1(y) := \begin{cases} f(y) & \text{if } y \notin C_1 \\ \epsilon & \text{if } y \in C_1 \end{cases}, \quad g_2(y) := \begin{cases} f(y) & \text{if } y \notin C_1 \\ -\epsilon & \text{if } y \in C_1 \end{cases}$$

Then, $f = \frac{1}{2}g_1 + \frac{1}{2}g_2$. Let us finally show that $g_1, g_2 \in \mathcal{O}(P, a)$, i.e. monotonicity. W.l.g. we prove it for g_1. Consider $x, y \in P$ such that $x \preceq y$. We have the following cases:

  - If $x, y \notin C_1$, then $g_1(x) = f(x) \leq f(y) = g_1(y)$.
  - Assume $x, y \in C_1$. Then, $g_1(x) = \epsilon = g_1(y)$.
  - If $x \notin C_1, y \in C_1$, then $x \in A_{-1}$ because $x \preceq y$ and $x \notin A_0$ (otherwise $x \in C_1$ because $y \in C_1$ and $x \preceq y$). Hence, $g_1(x) = f(x) = -1 \leq \epsilon = g_1(y)$.
  - If $x \in C_1, y \notin C_1$, this implies that $y \in A_1$. Hence, $g_1(x) = \epsilon \leq 1 = f(y) = g_1(y)$.

Hence, the result holds. □

Example 3. *(Continued Example 1) In this case, we can apply the previous proposition to obtain all the vertices of the polytope. It can be seen that we have eight vertices that are given in next table.*

Vertex	A_1	A_0	A_{-1}
$(1, 1, -1)$	x, y	a	z
$(1, 1, 0)$	x, y	a, z	$\emptyset$
$(-1, 1, 0)$	y	a, z	x
$(0, 0, 0)$	$\emptyset$	x, y, z, a	$\emptyset$
$(-1, 1, -1)$	y	a	x, z
$(0, 0, -1)$	$\emptyset$	x, y, a	z
$(-1, 0, -1)$	$\emptyset$	y, a	x, z
$(-1, 0, 0)$	$\emptyset$	z, y, a	x

Corollary 1. *Let (P,a) be a pointed order polytope such that there are posets P_1 and P_2 such that $P = P_1 \oplus a \oplus P_2$. Then, the vertices of $\mathcal{O}(P,a)$ can be identified to pairs (I,F) where I is a downset of P_1 and F is an upset of P_2.*

Remark 4. *Assume P is not connected, i.e. $P = P_1 \biguplus P_2 \biguplus ... \biguplus P_r$, where P_i are connected posets. According to the previous results, if say $a \in P_1$, for any vertex $f_{A_1,A_0,A_{-1}}$ of $\mathcal{O}(P,a)$, elements outside P_1 cannot be in A_0. Hence, they attain values $-1, 1$ and we conclude that $P_i = (A_{-1} \cap P_i) \cup (A_1 \cap P_i)$, $i = 2, ..., r$. In other words, $\mathcal{O}(P,a)$ behaves like an order polytope in $P_i, i \neq 1$ and*

$$\mathcal{O}(P,a) = \mathcal{O}(P_1,a) \times \mathcal{O}_{[-1,1]}(P_2) \times \cdots \mathcal{O}_{[-1,1]}(P_r).$$

Remark 5. *The characterization of vertices arising from Definition 5 might be seen as surprising if we compare it with the corresponding condition for order polytopes established in Theorem 1, where there is no A_0 and no connectivity is imposed on the sets. For understanding the underlying reasons of this condition, we have to turn to $\hat{P}$. Focusing on this poset, we see that a vertex of $\mathcal{O}(P)$ is just a partition $\{A_0, A_1\}$ of $\hat{P}$. In this case, A_1 (resp. A_0) is connected as subposet of $\hat{P}$ because $x \preceq \top, \forall x \in P$ (resp. $\bot \preceq x, \forall x \in P$). More insight about this fact is shown when studying the facial structure of $\mathcal{O}(P,a)$ in next subsection.*

3.2 Faces of the pointed order polytope

Let us now study the faces of $\mathcal{O}(P,a)$.

Proposition 5. *Let $\mathcal{O}(P,a)$ be a pointed order polytope. Then, the faces of $\mathcal{O}(P,a)$ are also pointed order polytopes $\mathcal{O}(P',a)$.*

Proof. Let us first prove the result for the facets, faces of dimension $|P|-2$. To obtain the facets, we have to turn an inequality $f(x) \leq f(y)$, $x \lessdot y$ into an equality, where $x, y \in \hat{P}$. Given a facet $\mathcal{F}$, let us define the poset $(\hat{P}', \preceq')$ where

$$\hat{P}' := (\hat{P} \backslash \{x,y\}) \cup \{z\},$$

and $\preceq'$ is given as:

$$\begin{cases} v \preceq' w & \Leftrightarrow & \begin{cases} v \preceq w, \text{ or} \\ v \preceq y, x \preceq w \end{cases} \forall v, w \in P \backslash \{x,y\} \\ z \preceq' w & \Leftrightarrow & x \preceq w \\ v \preceq' z & \Leftrightarrow & v \preceq y \end{cases}$$

Let us check that $\preceq'$ is an order relation:

- Reflexivity holds trivially.
- Suppose $v \preceq' w$ and $w \preceq' v, v \neq z, w \neq z$.

  If $v \preceq w$ and $w \preceq v$, we conclude $v = w$ by antisymmetry in $\preceq$.

  Another possibility is $v \preceq w$ and $w \preceq y, x \preceq v$. But in this case, we obtain $x \preceq v \preceq w \preceq y$, a contradiction with $x \lessdot y$. Similarly, if $v \preceq y, x \preceq w$, and $w \preceq v$, we conclude $x \preceq w \preceq v \preceq y$.

  Finally, if $v \preceq y, x \preceq w$ and $w \preceq y, x \preceq v$, we obtain $x \preceq w \preceq y$, again a contradiction.

  Let $v \neq z$ and assume $z \preceq' v$ and $v \preceq' z$. This implies $x \preceq v$ and $v \preceq y$, so that $x \preceq v \preceq y$, a contradiction.

- Suppose $u \preceq' v \preceq' w$ and $u \neq z, v \neq z, w \neq z$. Then, the possibilities are:

$$\begin{cases} u \preceq v \preceq w & \Rightarrow & u \preceq w & \Rightarrow & u \preceq' w \\ u \preceq y, x \preceq v, v \preceq w & \Rightarrow & u \preceq y, x \preceq w & \Rightarrow & u \preceq' w \\ u \preceq v, v \preceq y, x \preceq w & \Rightarrow & u \preceq y, x \preceq w & \Rightarrow & u \preceq' w \end{cases}$$

  The possibility $u \preceq y, x \preceq v, v \preceq y, x \preceq w$ leads to $x \preceq v \preceq y$, a contradiction.

  Let us now study the case when z appears. The possibilities are:

$$\begin{cases} z \preceq' v \preceq' w & \Rightarrow & x \preceq v \preceq w & \Rightarrow & x \preceq w & \Rightarrow & z \preceq' w \\ v \preceq' w \preceq' z & \Rightarrow & v \preceq w \preceq y & \Rightarrow & v \preceq y & \Rightarrow & v \preceq' z \\ v \preceq' z \preceq' w & \Rightarrow & \begin{cases} v \preceq y \\ x \preceq w \end{cases} & & & \Rightarrow & v \preceq' w \end{cases}$$

  The possibilities $z \preceq' v \preceq' w$ and $v \preceq' w \preceq' z$ where $v \preceq' w$ means $v \preceq y, x \preceq w$ are not possible because they lead to $x \preceq v \preceq y$.

Next, remark that $\hat{P}'$ has a top element and a bottom element. This is obvious if $x, y \notin \{\top, \bot\}$. If $y = \top$, this implies that $v \preceq' z, \forall v \in \hat{P}'$. Hence, z is the maximum in $\hat{P}'$ and we can rename z as $\top$. Similarly, if $x = \bot$ it follows that z is the minimum in $\hat{P}'$ and we can rename z as $\bot$.

Note that we can identify points of $\mathcal{O}(P', a)$ with the points of the facet $\mathcal{F}$ via $f(z) := f(x) = f(y)$, so that the map

$$(f(\bot), ..., f(x), f(y), ..., f(\top)) \leftrightarrow (f(\bot), ..., f(z), ..., f(\top))$$

is well-defined. Remark that if either $x = a$ or $y = a$, we can identify z to a and fix $f(z) = 0$.

Hence, facets are pointed order polytopes. But now, applying that a k-dimensional face is a facet of a $(k+1)$-dimensional face, we can reiterate the process to conclude that any face of $\mathcal{O}(P, a)$ is a pointed order polytope $\mathcal{O}(P', a)$ where a remains the same. $\square$

Let us now study the faces. We start with the problem of obtaining the facets of this polytope. For this, we transform an inequality of the system defining the pointed order polytope in an equality. We have then four possibilities:

- $x \in P \backslash \{a\}$ is maximal and we fix the value $f(x) = 1$.
- $x \in P \backslash \{a\}$ is minimal and we fix the value $f(x) = -1$.
- $x \in P \backslash \{a\}$ satisfying $x \lessdot a$ or $x \gtrdot a$ and we fix the value $f(x) = 0$.
- $x, y \in P \backslash \{a\}$ satisfying $x \lessdot y$ and we fix the condition $f(x) = f(y)$.

Thus, the following holds.

Proposition 6. *Let P be a poset and let us denote by M the number of maximal elements in P, m the number of minimal elements and r is the number of relations $x \lessdot y$ in P. The number of facets of $\mathcal{O}(P, a)$ is given by:*

- *$M + m + r$, when a is neither a maximal or a minimal element.*

- $M + m + r - 1$, *when a is a maximal or a minimal element, but not both.*
- $M + m + r - 2$, *when a is both a maximal and a minimal element (i.e. a is an isolated point in the Hasse diagram of P).*

We now turn to the problem of characterizing faces of $\mathcal{O}(P, a)$ different from vertices and facets. Following the same process as for order polytopes, we turn several inequalities into equalities. Hence, we obtain a partition of $\hat{P}$ into several blocks $A_\top, A_\perp, A_a, A_1, ..., A_r$, where $f(x) = f(y)$ whenever x, y are in the same block and $A_\top, A_\perp, A_a$ represent the blocks containing $\top, \perp$ and a, respectively. Therefore, a face can be given in terms of a partition of $\hat{P}$ and the problem relies in obtaining the conditions for a partition to determine a face.

Lemma 1. *Given a poset P and $a \in P$, we have*

$$\mathcal{O}(P, a) = \mathcal{O}_{[-1,1]}(P) \cap \{x_a = 0\}.$$

Proof. $\subseteq$) Let us consider $f \in \mathcal{O}(P, a)$. Hence, $f(a) = 0$ because $a \in A_0$. If $x \preceq y$, we have $f(x) \leq f(y)$ by monotonicity. Finally, $-1 \leq f(x) \leq 1$, so that $f \in \mathcal{O}_{[-1,1]}(P)$.

$\supseteq$) Consider $f \in \mathcal{O}_{[-1,1]}(P) \cap \{x_a = 0\}$. Hence, f satisfies monotonicity, $f(a) = 0$ and $f(x) \in [-1, 1]$. Therefore, $f \in \mathcal{O}(P, a)$. □

Now, for $\mathcal{O}_{[-1,1]}(P)$, Theorem 3 turns into:

Theorem 5. *A partition $\{A_\top, A_\perp, A_1, \ldots, A_r\}$ of $\widehat{P}$ is closed and determines a r-dimensional face of $\mathcal{O}_{[-1,1]}(P)$ if and only if it is compatible and connected.*

We will say that a partition is **closed** (for the pointed order polytope) if for every two blocks A_i, A_j, there exists $g \in \mathcal{O}(P, a)$ constant on each block such that $g(A_i) \neq g(A_j)$. The following holds.

Theorem 6. *A partition $\{A_\perp, A_\top, A_a, A_1, ..., A_r\}$ of $\hat{P}$ is closed and determines a r-dimensional face of $\mathcal{O}(P, a)$ if and only if it is compatible and connected.*

Proof. While it is possible to derive a proof similar to that of Theorem 3, we show here a proof in which we apply Theorem 5 to simplify the proof.

$\Leftarrow$) Let us assume that $\mathfrak{P} := \{A_\perp, A_\top, A_a, A_1, ..., A_r\}$ is a connected and compatible partition of $\hat{P}$. If we allow the value on A_a to oscillate between -1 and 1 instead of keeping it fixed to 0, we can apply Theorem 5 to conclude that $\mathfrak{P}$ is a closed partition for $\mathcal{O}_{[-1,1]}(P)$ determining a $(r + 1)$-dimensional face. Let us denote this face by $\mathcal{G}$ and consider

$$\mathcal{F} := \mathcal{G} \cap \{x_a = 0\}.$$

Note that $\mathcal{F}$ is not empty. To see this, consider

$$g_1(x) = \begin{cases} 1 & \text{if } x \in A_i, A_i \succeq_{\mathfrak{P}} A_a \\ 0 & \text{otherwise} \end{cases} \qquad g_2(x) = \begin{cases} -1 & \text{if } x \in A_i, A_i \preceq_{\mathfrak{P}} A_a \\ 0 & \text{otherwise} \end{cases}$$

Then, $g_1, g_2 \in \mathcal{G}$ so that $\frac{1}{2}g_1 + \frac{1}{2}g_2 \in \mathcal{G}$. Besides, $g_1(x) = 1, g_2(x) = -1$, for $x \in A_a$, so that $[\frac{1}{2}g_1 + \frac{1}{2}g_2](a) = 0$, and hence, $\mathcal{G} \cap \{x_a = 0\} \neq \emptyset$.

As $\mathcal{G}$ is a face of $\mathcal{O}_{[-1,1]}(P)$, there exists an hyperplane $\mathcal{H} := \{\vec{a}^t\vec{x} = b\}$ such that $\mathcal{H} \cap \mathcal{O}_{[-1,1]}(P) = \mathcal{G}$ and $\mathcal{O}_{[-1,1]}(P) \subseteq \mathcal{H}_\leq := \{\vec{a}^t\vec{x} \leq b\}$.

Consider then $\mathcal{H}' := \mathcal{H} \cap \{x_a = 0\}$ and let us define $\mathcal{H}'_\leq := \mathcal{H}_\leq \cap \{x_a = 0\}$. Then,

$$\mathcal{O}(P,a)=\mathcal{O}_{[-1,1]}(P)\cap\{x_a=0\}\subseteq\mathcal{H}_{\leq}\cap\{x_a=0\}=\mathcal{H}'_{\leq}.$$

Similarly, as $\mathcal{F}=\mathcal{G}\cap\{x_a=0\}$, we obtain

$$\mathcal{F}=\mathcal{G}\cap\{x_a=0\}=\mathcal{H}\cap\mathcal{O}_{[-1,1]}(P)\cap\{x_a=0\}=\mathcal{H}'\cap\mathcal{O}(P,a).$$

As $dim(\mathcal{H}')=dim(\mathcal{H})-1$, we conclude that $\mathcal{H}'$ is an hyperplane in $\mathbb{R}^{|P|-1}$ and hence $\mathcal{F}$ is a face of $\mathcal{O}(P,a)$. Moreover, since the elements of $\mathcal{F}$ are elements of $\mathcal{G}$, we conclude that elements of $\mathcal{F}$ are constant on each block. Besides, $f(x)=1$ if $x\in A_\top$, $f(x)=-1$ if $x\in A_\bot$ and $f(x)=0$ if $x\in A_a$. Hence, as these values are fixed and are the only values that are fixed, we conclude that $\mathcal{F}$ is a r-dimensional face.

Now let us see that $\mathfrak{P}$ is a closed partition in $\mathcal{O}(P,a)$. Consider two different subsets $A_i,A_j\in\mathfrak{P}$ both of them different from A_a. Let us represent the values of a point $f\in\mathcal{G}$ on (A_i,A_j,A_a) by a triple $(f(A_i),f(A_j),f(A_a))$. Since $\mathfrak{P}$ is the closed partition associated to the face $\mathcal{G}$ of $\mathcal{O}_{[-1,1]}(P)$, there is an extreme point $g_1\in\mathcal{G}$ taking different values at A_i and A_j. Let us suppose w.l.g. that the corresponding triple is $(1,-1,x_a)$. If $x_a=1$ (the case $x_a=-1$ is symmetric), as $\mathcal{G}\cap\{x_a=0\}\neq\emptyset$, there exists another vertex g_2 whose corresponding triple is $(y_1,y_2,-1)$. This way $f=\frac{1}{2}(g_1+g_2)$ takes values $(\frac{1+y_1}{2},\frac{-1+y_2}{2},0)$, and $f\in\mathcal{F}$. If $\frac{1+y_1}{2}\neq\frac{-1+y_2}{2}$, we are done. Otherwise, g_1 and g_2 take values $(1,-1,1)$ and $(-1,1,-1)$, respectively. Therefore, A_i and A_j form an antichain, and the same happens for A_j and A_a. Hence, there exists g_3 with values $(-1,-1,-1)$. Now, we take $f=\frac{1}{2}(g_1+g_3)$ whose corresponding triple is $(0,-1,0)$. Thus, $f\in\mathcal{F}$ and it separates A_i and A_j.

To differentiate between a block A_i and block A_a, we repeat the same procedure taking w.l.g. a vertex $g_1\in\mathcal{G}$ with values in (A_i,A_a) given by $(-1,1)$. Now, since $A_a\neq A_\top$ we have a vector $g_2\in\mathcal{G}$ with values $(y_1,-1)$. If $y_1=-1$ the vector $f=\frac{1}{2}(g_1+g_2)=(-1,0)$ differenciates A_i and A_a. If $y_1=1$ then A_i and A_a form an antichain, so there is a $g_3=(-1,-1)$ giving $f=\frac{1}{2}(g_1+g_3)=(-1,0)$ that differenciates A_i and A_a. Thus $\mathfrak{P}$ is closed in $\mathcal{O}(P,a)$.

$\Rightarrow$) Consider a partition $\mathfrak{P}=\{A_\bot,A_\top,A_a,A_1,...,A_r\}$ of $\hat{P}$ closed in $\mathcal{O}(P,a)$ and determining a r-dimensional face $\mathcal{F}$ of $\mathcal{O}(P,a)$. Since $\mathcal{O}(P,a)=\mathcal{O}_{[-1,1]}(P)\cap\{x_a=0\}$, $\mathcal{F}$ can be written as

$$\mathcal{F}=\mathcal{G}\cap\{x_a=0\},$$

where $\mathcal{G}$ is a $(r+1)$-dimensional face of $\mathcal{O}_{[-1,1]}(P)$. As $\mathcal{G}$ is a face, it is associated to a partition $\mathfrak{P}'=\{A'_\top,A'_\bot,A'_a,A'_1,...,A'_r\}$ of P. First, let us check that $\mathfrak{P}'=\mathfrak{P}$:

- If $x,y\in A'_i$, then $f(x)=f(y),\forall f\in\mathcal{G}$. Hence, $f(x)=f(y),\forall f\in\mathcal{F}$. So $A_i\subseteq A'_i$.
- If $x,y\in A_i$, then $f(x)=f(y),\forall f\in\mathcal{F}=\mathcal{G}\cap\{x_a=0\}$. Suppose there exist $x,y\in A_i$ and a vertex $g\in\mathcal{G}$ such that $g(x)\neq g(y)$. Then, $x\in A'_i,y\in A'_j$ and $g(x)=1,g(y)=-1$. Suppose w.l.g. that $g(a)=1$ and consider

$$g'(z)=\begin{cases}0 & \text{if } g(z)=1\\ -1 & \text{otherwise}\end{cases}$$

Thus, $g'\in\mathcal{G}$ and $g'(a)=0$. Hence, $g'\in\mathcal{F}$ and $g'(x)\neq g'(y)$, a contradiction. So $A'_i\subseteq A_i$.

Let us now show that $\mathfrak{P}$ is closed for $\mathcal{O}_{[-1,1]}(P)$. Consider $A_i,A_j\in\mathfrak{P}$. As the partition is closed for $\mathcal{O}(P,a)$, there exist $g\in\mathcal{O}(P,a)$ constant on each block such that $g(A_i)\neq g(A_j)$. Since $\mathcal{O}(P,a)=\mathcal{O}_{[-1,1]}(P)\cap\{x_a=0\}$, then $g\in\mathcal{O}_{[-1,1]}(P)$ and it is constant on each block. Therefore, $\mathfrak{P}$ is closed for $\mathcal{O}_{[-1,1]}(P)$. Since $\mathcal{G}$ is a face and $\mathfrak{P}$ is closed, we conclude by Theorem 5 that $\mathfrak{P}$ is connected and compatible for $\mathcal{O}_{[-1,1]}(P)$, with in turn implies that it is connected and compatible for $\mathcal{O}(P,a)$. $\square$

Remark 6. *When characterizing the vertices of $\mathcal{O}(P,a)$ we had obtained the condition of A_a being a connected subposet of P. The previous result gives an insight of the reason for this condition, as it imposes connectivity for each element of the partition.*

In particular, we have the following nice result for 2-dimensional faces.

Proposition 7. *The 2-dimensional faces of $\mathcal{O}(P,a)$ are triangles or quadrilaterals.*

Proof. Since any face of $\mathcal{O}(P,a)$ is again a pointed order polytope by Proposition 5, it suffices to analyze the 2-dimensional pointed order polytopes. These polytopes come from pointed order polytopes $\mathcal{O}(P,a)$ where $|P|=3$. In Figure 2, we can see all the non-isomorphic posets with three elements. Depending on the choice made for a we get different pointed order polytopes $\mathcal{O}(P,a)$. Let us study the different cases:

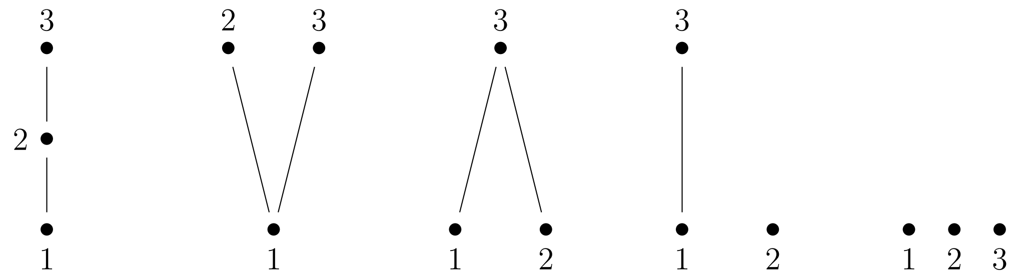

Figure 2: Non-isomorphic posets with 3 elements.

i) In the first poset we can set $a=1$ and then the corresponding pointed order polytope is a triangle with vertices $(0,0,0),(0,0,1)$ and $(0,1,1)$. If we set $a=2$ we get a square with vertices $(-1,0,0),(-1,0,1),(0,0,0),(0,0,1)$. The case $a=3$ is equivalent to case $a=1$ by duality.

ii) In the second poset we can set $a=1$ obtaining a square with vertices $(0,0,0),(0,0,1),(0,1,0)$ and $(0,1,1)$. If we choose $a=2$, we obtain the quadrilateral $(-1,0,-1),(-1,0,1),(0,0,0)$ and $(0,0,1)$. The case $a=3$ is equivalent to case $a=2$ by symmetry.

iii) This poset is the dual of the previous one, so we get the same conclusions.

iv) In the fourth poset, if we set $a=1$, we get a quadrilateral with vertices $(0,-1,0),(0,-1,1),(0,1,0)$ and $(0,1,1)$. In the case $a=2$, we obtain a triangle with vertices $(-1,0,-1),(-1,0,1)$ and $(1,0,1)$. The case $a=3$ is equivalent to case $a=1$ by duality.

v) In the last poset, if we set $a=1$, we obtain a square with vertices $(0,-1,-1),(0,-1,1),(0,1,-1)$ and $(0,1,1)$. The rest of cases $a=2$ and $a=3$ follow by symmetry.

Therefore, the result holds. □

3.3 Adjacency

Let us now deal with the problem of determining whether two vertices are adjacent.

Theorem 7. *Let $f_{A^1_\top,A^1_a,A^1_\bot}$ and $f_{A^2_\top,A^2_a,A^2_\bot}$ be two vertices of $\mathcal{O}(P,a)$. Then, these vertices are adjacent if and only if one of these subsets is common, there is a containing relation between the two other subsets and the difference is a connected subposet, i.e. $A^1_x=A^2_x$, $A^1_y\subseteq A^2_y$, (and hence, $A^1_z\supseteq A^2_z$) and $A^2_y\setminus A^1_y=A^1_z\setminus A^2_z$ is connected, where x,y,z are different elements of $\{\top,\bot,a\}$.*

Proof. From Theorem 6, we know that an edge in $\mathcal{O}(P,a)$ is given by a face partition of $\hat{P}$ in four connected subsets, say $\mathfrak{P} := \{A_\top, A_\bot, A_a, A_1\}$. From Proposition 5, we know that given a face, subfaces appear combining different blocks. As blocks for $\top, \bot$ and a are needed in any face partition, the only possibility comes from joining A_1 to other subset. Hence, we have three possible vertices whose corresponding partitions $\mathfrak{P}'$ are:

$$\{A_\top \cup A_1, A_a, A_\bot\},\ \{A_\top, A_a, A_\bot \cup A_1\},\ \{A_\top, A_a \cup A_1, A_\bot\}.$$

To determine which ones are vertices, we apply the conditions of Definition 5. Besides, by Theorem 6, we need to check that the new face partition $\mathfrak{P}'$ leads to an order relation $\preceq_{\mathfrak{P}'}$ between the subsets in the partition. More concretely, we have to see that $\preceq_{\mathfrak{P}'}$ is antisymmetric. Let us then study the different cases.

- **Case 1**: $A_a \preceq_{\mathfrak{P}} A_1$. This implies that block $A_1 \cup A_\bot$ is no longer possible as this would mean:

  $$\left\{ \begin{array}{lcl} A_\bot \preceq_{\mathfrak{P}} A_a & \Rightarrow & A_\bot \cup A_1 \preceq_{\mathfrak{P}'} A_a \\ A_a \preceq_{\mathfrak{P}} A_1 & \Rightarrow & A_\bot \cup A_1 \succeq_{\mathfrak{P}'} A_a \end{array} \right.$$

  Let us finally check that

  $$\{A_\top \cup A_1, A_a, A_\bot,\},\ \{A_\top, A_a \cup A_1, A_\bot\}$$

  are vertices of $\mathcal{O}(P,a)$. The first one satisfies all the conditions of Definition 5 as $A_\top \cup A_1$ is an upset. Indeed, for $x \in A_\top \cup A_1$ and $y \in \hat{P}, x \preceq y$, let us show that $y \in A_\top \cup A_1$. If $x \in A_\top \Rightarrow y \in A_\top$. If $x \in A_1$, as $A_a \preceq_{\mathfrak{P}} A_1$, then $y \notin A_\bot \cup A_a$. For the second one, remark that A_a and A_1 are connected by hypothesis. Moreover, since $A_a \preceq_{\mathfrak{P}} A_1$, there exist $x \in A_a, y \in A_1$ such that $x \preceq y$. Therefore, $A_a \cup A_1$ is a connected subposet.

- **Case 2**: $A_1 \preceq_{\mathfrak{P}} A_a$. Following the same steps as in the previous case, we conclude that the block $A_1 \cup A_\top$ is no longer possible and we obtain an edge whose vertices are given by $\{A_\top, A_a, A_\bot \cup A_1\}$ and $\{A_\top, A_a \cup A_1, A_\bot\}$.

- **Case 3**: Finally, assume that A_a and A_1 are not related. Hence, for $x \in A_a, y \in A_1$, we conclude that these elements are not related in P. Therefore, $A_a \cup A_1$ is not a connected subposet of P and the vertices of the edge are

  $$\{A_\top \cup A_1, A_a, A_\bot\},\ \{A_\top, A_a, A_\bot \cup A_1\}.$$

  Proving that $A_\top \cup A_1$ is an upset and $A_\bot \cup A_1$ a downset is done in the same way as in Case 1.

Thus, the vertices in an edge share one of the subsets in the partition. Note also that in each case, the difference between subsets is always A_1, that is a connected subposet of P by hypothesis. □

The condition of the previous result can be turned into the following one:

Corollary 2. *Let $f_{A^1_\top, A^1_a, A^1_\bot}$ and $f_{A^2_\top, A^2_a, A^2_\bot}$ be two vertices of $\mathcal{O}(P,a)$. Then, these vertices are adjacent if and only if one of these subsets is common and there is another pair of subsets with connected symmetric difference, i.e. $A^1_x = A^2_x$ and $A^1_y \Delta A^2_y := (A^1_y \backslash A^2_y) \cup (A^2_y \backslash A^1_y)$ is a connected subposet of P, where x, y are different elements of $\{\top, \bot, a\}$.*

Proof. Note that $A_y^1 \Delta A_y^2$ is connected if and only if $A_y^1 \subseteq A_y^2$ or $A_y^1 \supseteq A_y^2$ and the difference $A_y^2 \setminus A_y^1$ or $A_y^1 \setminus A_y^2$ is a connected subposet. If $A_x^1 = A_x^2$ and $A_y^1 \subseteq A_y^2$ (resp. $A_y^1 \supseteq A_y^2$), then we automatically get $A_z^1 \supseteq A_z^2$ (resp. $A_z^1 \subseteq A_z^2$). Moreover, if the difference $A_y^2 \setminus A_y^1$ (resp. $A_y^1 \setminus A_y^2$) is connected, then $A_z^2 \setminus A_z^1$ (resp. $A_z^1 \setminus A_z^2$) is also connected because it is the same subposet. □

Example 4. *(Continued Example 1) Let us consider $v_1 = (1, 1, -1)$. Hence, $A_1 = \{x, y\}, A_0 = \{a\}, A_{-1} = \{z\}$ and applying Theorem 7, we obtain that the vertices adjacent to v_1 are*

Vertex	A_1	A_0	A_{-1}
$(1, 1, 0)$	x, y	a, z	$\emptyset$
$(-1, 1, -1)$	y	a	x, z
$(0, 0, -1)$	$\emptyset$	x, y, a	z

Corollary 3. *Determining if two vertices of $\mathcal{O}(P, a)$ are adjacent can be solved in quadratic time.*

Proof. Given two vertices $f_{A_1^1, A_0^1, A_{-1}^1}, f_{A_1^2, A_0^2, A_{-1}^2}$, in order to be adjacent we need $A_1^1 = A_1^2$ or $A_0^1 = A_0^2$ or $A_{-1}^1 = A_{-1}^2$ and this can be checked in quadratic time in $|P|$ (where we consider as unit of time the comparison of elements of P). Next step is to check if $A_0^1 \subset A_0^2$ (or $A_0^2 \subset A_0^1$) and this can be done again in quadratic time. We finally need to check if $A_0^2 \setminus A_0^1$ is a connected subposet and this can be done in quadratic time, for example using Prim algorithm. □

Related to the problem of adjacency, we have the problem of determining the diameter of pointed order polytopes. Similar to the results of Theorem 4, we can state the following.

Theorem 8. *Let P be a finite poset, $a \in P$ and $\mathcal{O}(P, a)$ its associated pointed order polytope. Then:*

i) If $P = P_1 \uplus P_2$ and $a \in P_1$, then the diameter of $\mathcal{O}(P, a)$ is the sum of the diameter of $\mathcal{O}(P_1, a)$ and the diameter of $\mathcal{O}(P_2)$.

ii) If P has a maximum and a minimum different from a, the diameter of $\mathcal{O}(P, a)$ is at most 4.

iii) If $P = P_1 \oplus a \oplus P_2$ and the diameter of $\mathcal{O}(P_i)$ is d_i for $i \in \{1, 2\}$, then the diameter of $\mathcal{O}(P, a)$ is $d_1 + d_2$ and therefore this diameter is at most $w(P_1) + w(P_2)$.

Proof.

i) By Remark 4 we know that $\mathcal{O}(P, a) = \mathcal{O}(P_1, a) \times \mathcal{O}_{[-1,1]}(P_2)$. Since the adjacency graph of the product of polytopes is the cartesian product of its adjacency graphs and the diameter of the cartesian product of graphs is the sum of their diameters (see [6]), the result holds.

ii) Consider the vertex partition $\{A_1, A_0, A_{-1}\}$ associated to some vertex. Since P has maximum and minimum, the upset A_1 and the downset A_{-1} are connected subposets. Thus, we get the sequence of adjacent partitions:

$$\{A_1, A_0, A_{-1}\} - \{\emptyset, A_0 \cup A_1, A_{-1}\} - \{\emptyset, P, \emptyset\}.$$

This way the distance between any vertex and the vertex given by $A_1 = \emptyset, A_{-1} = \emptyset, A_0 = P$ is at most 2, so we can find a chain of lenght at most 4 passing through zero between any 2 vertices.

iii) By Remark 3 we know that $\mathcal{O}(P, a)$ is isometric to $\mathcal{O}(P_1) \times \mathcal{O}(P_2)$. Hence, the diameter is the sum of diameters. Finally, the upper bound arises by Theorem 4. □

4 The pointed order polytope of bi-capacities

As stated in Section 2, $\mathcal{BCAP}(X)$ can be seen as a convex polytope on $\mathbb{R}^{3^n-3}$. Consider the poset $(\mathcal{Q}^*(X), \preceq)$, where

$$\mathcal{Q}^*(X) := \mathcal{Q}(X) \backslash \{(X, \emptyset), (\emptyset, X)\}, \quad (A, B) \preceq (C, D) \Leftrightarrow A \subseteq C, B \supseteq D.$$

Example 5. *If $X = \{1, 2\}$, the Hasse diagram of $\mathcal{Q}^*(X)$ is given in Figure 3.*

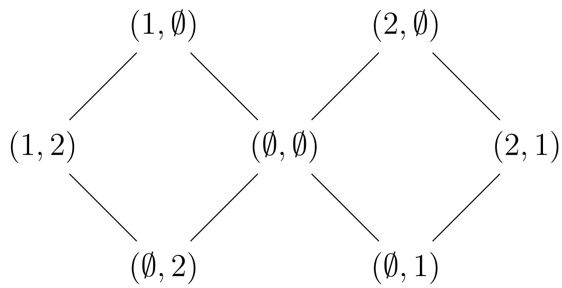

Figure 3: Hasse diagram of $\mathcal{Q}^*(X)$ when $|X| = 2$.

Now, from the definition of bi-capacities, the following holds:

Corollary 4. *The polytope $\mathcal{BCAP}(X)$ is the pointed order polytope $\mathcal{O}(\mathcal{Q}^*(X), (\emptyset, \emptyset))$.*

We aim to study the properties of this polytope at the light of the results of the previous section. First, we can find all vertices of this polytope applying Propositions 3 and 4. Consequently, for a bi-capacity ν to be a vertex, it is necessary that $\nu(A) \in \{-1, 0, 1\}, \forall A \in \mathcal{Q}^*(X)$. Then, we rename ν as $\nu_{A_1, A_0, A_{-1}}$, where A_1 (resp. A_0, A_{-1}) is the set of elements $(A, B) \in \mathcal{Q}^*(X)$ such that $\nu(A, B) = 1$ (resp. $\nu(A, B) = 0, \nu(A, B) = -1$). Now, the following can be established.

Corollary 5. *Consider a bi-capacity $\nu_{A_1, A_0, A_{-1}}$. Let us denote by*

$$\mathcal{Q}_1^*(X) := \{(A, B) \in \mathcal{Q}^*(X) : A \neq \emptyset\}, \quad \mathcal{Q}_{-1}^*(X) := \{(A, B) \in \mathcal{Q}^*(X) : B \neq \emptyset\}.$$

Then, $\nu_{A_1, A_0, A_{-1}}$ is a vertex if and only if

- *A_1 is an upset of $\mathcal{Q}_1^*(X)$.*
- *A_{-1} is a downset of $\mathcal{Q}_{-1}^*(X)$.*
- *A_0 is a connected subposet.*

Example 6. *(Continued Example 5). Let us obtain all vertices of bi-capacities when $|X| = 2$. We classify the vertices in terms of the different possibilities of A_1. Note that*

$$A_1 \subseteq \mathcal{Q}^*(X) \backslash \{(\emptyset, 2), (\emptyset, 1), (\emptyset, \emptyset)\} = \{(1, \emptyset), (1, 2), (2, \emptyset), (2, 1)\},$$

whose Hasse diagram is given in Figure 4.

As A_1 is an upset, we have the following cases for A_1:

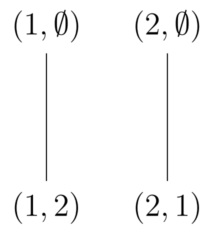

Figure 4: Hasse diagram of $\mathcal{Q}^*(X)\backslash\{(\emptyset,2),(\emptyset,1),(\emptyset,\emptyset)\}$.

$A_1=\emptyset$	$A_1=\{(1,\emptyset)\}$	$A_1=\{(2,\emptyset)\}$
$A_1=\{(1,\emptyset),(1,2)\}$	$A_1=\{(2,\emptyset),(2,1),\}$	$A_1=\{(1,\emptyset),(2,\emptyset)\}$
$A_1=\{(1,\emptyset),(2,\emptyset),(1,2)\}$	$A_1=\{(1,\emptyset),(2,\emptyset),(2,1)\}$	$A_1=\{(1,\emptyset),(2,\emptyset),(1,2),(2,1)\}$

Let us study two of them for the sake of clarity.

- $A_1=\emptyset$. *Hence,* $A_0\cup A_{-1}=\mathcal{Q}^*(X)$, *and its Hasse diagram is given in Figure 3. Now, by monotonicity,* $(1,\emptyset),(2,\emptyset)$ *are in* A_0*; and* $(\emptyset,\emptyset)\in A_0$ *by construction. Hence, the poset of possibilities for* A_{-1} *is given in Figure 5.*

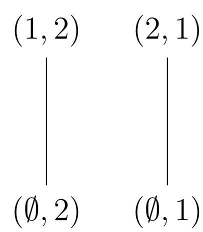

Figure 5: Hasse diagram of $\mathcal{Q}^*(X)\backslash\{(1,\emptyset),(2,\emptyset),(\emptyset,\emptyset)\}$.

The nine possible downsets for this poset are:

$A_{-1}=\emptyset$	$A_{-1}=\{(\emptyset,2)\}$	$A_{-1}=\{(\emptyset,1)\}$
$A_{-1}=\{(\emptyset,1),\emptyset,2)\}$	$A_{-1}=\{(1,2),(\emptyset,2)\}$	$A_{-1}=\{(2,1),(\emptyset,1)\}$
$A_{-1}=\{(1,2),(\emptyset,2),(\emptyset,1)\}$	$A_{-1}=\{(2,1),(\emptyset,1),(\emptyset,2)\}$	$A_{-1}=\{(1,2),(\emptyset,2),(\emptyset,1),(2,1)\}$

- $A_1=\{(1,\emptyset)\}$. *The Hasse diagram of poset* $A_0\cup A_{-1}=\mathcal{Q}^*(X)\backslash A_1$ *is given in Figure 6 left.*

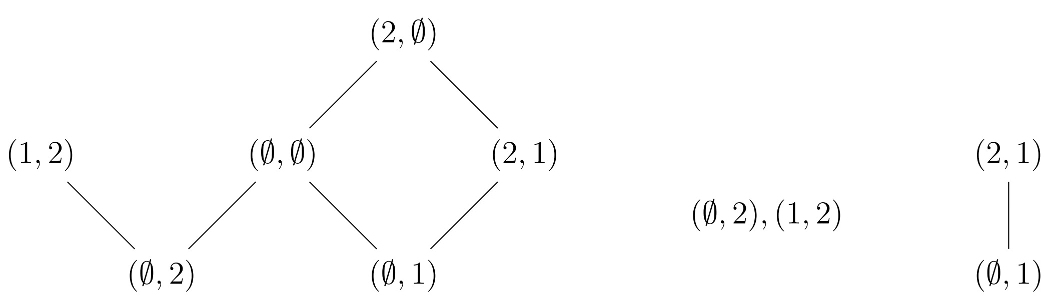

Figure 6: The case for $A_1=\{(1,\emptyset)\}$.

For this poset, note that $(\emptyset, \emptyset), (2, \emptyset) \in A_0$. Besides, $(\emptyset, 2), (1, 2)$ are both in A_0 or in A_{-1} because A_0 should be connected. Hence, the poset of possibilities for A_{-1} is given in Figure 6 right. The six possible downsets for this poset are:

$$\begin{array}{lll} A_{-1} = \emptyset & A_{-1} = \{(1,2), (\emptyset, 2)\} & A_{-1} = \{(\emptyset, 1)\} \\ A_{-1} = \{(1,2), (\emptyset, 2), (\emptyset, 1)\} & A_{-1} = \{(2,1), (\emptyset, 1)\} & A_{-1} = \{(1,2), (\emptyset, 2), (\emptyset, 1), (2,1)\} \end{array}$$

Proceeding this way for all possibilities, it can be seen that we have 49 vertices. Compare this value with the 4 vertices of the polytope of capacities over a referential of two points.

Indeed, the number of vertices for bi-capacities grows much faster than the corresponding number for capacities, and the latter already grows very fast. Table 1 shows the number of vertices for referentials of cardinality 1, 2 and 3.

$\lvert X \rvert$	1	2	3
# vertices of $\mathcal{FM}(X)$	1	4	18
# vertices of $\mathcal{BCAP}(X)$	4	49	56.843

Table 1: Comparison of vertices of $\mathcal{FM}(X)$ and $\mathcal{BCAP}(X)$ for small referentials X.

We can also deduce the number of facets applying Proposition 6.

Proposition 8. *Let $n = |X| > 1$. The number of facets of $\mathcal{BCAP}(X)$ is $2n3^{n-1}$.*

Proof. Since $(\emptyset, \emptyset)$ is not a maximal element nor a minimal of $\mathcal{Q}^*(X)$, the number of facets is given by (see Proposition 2) $M+m+r$. This number is equal to the number of relations in the poset $\mathcal{Q}(X)$ keeping the maximum $(X, \emptyset)$ and the minimum $(\emptyset, X)$. If we denote $|A| = i, |B| = j$, let us compute the number of elements covering (A, B). If we focus on B we can remove j elements, so we find j elements of the form $(A, B \setminus \{k\})$ covering (A, B). Now, if we focus on A we can add $n - (i + j)$ elements to get an element $(A \cup \{k\}, B)$ covering (A, B). This way we obtain $n - (i + j) + j = n - i$ elements covering (A, B). Finally, we add all the covering relations:

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} (n-i) \binom{n}{i} \binom{n-i}{j} = \sum_{i=0}^{n} (n-i) \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} =$$

$$= \sum_{i=0}^{n} (n-i) 2^{n-i} \binom{n}{i} = n \sum_{i=0}^{n} 2^{n-i} \binom{n}{i} - \sum_{i=0}^{n} i 2^{n-i} \binom{n}{i}.$$

Note that $n \sum_{i=0}^{n} 2^{n-i} \binom{n}{i} = n3^n$. On the other hand:

$$\sum_{i=0}^{n} i 2^{n-i} \binom{n}{i} = \sum_{i=1}^{n} 2^{n-i} i \binom{n}{i} = n \sum_{i=1}^{n} 2^{n-i} \binom{n-1}{i-1} = n \sum_{i=0}^{n-1} 2^{n-1-i} \binom{n-1}{i} = n3^{n-1}.$$

Hence, $n3^n - n3^{n-1} = 2n3^{n-1}$. □

Next, given two vertices of $\mathcal{BCAP}(X)$, adjacency can be checked via Theorem 7, as next example shows.

Example 7. *(Continued Example 6) Consider the vertex* $f_{A_1,A_0,A_{-1}}$ *where*

$$A_1 = \{(1,\emptyset),(1,2)\}, \quad A_0 = \{(2,\emptyset),(2,1),(\emptyset,\emptyset),(\emptyset,1)\}, \quad A_{-1} = \{(\emptyset,2)\}.$$

Hence, for $f_{A'_1,A'_0,A'_{-1}}$ *given as*

$$A'_1 = \{(1,\emptyset),(1,2),(2,\emptyset)\}, \quad A_0 = \{(2,1),(\emptyset,\emptyset),(\emptyset,1)\}, \quad A_{-1} = \{(\emptyset,2)\},$$

we conclude applying Theorem 7 that these vertices are adjacent because $A_{-1} = A'_{-1}$, $A'_1\backslash A_1 = A_0\backslash A'_0 = \{(2,\emptyset)\}$, *that is a connected subposet. Moreover, Theorem 7 allows to obtain the set of vertices* $f_{A'_1,A'_0,A'_{-1}}$ *adjacent to* $f_{A_1,A_0,A_{-1}}$. *This set is given by*

- **Case 1:** $A_1 = A'_1$. *In this case, we need to find the possible* A'_0 *such that* $A'_0 \subset A_0$ *and* $A_0\backslash A'_0$ *connected or viceversa. There are three possibilities:*

$A'_0 = \{(2,\emptyset),(2,1),(\emptyset,\emptyset)\}$	$A_0\backslash A'_0 = \{(\emptyset,1)\}$
$A'_0 = \{(2,\emptyset),(2,1),(\emptyset,\emptyset),(\emptyset,1),(\emptyset,2)\}$	$A'_0\backslash A_0 = \{(\emptyset,2)\}$
$A'_0 = \{(2,\emptyset),(\emptyset,\emptyset)\}$	$A_0\backslash A'_0 = \{(2,1),(\emptyset,1)\}$

- **Case 2:** $A_0 = A'_0$. *In this case, we need to find the possible* A'_1 *such that* $A'_1 \subset A_1$ *and* $A_1\backslash A'_1$ *connected or viceversa. There is only one possibility:*

$$A'_1 = \{(1,\emptyset)\}, \quad A_1\backslash A'_1 = \{(1,2)\}.$$

- **Case 3:** $A_{-1} = A'_{-1}$. *In this case, we need to find the possible* A'_1 *such that* $A'_1 \subset A_1$ *and* $A_1\backslash A'_1$ *connected or viceversa. There are three possibilities:*

$A'_1 = \emptyset$	$A_1\backslash A'_1 = \{(1,\emptyset),(1,2)\}$
$A'_1 = \{(1,\emptyset),(1,2),(2,\emptyset)\}$	$A'_1\backslash A_1 = \{(2,\emptyset)\}$
$A'_1 = \{(1,\emptyset),(2,\emptyset),(1,2),(2,1)\}$	$A'_1\backslash A_1 = \{(2,1),(2,\emptyset)\}$

In the last part of the section, we study the diameter of $\mathcal{BCAP}(X)$. First, note that we are not in the conditions of any case of Theorem 8, so that we have to study this case separately. In order to make the proof more readable, we divide it into several lemmas.

Lemma 2. [1] *Let* $f_{A_1,A_0,A_{-1}}$ *be a vertex of* $\mathcal{BCAP}(X)$ *and consider a connected component* C *of* A_1 *(resp.* A_{-1}*). Then,* C *is an upset and there exists* $i \in X$ *such that* $(X\backslash i,\emptyset) \in C$ *(resp.* $(\emptyset, X\backslash i) \in C$*).*

Proof. Given C a connected component of A_1, let us first show that C is a filter. Consider $x \in C$ and $y \succeq x$. As A_1 is an upset, $y \in A_1$ and it is possible to find a chain $x = z_0 - z_1 - ... - z_r = y$ included in A_1. Hence, $y \in C$. Now, for $(A,B) \in C$, we conclude that $(X\backslash i,\emptyset) \in C$ for $X\backslash i \supseteq A$. □

Corollary 6. *Let* $f_{A_1,A_0,A_{-1}}$ *be a vertex of* $\mathcal{BCAP}(X)$. *Consider a connected component* C *of* A_1 *(resp.* A_{-1}*). The maximal (resp. minimal) elements of* C *are of type* $(X\backslash i,\emptyset)$ *(resp.* $(\emptyset, X\backslash i)$*).*

[1] It is possible to derive general versions of this result applying for general pointed order polytopes. The same happens for Corollary 6, Lemma 3 and Corollary 7 below.

Besides, as the different connected components are disjoint sets, so are the corresponding maximal elements. Note however that they do not necessarily form a partition of $\{(X\backslash i, \emptyset) : i \in X\}$, as it could be the case that some of these elements could lay in A_0.

Remark that for a connected component C of A_1 whose corresponding maximal elements are $\{(X\backslash i, \emptyset) : i \in I \subseteq X\}$, it follows that

$$C = \left(\bigcup_{i \in I} \downarrow (X\backslash i, \emptyset) \right) \cap A_1,$$

where $\downarrow (X\backslash i, \emptyset)$ denotes the set of elements (A, B) s.t. $(X\backslash i, \emptyset) \succeq (A, B)$. Consequently, we can identify C with its maximal elements.

Lemma 3. *Let $f_{A_1,A_0,A_{-1}}$ be a vertex $\mathcal{BCAP}(X)$ and consider a connected component C of A_1 (resp. A_{-1}) whose corresponding maximal (resp. minimal) elements are*

$$\{(X\backslash i, \emptyset) : i \in I \subseteq X\} \quad (resp. \ \ \{(\emptyset, X\backslash i) : i \in I \subseteq X\}).$$

Then, for any $(A, B) \in C$,

$$A \supseteq \bigcap_{i \in I} (X\backslash i) \qquad (resp. \quad B \supseteq \bigcap_{i \in I} (X\backslash i)).$$

Proof. Take $(A, B) \in C$ and consider $i \notin A$. Then, $A \subseteq X\backslash i$ and hence $(A, B) \preceq (X\backslash i, \emptyset)$. Hence, $(X\backslash i, \emptyset) \in C, \forall i \notin A$. Moreover,

$$A = \bigcap_{i \notin A} (X\backslash i).$$

Consequently, $A \supseteq \bigcap_{i \in I} (X\backslash i)$. □

Corollary 7. *Let $f_{A_1,A_0,A_{-1}}$ be a vertex of $\mathcal{BCAP}(X)$ and consider a connected component C of A_1 (resp. A_{-1}) whose corresponding maximal (resp. minimal) elements are*

$$\{(X\backslash i, \emptyset) : i \in I \subseteq X\} \quad (resp. \ \ \{(\emptyset, X\backslash i) : i \in I \subseteq X\}).$$

Then, the minimal (resp. maximal) elements of C are of type (A, B), with

$$A \supseteq \bigcap_{i \in I} (X\backslash i) \qquad (resp. \quad B \supseteq \bigcap_{i \in I} (X\backslash i)).$$

Let us define:

$$P_1^* = \{(A, \emptyset) : A \subseteq X, A \neq \emptyset, X\},\ P_0^* = \{(A, B) : A \neq X, \emptyset, B \neq X, \emptyset\},\ P_{-1}^* = \{(\emptyset, B) : B \subseteq X, B \neq \emptyset, X\}.$$

Hence, $\mathcal{Q}^*(X) = P_1^* \cup P_0^* \cup P_{-1}^* \cup \{(\emptyset, \emptyset)\}$ and P_1^*, P_0^*, P_{-1}^* are pairwise disjoint.

Lemma 4. *Assume $n \geq 4$ and let $f_{A_1,A_0,A_{-1}}$ be a vertex of $\mathcal{BCAP}(X)$. Suppose that A_1 (resp. A_{-1}) has at least three connected components. Then, $A_0 \cap P_1^*$ (resp. $A_0 \cap P_{-1}^*$) is connected.*

Proof. As A_1 has at least three connected components, we conclude by Corollary 6 that none of these connected components has $n-1$ maximal elements. Hence, applying Corollary 7, we conclude that $\{(i, \emptyset) : i \in X\} \subseteq A_0$. Moreover, if $n \geq 4$, there is at most one connected component with $n-2$ maximal elements and thus, there is at most one subset type $(i_0 j_0, \emptyset) \in A_1$ by Lemma 3.

Let us see that it is possible to connect in $A_0 \cap P_1^*$ any pair $(i, \emptyset), (j, \emptyset)$. If $\{i, j\} \neq \{i_0, j_0\}$, then $(ij, \emptyset) \in A_0$ and we are done. Let us then consider the (possible) case $(i_0, \emptyset), (j_0, \emptyset)$. As $n \geq 4$, there exists $k \in X \backslash \{i_0, j_0\}$ and hence we have the path

$$(i_0, \emptyset) - (i_0 k, \emptyset) - (k, \emptyset) - (j_0 k, \emptyset) - (j_0, \emptyset).$$

Consider now $(A, \emptyset), (B, \emptyset) \in A_0$. Then, $(A', \emptyset), (B', \emptyset) \in A_0$, for any $A' \subset A, B' \subset B$, as otherwise $(A', \emptyset) \in A_1$ and thus $(A, \emptyset) \in A_1$ because A_1 is an upset. Take $i_0 \in A, j_0 \in B$. Then, we can build the path in $A_0 \cap P_1^*$ given by

$$(A, \emptyset) - (i_0, \emptyset) - (j_0, \emptyset) - (B, \emptyset)$$

and the result follows. □

Lemma 5. *Assume $n \geq 4$ and let $f_{A_1, A_0, A_{-1}}$ be a vertex of $\mathcal{BCAP}(X)$. Suppose that A_1 (resp. A_{-1}) has at least three connected components. Then, $A_0 \backslash (P_{-1}^* \cup \{(\emptyset, \emptyset)\})$ (resp. $A_0 \backslash (P_1^* \cup \{(\emptyset, \emptyset)\})$) is connected.*

Proof. Applying Lemma 4, it suffices to show that any $(A, B) \in A_0$ such that $A \neq \emptyset, B \neq \emptyset$ can be connected to some $(C, \emptyset) \in A_0$ without leaving $A_0 \backslash (P_{-1}^* \cup \{(\emptyset, \emptyset)\})$. For this, note that as A_0 is connected, (A, B) can be connected to $(\emptyset, \emptyset)$. Consider a path

$$(A, B) =: (A_1, B_1) - (A_2, B_2) - ... - (A_{r-1}, B_{r-1}) - (A_r, B_r) := (\emptyset, \emptyset)$$

such that $(A_i, B_i) \lessdot (A_{i+1}, B_{i+1})$ or $(A_i, B_i) \gtrdot (A_{i+1}, B_{i+1})$ for $i = 1, ..., r-1$. This means that at each step we are adding or removing an element of X from either A_i or B_i. Note that in these conditions, (A_{r-1}, B_{r-1}) adopts the form $(i, \emptyset)$ or $(\emptyset, i)$. Hence, at a certain step, the path crosses from $P_0^* \cap A_0$ to $P_1^* \cap A_0$ or $P_{-1}^* \cap A_0$. Let us consider the first (A_i, B_i) where this happens.

- If $(A_i, B_i) \in P_1^* \cap A_0$, then we can take (A_i, B_i) as $(C, \emptyset)$ and we are done.
- If $(A_i, B_i) \in P_{-1}^* \cap A_0$, this means that (A_{i-1}, B_{i-1}) can be written as (j, B') for some $j \in X$. Now, as $(j, B') \preceq (j, \emptyset) \in A_0$, it follows that $(j, C) \in A_0, \forall C \subseteq B'$, and we can build the chain

$$(A, B) =: (A_1, B_1) - (A_2, B_2) - ... - (A_{i-1}, B_{i-1}) - ... - (j, \emptyset).$$

Hence, the result holds. □

Lemma 6. *Assume $n \geq 4$ and let $f_{A_1, A_0, A_{-1}}$ be a vertex of $\mathcal{BCAP}(X)$ and suppose A_{-1} (resp. A_1) has at least three connected components. Then,*

$$(A_1 \cup A_0) \cap \{(A, B) : B \neq \emptyset\} = (A_1 \cup A_0) \backslash (P_1^* \cup \{(\emptyset, \emptyset)\})$$

(resp. $(A_{-1} \cup A_0) \cap \{(A, B) : A \neq \emptyset\} = (A_{-1} \cup A_0) \backslash (P_{-1}^* \cup \{(\emptyset, \emptyset)\})$) is connected.

Proof. Consider $(A,B),(C,D) \in (\mathcal{A}_0 \cup \mathcal{A}_1) \setminus (P_1^* \cup \{(\emptyset,\emptyset)\})$. Then, $B \neq \emptyset, D \neq \emptyset$. Note that as $(A,B) \in \mathcal{A}_0 \cup \mathcal{A}_1$, so are all elements (A,B') such that $B' \subseteq B$. Hence, there is a chain in $(\mathcal{A}_0 \cup \mathcal{A}_1) \setminus (P_1^* \cup \{(\emptyset,\emptyset)\})$ connecting (A,B) and (A,i) with $i \in B$. Next, $(\emptyset,i) \in \mathcal{A}_0 \cup \mathcal{A}_1$ by Lemma 5. Hence, so is any (A',i) with $A' \subseteq X \setminus i$. Then, there is a chain in $(\mathcal{A}_0 \cup \mathcal{A}_1) \setminus (P_1^* \cup \{(\emptyset,\emptyset)\})$ connecting (A,i) and $(\emptyset,i)$. Finally, proceeding the same way for (C,D) and $(\emptyset,j), j \in D$, and applying that $(\emptyset,i)$ can be connected to $(\emptyset,j)$ without leaving $\mathcal{A}_0 \setminus (P_1^* \cup \{(\emptyset,\emptyset)\})$ by Lemma 4, we have a sequence in $(\mathcal{A}_0 \cup \mathcal{A}_1) \setminus (P_1^* \cup \{(\emptyset,\emptyset)\})$ given by

$$(A,B) - (A,i) - (\emptyset,i) - (\emptyset,j) - (C,j) - (C,D).$$

Hence, the result holds. □

Lemma 7. *Assume $n \geq 3$ and let $f_{\mathcal{A}_1,\mathcal{A}_0,\mathcal{A}_{-1}}$ be a vertex of the polytope of bi-capacities. Then, $\mathcal{A}_1 \cup P_1^*$ (resp. $\mathcal{A}_{-1} \cup P_{-1}^*$) is connected.*

Proof. As P_1^* is connected when $n \geq 3$, it suffices to show that any $(A,B) \in \mathcal{A}_1 \setminus P_1^*$ can be connected to some $(C,\emptyset)$. But this holds because as $(A,B) \in \mathcal{A}_1$, so are all elements (A,B') such that $B' \subseteq B$ and hence, there is a chain in $\mathcal{A}_1 \cup P_1^*$ connecting (A,B) and $(A,\emptyset)$ □

Lemma 8. *Assume $n \geq 3$ and let $f_{\mathcal{A}_1,\mathcal{A}_0,\mathcal{A}_{-1}}$ be a vertex of the set of bi-capacities. Then,*

$$\mathcal{A}_1 \cup [(P_1^* \cup P_0^*) \cap \mathcal{A}_0] = P_1^* \cup [(\mathcal{A}_1 \cup \mathcal{A}_0) \cap P_0^*]$$

(resp. $\mathcal{A}_{-1} \cup \left[(P_{-1}^* \cup P_0^*) \cap \mathcal{A}_0\right] = P_{-1}^* \cup [(\mathcal{A}_{-1} \cup \mathcal{A}_0) \cap P_0^*]$) is connected.

Proof. As before, it suffices to prove that any $(A,B) \in [(\mathcal{A}_1 \cup \mathcal{A}_0) \cap P_0^*]$ can be connected to some $(C,\emptyset)$. As $(A,B) \in \mathcal{A}_1 \cup \mathcal{A}_0$, so are all elements (A,B') such that $B' \subseteq B$ because $\mathcal{A}_0 \cup \mathcal{A}_1$ is an upset (as $\mathcal{A}_{-1}$ is a downset). Hence, there is a chain in $\mathcal{A}_1 \cup \mathcal{A}_0$ connecting (A,B) and $(A,\emptyset)$. □

We state now the main result about the diameter of the polytope of bi-capacities.

Theorem 9. *Let us consider the polytope of bi-capacities over a referential set X such that $|X| \geq 4$. Then, the diameter of this polytope is bounded by 8.*

Proof. Let $f_{\mathcal{A}_1,\mathcal{A}_0,\mathcal{A}_{-1}}$ be a vertex of the set of bi-capacities. Our strategy is to show that it is possible to connect this vertex to $f_{\emptyset,\mathcal{Q}^*(X),\emptyset}$ in at most four steps. We have to consider several cases.

- **Case 1: $\mathcal{A}_1$ has one or two connected components.**

  If $\mathcal{A}_1$ is connected, then $f_{\mathcal{A}_1,\mathcal{A}_0,\mathcal{A}_{-1}}$ and $f_{\emptyset,\mathcal{A}_1 \cup \mathcal{A}_0,\mathcal{A}_{-1}}$ are adjacent vertices. To see this, it suffices to show that $f_{\emptyset,\mathcal{A}_1 \cup \mathcal{A}_0,\mathcal{A}_{-1}}$ is a vertex, i.e. $\mathcal{A}_1 \cup \mathcal{A}_0$ is a connected subposet of $\mathcal{Q}^*(X)$. But this holds because the maximal elements of $\mathcal{A}_1$ are of type $(X \setminus i, \emptyset)$ that are related to $(\emptyset,\emptyset)$.

  Similarly, if $\mathcal{A}_1$ has two connected components $\mathcal{C}_1, \mathcal{C}_2$, applying the same argument we have the sequence of adjacent vertices given by

  $$f_{\mathcal{A}_1,\mathcal{A}_0,\mathcal{A}_{-1}} - f_{\mathcal{C}_2,\mathcal{C}_1 \cup \mathcal{A}_0,\mathcal{A}_{-1}} - f_{\emptyset,\mathcal{A}_1 \cup \mathcal{A}_0,\mathcal{A}_{-1}}.$$

  - **Case 1.1: $\mathcal{A}_{-1}$ has one or two connected components.**

    Applying the same argument, we conclude that in at most two steps, it is possible to connect $f_{\emptyset,\mathcal{A}_1 \cup \mathcal{A}_0,\mathcal{A}_{-1}}$ and $f_{\emptyset,\mathcal{Q}^*(X),\emptyset}$.

- **Case 1.2: A_{-1} has more than two connected components.**
  In this case, we can apply Lemma 6 and conclude that $f_{\emptyset,A_0\cup A_1,A_{-1}}$ and $f_{\emptyset,P_1^*\cup\{(\emptyset,\emptyset)\},P_0^*\cup P_{-1}^*}$ are adjacent. And finally, $f_{\emptyset,P_1^*\cup\{(\emptyset,\emptyset)\},P_0^*\cup P_{-1}^*}$ and $f_{\emptyset,\mathcal{Q}^*(X),\emptyset}$ are adjacent.

- **Case 2: A_1 has more than two connected components.**

  - **Case 2.1: A_{-1} has one or two connected components.**
    This case is symmetrical to Case 1.2.
  - **Case 2.2: A_{-1} has more than two connected components.**
    Consider $f_{A_1,A_0,A_{-1}}$. By Lemma 5, we know that $(P_0^*\cup P_1^*)\cap A_0$ is connected, so that $f_{A_1,A_0,A_{-1}}$ and $f_1^1 := f_{A_1\cup\left[(P_1^*\cup P_0^*)\cap A_0\right],(A_0\cap P_{-1}^*)\cup\{(\emptyset,\emptyset)\},A_{-1}}$ are adjacent.
    Now, by Lemma 4, we obtain that $A_0\cap P_{-1}^*$ is connected. Hence, f_1^1 is adjacent to $f_1^2 := f_{A_1\cup\left[(P_1^*\cup P_0^*)\cap A_0\right],\{(\emptyset,\emptyset)\},A_{-1}\cup P_{-1}^*}$.
    Finally, Lemmas 7 and 8 show that we can get $f_{\emptyset,\mathcal{Q}^*(X),\emptyset}$ from f_1^2 in two steps.

Hence, for any pair of vertices, it is possible to connect them passing through $f_{\emptyset,\mathcal{Q}^*(X),\emptyset}$ in at most eight steps, so that the diameter of the set of bicapacities on X for $n\geq 4$ is bounded by eight. □

It rests to study the cases for $n=2$ and $n=3$.

Lemma 9. *The diameter of $\mathcal{BCAP}(X)$ when $|X|=2$ is 4.*

Proof. Consider two vertices $f_{A_1^1,A_0^1,A_{-1}^1}$ and $f_{A_1^2,A_0^2,A_{-1}^2}$. Then, it can be seen that $A_1^1\triangle A_1^2$ has at most two connected components, and the same happens for $A_{-1}^1\triangle A_{-1}^2$. Hence, it is possible to find a path between $f_{A_1^1,A_0^1,A_{-1}^1}$ and $f_{A_1^2,A_0^2,A_{-1}^2}$ of length bounded by four. Indeed, this bound is achieved for

$$A_1^1=\{(1,\emptyset)\}, A_{-1}^1=\{(\emptyset,1)\}, A_0^1=\{(2,\emptyset),(2,1),(1,2),(\emptyset,\emptyset),(\emptyset,2)\},$$

$$A_1^2=\{(2,\emptyset)\}, A_{-1}^2=\{(\emptyset,2)\}, A_0^2=\{(1,\emptyset),(2,1),(1,2),(\emptyset,\emptyset),(\emptyset,1)\}.$$

□

Proposition 9. *The diameter of $\mathcal{BCAP}(X)$ when $|X|=3$ is bounded by 8.*

Proof. Consider two vertices $f_{A_1^1,A_0^1,A_{-1}^1}$ and $f_{A_1^2,A_0^2,A_{-1}^2}$. The idea of the proof consits in finding a bound for the number of steps necessary to pass from A_1^1 to A_1^2 and from A_{-1}^1 to A_{-1}^2. Note that as a consequence of Corollary 6, the number of connected components of A_1^1, A_1^2, A_{-1}^1 and A_{-1}^2 is bounded by 3. We have to consider several cases.

- **Case 1:A_1^1, A_1^2, A_{-1}^1 and A_{-1}^2 have all of them less than 3 connected components.** In this case, we can proceed as in Case 1.1 of Theorem 9 and conclude that the distance between $f_{A_1^1,A_0^1,A_{-1}^1}$ and $f_{A_1^2,A_0^2,A_{-1}^2}$ is bounded by 8.

- **Case 2: Some of A_1^1, A_1^2, A_{-1}^1 and A_{-1}^2 have 3 connected components.** Suppose w.l.g. that A_1^1 has three connected components. Then, A_1^1 adopts the form

$$A_1^1=\{(12,\emptyset),(13,\emptyset),(23,\emptyset)\}\cup AUX,$$

  where $AUX\subseteq\{(12,3),(13,2),(23,1)\}$. Let us now consider A_1^2. We have the following subcases:

- **Case 2.1: $A_1^2 = \emptyset$ or A_1^2 has one connected component.** In this case, we can proceed as in Case 1.1 of Theorem 9 and conclude that it is possible to pass from A_1^1 to A_1^2 in at most four steps.
- **Case 2.2: A_1^2 has two connected components.** Then, up to a permutation, A_1^2 has the two following forms:

$$A_1^2 = \{(12, \emptyset), (13, \emptyset)\} \cup AUX',$$

where $AUX' \subseteq \{(12, 3), (13, 2)\}$. In this case, comparing the corresponding connected components generated by $(ij, \emptyset)$, we conclude that it is possible to pass from A_1^1 to A_1^2 in at most three steps. The other possibility is

$$A_1^2 = \{(1, \emptyset), (12, \emptyset), (13, \emptyset), (23, \emptyset)\} \cup AUX'',$$

where $AUX'' \subseteq \{(12, 3), (13, 2), (1, 23), (1, 2), (1, 3), (23, 1)\}$. Then, the symmetric difference between the connected components generated by $(12, \emptyset)$ and $(13, \emptyset)$ in A_1^1 and the connected component generated by $(1, \emptyset)$ in A_1^2 has at most three connected components. Thus, it is possible to pass from one to another in at most three steps. Finally, comparing the connected component generated by $(23, \emptyset)$, we conclude that it is possible to pass from A_1^1 to A_1^2 in at most four steps.
- **Case 2.3: A_1^2 has three connected components.** In this case, A_1^2 has the same form as A_1^1. Hence, it suffices to compare the corresponding connected components to conclude that it is possible to pass from A_1^1 to A_1^2 in at most three steps.

This finishes the proof. □

5 Conclusions and open problems

In this paper we have studied the set of bi-capacities seen as a polytope. Bi-capacities arise when dealing with Decision Making with bipolar scales. They also appear in Game Theory when there is a coalition of players, a coalition of players against it and some other neutral players. To tackle this problem, we have defined the concept of pointed order polytope. This concept is based on a poset P and a special element a in the poset. In the case of bi-capacities, the poset is $\mathcal{Q}^*(X)$ and the special element is $(\emptyset, \emptyset)$. What makes pointed order polytopes an appealing object is that they rely on the subjacent poset and thus, they can be studied via this poset, a problem usually easier to handle.

We have derived the set of vertices of a general pointed order polytope, and the general form of its faces. Besides, we have solved the problem of whether two vertices of the pointed order polytope are adjacent in a simple way. From these general results, we have derived some results about the polytope of bi-capacities. In particular, we have obtained a bound for the diameter.

We feel that pointed order polytopes can be an interesting tool for studying in a systematic way polytopes appearing in Decision Making when using a bipolar scale. Of course, there are many aspects of pointed order polytopes that remain open problems and need more research. One of these problems is deriving the volume of $\mathcal{O}(P, a)$. In the case of order polytopes, this volume is given in terms of linear extensions of P, and this characterization also provides a triangulation of the order polytope. However, the result does not longer hold for pointed order polytopes. This problem, together with the problem of deriving a triangulation of pointed order polytopes, are problems that we intend to study in the future. These problems seem specially interesting when we restrict to bi-capacities or subfamilies of bi-capacities being pointed order polytopes.

We have considered in this paper an application to the set of bi-capacities. However, there are other situations in MCDM and Game Theory in which pointed order polytopes could be useful:

- An interesting case appearing specially in the field of Game Theory arises when some coalitions fail to form. This can be also extended for bipolar scales. This situation can be modelled again via pointed order polytopes, where the subjacent poset is no longer $\mathcal{Q}^*(X)$ but a proper subset $\mathcal{FC}(X)$ of $\mathcal{Q}^*(X)$. Depending on the structure of $\mathcal{FC}(X)$, many properties could be derived.
- In the field of Game Theory, it is unusual to consider fixed values for $\nu(X, \emptyset)$ and $\nu(\emptyset, X)$. This situation can be studied in a similar way to that of pointed order polytopes in which the condition $-1 \leq f(x) \leq 1$ is no longer valid. We thus obtain a non-bounded polytope. We feel that the properties of this polytope could be deeply related to those of the corresponding pointed order polytope (see [21] for the comparison in the case of order polytopes).

Next, there are other problems that seem interesting but not evident. Among them, we would like to focus the attention on the number of vertices and, especially, if this value is in some way related to the Dedekind numbers [10] that lead to the number of vertices of the capacities.

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References

[1] J. Abbas. The bipolar Choquet integral based on ternary-element sets. *Journal of Artificial Intelligence and Soft Computing Research*, 6(1):13–21, 2016.

[2] G. Beliakov and J.Z. Wu. Learning fuzzy measures from data: simplifications and optimisation strategies. *Information Sciences*, 494:100–113, 2019.

[3] O.V. Borodin and A.O. Ivanova. Describing faces in 3-polytopes with no vertices of degree from 5 to 7. *Discrete Mathematics*, 342(11):3208–3215, 2019.

[4] D. Candeloro, R. Mesiar, and A.R. Sambucini. A special class of fuzzy measures. Choquet integral and applications. *Fuzzy Sets and Systems*, (355):83–99, 2019.

[5] D. Catanzaro and R. Pesenti. Enumerating vertices of the balanced minimum evolution polytope. *Computers and Operations Research*, (109):209–217, 2019.

[6] M. R. Chithra and A. Vijayakumar. The diameter variability of the cartesian product of graphs. *Discrete Mathematics, Algorithms and Applications*, 6(1), 2014.

[7] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, (5):131–295, 1953.

[8] E. F. Combarro and P. Miranda. Identification of fuzzy measures from sample data with genetic algorithms. *Computers and Operations Research*, 33(10):3046–3066, 2006.

[9] E. F. Combarro and P. Miranda. Adjacency on the order polytope with applications to the theory of fuzzy measures. *Fuzzy Sets and Systems*, 180:384–398, 2010.

[10] R. Dedekind. Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler. *Festschrift Hoch Braunschweig Ges. Werke*, II:103–148, 1897. In German.

[11] D. Denneberg. *Non-additive measures and integral.* Kluwer Academic, Dordrecht (The Netherlands), 1994.

[12] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 5:587–607, 1997.

[13] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, (92):167–189, 1997.

[14] M. Grabisch. *Set functions, games and capacities in Decision Making*, volume 46 of *Theory and Decision Library.* Springer, 2016.

[15] M. Grabisch and C. Labreuche. Bi-capacities. I. the Choquet integral. *Fuzzy Sets and Systems*, 151(2):236–259, 2005.

[16] M. Grabisch and C. Labreuche. Bi-capacities. II. Definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151(2):211–236, 2005.

[17] S. Greco, B. Matarazzo, and S. Giove. The Choquet integral with respect to a level dependent capacity. *Fuzzy Sets and Systems*, 175(1):1–35, 2011.

[18] I. Kojadinovic and J.L. Marichal. Entropy of bi-capacities. *European Journal of Operational Research*, 178(1):168–184, 2007.

[19] F. Lange and M. Grabisch. New axiomatizations of Shapley interaction index for bi-capacities. *Fuzzy Sets and Systems*, 176:64–75, 2011.

[20] J. Li, X. Yao, X. Sun, and D. Wu. Determining the fuzzy measure in multiple criteria decision aiding from the tolerance perspective. Representation of associative functions. *European Journal of Operational Research*, 264:428–439, 2018.

[21] P. Miranda and P. García-Segador. Order cones: A tool for deriving k-dimensional faces of cones of subfamilies of monotone games. *Annals of Operational Research*, 295(1):117–137, 2020.

[22] P. Miranda and M. Grabisch. p-symmetric bi-capacities. *Kybernetica*, 40(4):421–440, 2004.

[23] Y. Narukawa, V. Torra, and M. Sugeno. Choquet integral with resepct to a symmetric fuzzy measure of a function on the real line. *Annals of Operations Research*, 244:571–581, 2016.

[24] C.E. Osgood, G.J. Suci, and P.H. Tannenbaum. *The measurement of meaning.* University of Illinois Press, Urbana (Il.), 1957.

[25] R. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

[26] M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, 1974.

[27] L. Xie and M. Grabisch. The core of bicapacities and bipolar games. *Fuzzy Sets and Systems*, 158(9):1000–1012, 2007.