International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems

On the polytope of 3-tolerant fuzzy measures

P. García-Segador

National Statistics Institute
Paseo de la Castellana, 183, 28071 Madrid (Spain)
pedro.garcia.segador@ine.es

P. Miranda

Interdisciplinary Mathematical Institute
Complutense University of Madrid
Plaza de Ciencias, 3, 28040 Madrid (Spain)
pmiranda@mat.ucm.es

Received (received date)
Revised (revised date)

In this paper we study some geometrical properties of the polytope of 3-tolerant fuzzy measures. To achieve this task, we profit that this polytope is an order polytope and hence we can extract many properties from the subjacent poset. The main result in the paper is a straightforward procedure for obtaining a random 3-tolerant fuzzy measure. We also compute the volume and obtain some other properties of this polytope. These results can be also applied by duality to the polytope of 3-intolerant measures and they can also be easily extended to other subfamilies of fuzzy measures.

Keywords: 3-tolerant fuzzy measures; 3-interactive fuzzy measures; order polytope; random generation

1. Introduction

Fuzzy measures[28] (also known as capacities[8] or non-additive measures[13]) and Choquet integral[8] have revealed themselves as a very important tool for dealing with situations appearing in Multicriteria Decision Making and other fields.[17] This is due to the fact that they are very flexible and thus able to model situations appearing in real problems. For example, in Multicriteria Decision Making, fuzzy measures can deal with the existence of interactions among criteria, or situations of veto and favor.[15,16]

On the other hand, this ability has its counterpart in the computational complexity. In this sense, $2^n - 2$ coefficients are needed to define a fuzzy measure over a referential of n elements, while only $n-1$ suffice to define a probability. Hence, the complexity grows exponentially and this reduces the practical applicability of fuzzy measures when the referential set is large. In order to reduce this complexity, several attemps have been made for different contexts, defining subfamilies of fuzzy measures with a reduced complexity. Examples of these subfamilies are k-additive measures,[16] λ-measures,[28] k-symmetric measures[25] or k-interactive measures,[4] to cite a few. From a geometrical point of view, the set of fuzzy measures and the set of all fuzzy measures inside many of these subfamilies can be seen as (convex) polytopes.[9, 10]

In this paper we deal with the set of k-tolerant (and k-intolerant measures) defined in.[21] This subfamily models situations in which the decision maker is more or less tolerant to bad scores. Hence, k-tolerant measures mean that the decision maker shall be satisfied (and hence the corresponding Choquet integral shall attend a high score) with an alternative that is good for at least k criteria. Similarly, k-intolerant measures imply that she shall not be satisfied if an alternative is not good for at least $n-k+1$ criteria. It can be seen that k-tolerant measures for different values of k determine a partition of the set of fuzzy measures. Moreover, the set of fuzzy measures being at most k-tolerant measures over a finite referential set is a convex polytope. Specially appealing is the case for $k=3$, that allows to model many situations arising in practice while keeping a reduced complexity.

In this paper, we study some of the properties of the polytope of 3-tolerant measures. In particular, we develop a procedure to generate randomly points in this polytope. Besides, we obtain the number of vertices, compute its hypervolume and derive some other properties regarding the combinatorial structure of this polytope.

Interesting from a mathematical point of view, these properties are also interesting from a practical point in view, more concretely in the practical identification of a 3-tolerant measure from sample data (see[3, 5, 9] for papers dealing with other cases in the field of fuzzy measures).

It should be noted that generating points in a random way in a polytope is a complex problem that has not been completely solved in a satisfactory way. There are several methods to deal with it, as using Markov chains,[18] the sweep-plane method,[19] the grid method,[14] and so on. The method that we will apply for 3-tolerant measures is the method of *triangulation*.[14] Basically, it consists in dividing the polytope in simplices, i.e. extensions of triangles in dimensions higher than 2 (triangulate), and apply that generating points in a simplex is very easy. Hence, the method divides the polytope in simplices, choose one of them with a probability proportional to its volume and then generate a point in the selected simplex. On the other hand, it is not easy in general to triangulate a polytope, and it is difficult to compute the volumes of the corresponding simplices. However, we will see in the paper that these problems can be solved for the polytope of 3-tolerant measures in a satisfactory way.

To achieve these tasks, we apply that this polytope belongs to a special class of polytopes known as order polytopes.[27] These polytopes have the advantage that they are defined in terms of a partially ordered set (poset) and thus, it is possible to study different properties from this poset, a problem usually simpler to solve. Order polytopes have been applied in several papers dealing with subfamilies of fuzzy measures, see for example[2] where order polytopes and their properties are applied to derive two alternative ways to generate random k-interactive measures.

Finally, note that determining the combinatorial structure of a family of polytopes is a difficult an interesting problem and many papers have been devoted just treating special cases.[1, 23, 29] And the same happens for the class of order polytopes. While there are general results for, say, triangulate this kind of polytopes, the practical use of these results is limited, as they are based on generating a linear extension of the poset in a random fashion, a problem that is known to be $\sharp$P-complete.[7] In this paper we give a simple and easy-to-apply algorithm to generate 3-tolerant measures that can be applied to any cardinality with a reduced (polynomial) complexity.

The rest of the paper goes as follows. In next section, we introduce the basic facts, results and notation. In Section 3 we give a simple way to generate randomly a 3-tolerant measure. Section 4 studies other properties of this polytope. In Section 5 we extend these results for other subfamilies of fuzzy measures. We finish with the conclusions and open problems.

2. Basic concepts

In this section we introduce the basic concepts that will be needed throughout the paper.

2.1. *k-tolerant measures*

Let us consider a finite referential set $X = \{x_1, ..., x_n\}$ of n elements (criteria, players, ...). Subsets of X are denoted by capital letters A, B and so on. We will denote by $\mathcal{P}(X)$ the set of subsets of X and by

$$\mathcal{P}^k(X) := \{A \subseteq X : |A| \leq k\}, \quad \mathcal{P}^k_*(X) := \mathcal{P}^k(X) \backslash \{\emptyset\}.$$

Definition 1.[28] A **fuzzy measure** is a map $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying

- $\mu(\emptyset) = 0, \mu(X) = 1$ (boundary conditions).
- $\mu(A) \leq \mu(B)$ when $A \subseteq B$ (monotonicity).

We will denote by $\mathcal{FM}(X)$ the set of all fuzzy measures over X. To define a fuzzy measure, it is necessary to give $2^n - 2$ coefficients. In order to cope with the problem of complexity while keeping a rich structure, several subfamilies of fuzzy measures have been proposed. One of these subfamilies is the family of k-tolerant measures (resp. k-intolerant measures).

Definition 2.[21] A fuzzy measure μ is k-**tolerant** if $\mu(A) = 1$ when $|A| \geq k$ and there exists A such that $|A| = k - 1$ and $\mu(A) < 1$.

Let us denote by $\mathcal{TOL}^k(X)$ the set of fuzzy measures on X being k'-tolerant, with $k' \leq k$. As $\mu(A) = 1$ if $|A| > k$ for $\mu \in \mathcal{TOL}^k(X)$, it follows that the number of coefficients needed to define μ reduces to

$$d := \sum_{i=1}^{k} \binom{n}{i}.$$

In other words, the sets that are needed to define a k-tolerant measure are those in $\mathcal{P}_*^{k-1}(X)$. Moreover, remark that the convex combination of two measures in $\mathcal{TOL}^k(X)$ is still in $\mathcal{TOL}^k(X)$, so that if we identify $\mu \in \mathcal{TOL}^k(X)$ with the point $(\mu(A))_{|A|<k} \in \mathbb{R}^d$, then $\mathcal{TOL}^k(X)$ is a convex polytope.

2.2. *Posets*

For a general reference about this subsection, see.[12] A **finite partially ordered set** (brief **poset**) is a pair $(P, \preceq)$ (or P if $\preceq$ is known) where P is a finite set and $\preceq$ is a relation on P that is reflexive, antisymmetric and transitive. We will denote the elements of P as $x, y, ...$ Posets can be represented through *Hasse diagrams* (see Figure 1 left).

If given $x, y \in P$, either $x \preceq y$ or $y \preceq x$, we say that $\preceq$ is a *total order* and P is said to be a *chain*. The chain of n elements is denoted $\boldsymbol{n}$. If none of the elements of P are related, we say that P is an *antichain* and we denote the antichain of n elements by $\overline{\boldsymbol{n}}$.

The elements $x \in P$ satisfying that $y \not\preceq x$, $\forall y \neq x$, are called *minimal elements* of P. The set of minimal elements of P will be denoted by $\mathcal{MIN}(P)$.

Given a poset, a **filter** or **upset** F is a subset of P such that for $x, y \in P$, if $x \in F$ and $x \preceq y$ this implies $y \in F$. Similarly, an **ideal** or **downset** I of P is a subset such that $x \in I$ and $y \preceq x$ implies $y \in I$. The set of all filters (resp. ideals) of poset P is denoted by $\mathcal{F}(P)$ (resp. $\mathcal{I}(P)$). Note that for $I \in \mathcal{I}(P)$, it follows that $P \setminus I \in \mathcal{F}(P)$. Hence,

$$|\mathcal{I}(P)| = |\mathcal{F}(P)|. \tag{1}$$

It is well-known that $(\mathcal{I}(P), \subseteq)$ is a lattice. An example of the Hasse diagram of $(\mathcal{I}(P), \subseteq)$ is given in Figure 1 right.

Given two disjoint posets, $(P, \preceq_P)$ and $(Q, \preceq_Q)$, we define the **disjoint union**, $P \biguplus Q$ as the poset $(P \cup Q, \preceq_{P \biguplus Q})$, where $x \preceq_{P \biguplus Q} y$ if $x, y \in P$ and $x \preceq_P y$ or $x, y \in Q$ and $x \preceq_Q y$. And example of this operation is given in Figure 2.

Definition 3. Given a poset $(P, \preceq)$, a **linear extension** of this poset is a total order $(P, \leq)$ such that if $x \preceq y$, then $x \leq y$.

A linear extension is a total order on P extending $\preceq$. We will denote by $\mathcal{L}(P)$ the set of all linear extensions of poset P and $e(P) := |\mathcal{L}(P)|$.

When dealing with disjoint union of posets, the following result holds for the number of linear extensions:

Lemma 1. *Given two posets P, Q, it follows that*

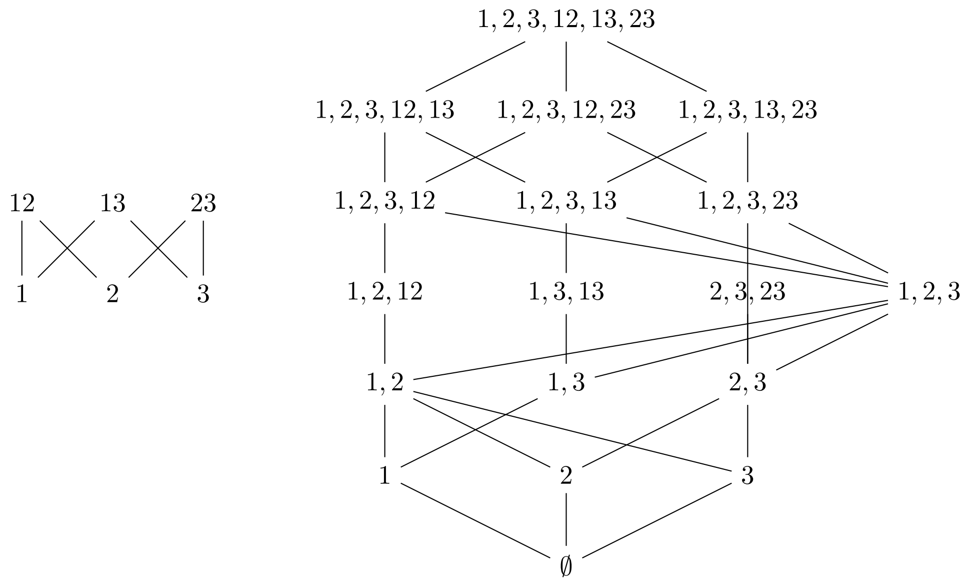

Fig. 1. Hasse diagram of $(\mathcal{P}^2_*(X), \subseteq)$ for a referential of three elements (left) and the corresponding ideal lattice $(\mathcal{I}(\mathcal{P}^2_*(X)), \subseteq)$ (right).

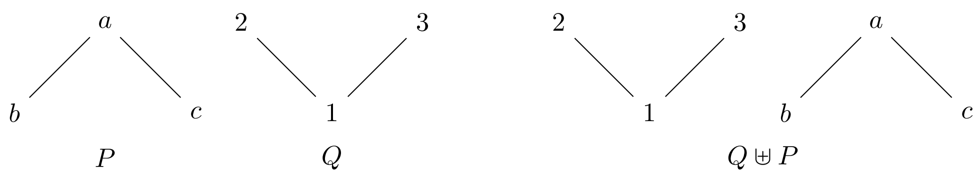

Fig. 2. Disjoint union of posets.

$$e(P \biguplus Q) = \binom{|P| + |Q|}{|P|} e(P)e(Q).$$

2.3. *Order polytopes*

Let us now deal with the notion of order polytope.

Definition 4.[27] Let $(P, \preceq)$ be a poset with p elements. We define the **order polytope** associated to P as the set of points $f \in \mathbb{R}^p$ ordered by the elements of P satisfying

- $0 \leq f(x) \leq 1, \forall x \in P$.
- $f(x) \leq f(y)$ if $x \preceq y$.

There are many polytopes appearing in the Theory of Fuzzy Measures that are order polytopes. For example, it has been shown in[11] that $\mathcal{FM}(X)$ is the order polytope $\mathcal{O}(\mathcal{P}(X)\backslash\{X, \emptyset\})$, where $A \preceq B$ if and only if $A \subseteq B$. Similarly, the set of normalized monotone games with restricted cooperation is an order polytope, no matter the set of feasible coalitions.[24] And the set of fuzzy measures being at most k-symmetric, too.[11]

Now, consider a fuzzy measure $\mu \in \mathcal{TOL}^k(X)$. Then, as it suffices to define μ on subsets in $\mathcal{P}_*^{k-1}(X)$ (see Fig. 1 left), it follows that μ is characterized as a set function satisfying

- $0 \leq \mu(A) \leq 1, \quad \forall A \in \mathcal{P}_*^k(X)$.
- $\mu(A) \leq \mu(B)$ if $A \subseteq B, A, B \in \mathcal{P}_*^k(X)$.

Therefore,

$$\mathcal{TOL}^k(X) = \mathcal{O}(\mathcal{P}_*^{k-1}(X)).$$

We will study $\mathcal{TOL}^3(X)$ and other related polytopes at the light of this result. Order polytopes have the advantage that the combinatorial structure of the polytope can be studied in terms of the subjacent polytope, usually a simpler problem. For example, vertices of an order polytope are characterized as follows.

Theorem 1.[27] *The vertices of the order polytope $\mathcal{O}(P)$ are the characteristic functions of filters of P.*

For determining a face of a polytope, it is convenient to add to P a top and bottom elements $\top, \bot$. Let us denote by $\hat{P}$ this extended poset. Hence, an order polytope can be written as the set of points satisfying

- $f(\top) = 1$, $f(\bot) = 0$.
- If $x \preceq y$, then $f(x) \leq f(y)$, $\forall x, y \in \hat{P}$.

Next, to determine a face of a polytope we need to turn some inequalities defining the polytope into equalities. For order polytopes, this means to consider some $f(x) = f(y)$. Hence, a face is characterized by a partition $\mathfrak{B} := \{A_\top, A_\bot, A_1, ..., A_r\}$ of $\hat{P}$ such that if f belongs to the face, then $f(x) = f(y)$ whenever x, y are in the same block. $A_\top, A_\bot$ represent the blocks containing $\top$ and $\bot$ and f on $A_\top$ (resp. $A_\bot$) is fixed to $f(A_\top) = 1$ (resp. $f(A_\bot) = 0$). However, it should be noted that it is not true that any partition determines a face and it is necessary to impose additional conditions.[27]

A partition $\mathfrak{P} = \{A_\top, A_\bot, A_1, ..., A_r\}$ of P is **connected** if all A_i are connected suposets of $\hat{P}$.

Let us define the relation $\preceq_{\mathfrak{P}}$ on $\{A_\top, A_\bot, A_1, ..., A_r\}$ by

$$A_i \preceq_{\mathfrak{P}} A_j \Leftrightarrow \exists x \in A_i, y \in A_j, x \preceq y.$$

A partition $\mathfrak{P}$ is **compatible** if $\preceq_{\mathfrak{P}}$ is reflexive, antisymmetric and transitive (i.e. a partial order). Finally, a partition $\mathfrak{P}$ is **closed** if for any $A_i, A_j, i \neq j$, there exists $f \in \mathcal{O}(P)$ constant on each block such that $f(A_i) \neq f(A_j)$.

Theorem 2.[27] *A partition $\{A_\top, A_\perp, A_1, \ldots, A_r\}$ of $\widehat{P}$ is closed and determines a r-dimensional face of $\mathcal{O}(P)$ if and only if it is compatible and connected.*

Moreover, it can be proved that faces of order polytopes are affinely isomorphic to order polytopes (see[27]). Indeed, the following holds.

Theorem 3. *Let P be a finite poset and $\mathcal{F}$ be a face of $\mathcal{O}(P)$ with associated face partition $\mathcal{B}(\mathcal{F}) = \{B_\top, B_\perp, B_1, \cdots, B_r\}$. Then $(\mathcal{B}(\mathcal{F}), \preceq_{\mathcal{B}(\mathcal{F})})$ is a poset and $\mathcal{F} \cong \mathcal{O}\left((\mathcal{B}(\mathcal{F}) \setminus \{B_\top, B_\perp\}, \preceq_{\mathcal{B}(\mathcal{F})})\right)$.*

Similarly, it is possible to find an appealing condition to determine if two vertices are adjacent in an order polytope.

Theorem 4.[11] *Given two vertices of $\mathcal{O}(P)$ whose corresponding filters are F_1 and F_2, they are adjacent if and only if $F_1 \subseteq F_2$ and $F_2 \backslash F_1$ is a connected subposet of P.*

A final result on faces will be applied in the paper.

Lemma 2. *Poset $(P, \preceq)$ is a chain if and only if $\mathcal{O}(P)$ is a simplex. Similarly, $(P, \preceq)$ is an antichain if and only if $\mathcal{O}(P)$ is a hypercube.*

Let us finally treat the problem of triangulating an order polytope. For $\mathcal{O}(P)$ it is possible to build a triangulation based on the following result (see,[22] pag. 304):

Theorem 5. *Let $(P, \preceq)$ be a poset of p elements.*

- *If $\preceq$ is a total order on P, then the corresponding order polytope is a simplex of volume $\frac{1}{p!}$.*
- *For any partial ordering $\preceq$ on P, the simplices of the order polytope of $(P, \leq)$, where $\leq$ is a linear extension of $\preceq$, cover $\mathcal{O}(P)$ and have disjoint interiors. Consequently, $vol(\mathcal{O}(P)) = \frac{1}{p!}e(P)$.*

These results are also outlined in.[27] Consequently, in order to generate randomly a point in an order polytope, it suffices to generate randomly a linear extension of $(P, \preceq)$ and then generate a point in the corresponding simplex. Note however that the problem of generating a random linear extension of a general poset is a $\sharp$ P-complete problem.[6]

3. Random generation on $\mathcal{TOL}^3(X)$

As stated in Theorem 5, triangulating an order polytope is deeply related to the problem of generating random linear extensions of the subjacent poset. Now, given a linear extension $(x_1, ..., x_p)$ remark that $x_1 \in \mathcal{MIN}(P)$; next, $x_2 \in \mathcal{MIN}(P \backslash \{x_1\})$

and in general, $x_k \in \mathcal{MIN}(P\backslash\{x_1, ..., x_{k-1}\})$. As a consequence, $\{x_1, ..., x_k\} \in \mathcal{I}(P), \forall k = 1, ..., n$. Therefore, a linear extension can be identified to paths from the empty ideal $\emptyset$ to the whole ideal P in the Hasse diagram of $\mathcal{I}(P)$. This is the idea for the algorithm proposed in[20] for generating a random linear extension. Hence, starting with the empty ideal, and assuming that $I = \{x_1, ..., x_{i-1}\}$ is the ideal of elements that have been already selected, the algorithm picks as next element in the linear extension a minimal element of $P\backslash I$.

However, in order to get a random linear extension, this new element has to be selected in a way that all linear extensions are equally probable. To compute these probabilities we note that

$$e(P) = \sum_{x \in \mathcal{MIN}(P)} e(P \setminus \{x\}). \tag{2}$$

For $x_i \in \mathcal{MIN}(P \setminus I)$ and denoting $P(x_i|I)$ the probability of selecting x_i when I is the ideal of elements already selected, this leads to

$$P(x_i|I) = \frac{e(P\backslash(I \cup x_i))}{e(P\backslash I)}. \tag{3}$$

Hence, the probabilities $P(x_i|I)$ can be stated in terms of the number of linear extensions. In,[20] a recursive way of obtaining $e(F), \forall F \in \mathcal{F}(P)$ is given. Remark that the whole procedure relies on the fact that the lattice of ideals is known. And for this, it is necessary to compute all ideals in $\mathcal{I}(P)$. However, $|\mathcal{I}(P)|$ usually grows very fast with the cardinality of P. Therefore, this procedure is unfeasible in general for big posets.

Related to $\mathcal{TOL}^3(X)$, in next result we compute $|\mathcal{I}(P)|$ for $P = \mathcal{P}_*^2(X)$, showing that this number grows exponentially with $n = |X|$.

Proposition 1. *The number of ideals of $\mathcal{TOL}^3(X)$ for $|X| = n$ is given by*

$$|\mathcal{I}(P)| = \sum_{i=0}^{n} 2^{\binom{n-i}{2}} \binom{n}{i}.$$

Proof. Applying Eq. (1), it suffices to compute the number of filters of $\mathcal{P}_*^2(X)$.

Let F be a filter and assume that there are exactly k singletons in $F, k = 0, 1, ..., n$. Hence, all pairs containing any of these singletons are in F. There are $\binom{n-k}{2}$ pairs that do not contain any of them and hence, for fixed singletons, there are $2^{\binom{n-k}{2}}$ possible filters containing exactly these k singletons, depending on if any of the $\binom{n-k}{2}$ pairs is in the filter or not. Hence, varying the set of k singletons, the number of filters in this conditions is given by

$$2^{\binom{n-k}{2}} \binom{n}{k}.$$

Summing up all these quantities for $k = 0, 1, ...n$, the result holds. □

Table 1 provides the number of ideals of $\mathcal{P}_*^2(X)$ for the first values of $|X|$.

$\|X\|$	$\|\mathcal{F}(\mathcal{P}_*^2(X))\|$
1	2
2	5
3	18
4	113
5	1 450
6	40 069
7	2 350 594
8	286 192 504
9	71 213 783 696
10	35 883 905 263 770

Table 1. Number of filters of $\mathcal{P}_*^2(X)$ for several values of $|X|$.

Hence, we need an alternative way to generate linear extensions in $\mathcal{P}_*^2(X)$. For this, the cornerstone result is the following:

Lemma 3. *Let P be a finite poset, $I \in \mathcal{I}(P)$, and $x, y \in \mathcal{MIN}(P \setminus I)$. Then,*

$$P(x|I) \cdot P(y|(I \cup \{x\})) = P(y|I) \cdot P(x|(I \cup \{y\})). \tag{4}$$

Proof. It follows almost trivially applying Eq. (3):

$$\begin{aligned} P(x|I) \cdot P(y|I \cup \{x\}) &= \frac{e\left(P \setminus (I \cup \{x\})\right)}{e(P \setminus I)} \cdot \frac{e\left(P \setminus (I \cup \{x,y\})\right)}{e(P \setminus (I \cup \{x\})} = \frac{e\left(P \setminus (I \cup \{x,y\})\right)}{e(P \setminus I)} = \\ &= \frac{e\left(P \setminus (I \cup \{y\})\right)}{e(P \setminus I)} \cdot \frac{e\left(P \setminus (I \cup \{x,y\})\right)}{e(P \setminus I \cup \{y\})} = P(y|I) \cdot P(x|I \cup \{y\}). \end{aligned}$$

Therefore, the result holds. □

As a consequence, given $I \in \mathcal{I}(P)$ and $\{x_1, ..., x_r\} =: \mathcal{MIN}(P \setminus I)$, we can compute the values $P(x_i|I), i = 1, ..., r$ if we know the probabilities $P(x_1|(I \cup \{x_j\}))$ and $P(x_j|(I \cup \{x_1\})), j \neq 1$ by solving the linear system

$$P(x_1|I) \cdot P(x_i|(I \cup \{x_1\})) = P(x_i|I) \cdot P(x_1|(I \cup \{x_i\})) \tag{5}$$

$$\sum_{i=1}^{r} P(x_i|I) = 1 \tag{6}$$

Moreover, as $P(x|(P \setminus \{x\})) = 1$ if $x \in \mathcal{MAX}(P)$, we can solve the previous systems in a recursive way.

Although the previous result seems to solve the problem, remark that it is necessary to solve a system for each $I \in \mathcal{I}(P)$ and by Proposition 1, we know that $|\mathcal{I}(P)|$ grows exponentially for $\mathcal{P}^2_*(X)$.

The other main idea of our procedure comes from the fact that, although there are many ideals in $\mathcal{P}^2_*(X)$, many of them are isomorphic, and hence, the number of situations that *need* to be considered for $\mathcal{P}^2_*(X)$ is very reduced. For convenience, we will treat the case considering filters instead of ideals.

Lemma 4. *Let F be a filter of $\mathcal{P}^2_*(X)$ and consider two singletons $\{x\}, \{y\}$ in F. Then,*

$$F \setminus \{x\} \cong F \setminus \{y\},$$

where $\cong$ denotes that the posets are isomorphic. Similarly, for two pairs $\{x, y\}, \{z, t\} \in \mathcal{MIN}(F)$,

$$F \setminus \{x, y\} \cong F \setminus \{z, t\}.$$

Proof. We will make the proof for singletons. The proof for pairs is similar. Let $\phi : F \setminus \{x\} \to F \setminus \{y\}$ be a bijective map such that it interchanges x and y, i.e. for $A \in F \setminus \{x\}$, it is defined by

$$\phi(A) = \begin{cases} A, & \text{ify} \not\in \text{A} \\ (A \setminus \{y\}) \cup \{x\}, & \text{otherwise} \end{cases}$$

Then, ϕ is well-defined and bijective. As $A \subseteq B \Leftrightarrow \phi(A) \subseteq \phi(B)$, then ϕ is an isomorphism and we conclude that $F \setminus \{x\} \cong F \setminus \{y\}$. □

Corollary 1. *Let F be a filter of $\mathcal{P}^2_*(X)$, and consider two singletons $\{x\}$ and $\{y\}$ in F. Then,*

$$P(\{x\}|(P \setminus F)) = P(\{y\}|(P \setminus F)).$$

Similarly, for two pairs $\{x, y\}, \{z, t\} \in \mathcal{MIN}(F)$, it holds

$$P(\{x, y\}|(P \setminus F)) = P(\{z, t\}|(P \setminus F)).$$

Proof. It suffices to remark that by Lemma 4, $F \setminus \{x\} \cong F \setminus \{y\}$ and $F \setminus \{x, y\} \cong F \setminus \{z, t\}$, so it follows

$$e(F \setminus \{x\}) = e(F \setminus \{y\}), \qquad e(F \setminus \{x, y\}) = e(F \setminus \{z, t\}).$$

Now, the result holds applying Eq. (3). □

Next, let us determine the form of the filters leading to *different* systems.

Lemma 5. *Let F_1 and F_2 be two filters of $\mathcal{P}^2_*(X)$. Then $F_1 \cong F_2 \Leftrightarrow F_1$ and F_2 have the same number of singletons and pairs.*

Proof. Let us denote

$$F_1 = \{x_1, \ldots, x_s, A_1, \ldots, A_l, B_1, \ldots, B_p\},$$

where $x_1, \ldots, x_s$ are the singletons in F_1, $A_1, \ldots, A_l$ are the pairs in F_1 such that $A_i \cap \{x_1, \ldots, x_s\} \neq \emptyset$, and $B_1, \ldots, B_p$ are the pairs in F_1 such that $B_i \cap \{x_1, \ldots, x_s\} = \emptyset$.

Thus defined, denoting $\uparrow x_i := \{\{x_i\}, \{x_i, x_j\} : j \neq i\}$, it follows that filter F_1 can be written as $F_1 = G_1 \uplus H_1$, where

$$G_1 := \uparrow x_1 \cup \uparrow x_2 \cup \cdots \cup \uparrow x_s = \{x_1, \ldots, x_s, A_1, \ldots, A_l\}$$

and $H_1 := \{B_1, \ldots, B_p\}$. Similarly, $F_2 = G_2 \uplus H_2$.

Now, $F_1 \cong F_2 \Leftrightarrow G_1 \cong G_2$ and $H_1 \cong H_2$. And as H_1 and H_2 are antichains, it follows that $H_1 \cong H_2 \Leftrightarrow |H_1| = |H_2|$.

Let us now show that $G_1 \cong G_2$ if and only if they have the same number of singletons.

$\Rightarrow$) If $G_1 \cong G_2$, then they have exactly the same number of minimal elements, and minimal elements for G_1 and G_2 are exactly the singletons.

$\Leftarrow$) Let us denote by $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_s\}$ the singletons in G_1 and G_2, respectively. Hence, we can build a bijective map in X applying $\{x_1, \ldots, x_s\}$ into $\{y_1, \ldots, y_s\}$. For example, consider $f : X \to X$ defined as

$$f(z) := \begin{cases} y_i & \text{if } z = x_i \\ x_i & \text{if } z = y_i \\ z & \text{otherwise} \end{cases}$$

Hence, for $A \in G_1$, it follows that $f(A) \in G_2$ and f is a bijection. Besides, f keeps the containing condition. Thus, f is an isomorphism and $G_1 \cong G_2$.

As $G_1 \cong G_2$ implies that the number of pairs is the same for both posets, the result holds. □

As all filters having the same number of singletons and the same number of pairs are isomorphic, from now on we will denote by $F(t_1, t_2)$ a (general) filter having t_1 singletons and t_2 pairs. Next proposition computes the number of linear systems to be solved.

Proposition 2. *The number of non-isomorphic filters of $\mathcal{P}^2_*(X)$ when $|X| = n$ is*

$$\binom{n+1}{3} + n + 1.$$

Proof. It suffices to study the number of different filters $F(t_1, t_2)$. Obviously, $0 \leq t_1 \leq n$. Now, for fixed t_1, if we consider all the possible $\binom{n}{2}$ pairs and take away the $\binom{n-t_1}{2}$ pairs without any of the t_1 singletons, we get that the number of pairs covering at least one of the t_1 singletons is $\binom{n}{2} - \binom{n-t_1}{2}$. Consequently,

$$0 \leq t_1 \leq n, \quad \binom{n}{2} - \binom{n-t_1}{2} \leq t_2 \leq \binom{n}{2}.$$

Now, since all combinations (t_1, t_2) satisfying these bounds are valid, the number of possible $F(t_1, t_2)$ is given by

$$\begin{aligned}\sum_{t_1=0}^{n} \left[\binom{n}{2} - \left(\binom{n}{2} - \binom{n-t_1}{2} \right) + 1 \right] &= \sum_{t_1=0}^{n} \binom{n-t_1}{2} + n + 1 \\ &= \sum_{h=2}^{n} \binom{h}{2} + n + 1 \\ &= \binom{n+1}{3} + n + 1.\end{aligned}$$

Hence, the result holds. □

Table 2 provides the first values for the number of non-isomporphic filters. Compare its cubic growth with the exponential growth of the number of filters given in Table 1.

$\lvert X\rvert$	2	3	4	5	6	7	8	9	10
Filters	4	8	15	26	42	64	93	130	176

Table 2. Number of non-isomorphic filters of $\mathcal{P}^2_*(X)$ for several values of $|X|$.

Finally, let us show that the solution of any system for each possibility (t_1, t_2) can be written explicitly in terms of t_1 and t_2.

Theorem 6. *Let $F(t_1, t_2)$ be a filter of $\mathcal{P}^2_*(X)$, and let us denote $I(t_1, t_2) := \mathcal{P}^2_*(X) \setminus F(t_1, t_2)$ its associated ideal. Let $\{x\}$ be a singleton of $F(t_1, t_2)$ and $\{y, z\}$ a pair of $\mathcal{MIN}(F(t_1, t_2))$. Then:*

$$P(\{x\}|I(t_1, t_2)) = \frac{t_1 + \binom{n}{2} - \binom{n-t_1}{2}}{t_1(t_1 + t_2)}, \qquad P(\{y, z\}|I(t_1, t_2)) = \frac{1}{t_1 + t_2}.$$

Proof. As we know by last results, $F(t_1, t_2)$ has t_1 interchangeable singletons, all of them minimals. A pair is minimal for $F(t_1, t_2)$ if and only if it does not contain any singleton of $F(t_1, t_2)$. Hence, there are $T := t_2 + \binom{n-t_1}{2} - \binom{n}{2}$ minimal interchangeable pairs. Denote $\alpha := P(\{x\}|I(t_1,t_2))$ and $\beta := P(\{y,z\}|I(t_1,t_2))$ where $\{y,z\} \in \mathcal{MINF}(t_1,t_2)$. Hence,

$$t_1\alpha + T\beta = 1.$$

As $F(t_1,t_2) \cong F(t_1,t_2-T) \uplus \overline{\boldsymbol{T}}$, it follows by Eq. (2) that

$$\beta = P(\{y,z\}|I(t_1,t_2)) = \frac{e(F(t_1,t_2)\setminus\{\{y,z\}\})}{e(F(t_1,t_2))} = \frac{e(F(t_1,T-t_2)\uplus\overline{\boldsymbol{T-1}})}{e(F(t_1,T-t_2)\uplus\overline{\boldsymbol{T}})}.$$

Now, applying Lemma 1, we obtain

$$\beta = \frac{\binom{|F(t_1,T-t_2)|+T-1}{T-1}e(F(t_1,T-t_2))(T-1)!}{\binom{|F(t_1,T-t_2)|+T}{T}e(F(t_1,T-t_2))T!} = \frac{1}{|F(t_1,T-t_2)|+T}.$$

Since $T + |F(t_1,t_2-T)| = t_1+t_2$, we conclude

$$\beta = \frac{1}{t_1+t_2}.$$

Finally, from $t_1\alpha + T\beta = 1$ we obtain the value of α,

$$\alpha = \frac{1-\beta T}{t_1} = \frac{1-\frac{1}{t_1+t_2}\left(t_2+\binom{n-t_1}{2}-\binom{n}{2}\right)}{t_1} = \frac{t_1+\binom{n}{2}-\binom{n-t_1}{2}}{t_1(t_1+t_2)}.$$

Therefore, the result holds. □

Theorem 6 provides us with a general way to generate a random linear extension of $\mathcal{P}_*^2(X)$. This is Algorithm 1.

And applying Theorem 5, generating linear extensions of $\mathcal{P}_*^2(X)$ allows us to generate a random measure in $\mathcal{TOL}^3(X)$. This is Algorithm 2.

Proposition 3. *Let $|X| = n$, then the computational complexity of Algorithm 1 and Algorithm 2 is $O(n^4)$.*

Proof. Let us start computing the complexity of each part of Algorithm 1. The number of minimals in $\mathcal{P}_*^2(X)\setminus I$ will be in the worst case $\binom{n}{2}$ (taking all the pairs) and therefore the complexity of this part is quadratic at most. Calculating the probabilities and selecting an element at random has complexity $O(1)$. Now we must repeat this procedure for each element in the path, i.e. a quadratic number of times. Therefore, the complexity of Algorithm 1 will be $O(n^2 \cdot n^2) = O(n^4)$.

Algorithm 1 RANDOM GENERATION OF LINEAR EXTENSIONS OF $\mathcal{P}^2_*(X)$

INIZIALIZATION

$I = \emptyset, \quad index = 1, \quad nsing = n, \quad npair = \binom{n}{2}.$

while $I \neq P$ **do**

COMPUTING MINIMALS

Compute $\mathcal{MIN}(P \backslash I)$.

Assign $k \equiv$ number of singletons in $\mathcal{MIN}(P \backslash I) = nsing$.

Assign $q \equiv$ number of pairs in $\mathcal{MIN}(P \backslash I) = npair + \binom{n-k}{2} - \binom{n}{2}$.

COMPUTING PROBABILITIES

A pair is chosen with probability

$$P(pair) = q \times \frac{1}{npair + nsing}.$$

A singleton is chosen with probability $P(sing) = 1 - P(pair)$.

SELECTING ELEMENT

If a singleton is selected, select randomly a singleton $\{x\}$ in $\mathcal{MIN}(P \backslash I)$.

$$e(index) = \{x\}, \quad I = I \cup \{x\}, \quad nsing = nsing - 1.$$

If a pair is selected, select randomly a pair $\{x, y\}$ in $\mathcal{MIN}(P \backslash I)$.

$$e(index) = \{x, y\}, \quad I = I \cup \{x, y\}, \quad npair = npair - 1.$$

Actualize $index = index + 1$.

end

return e.

Algorithm 2 starts using Algorithm 1. Next, a quadratic number of random numbers is generated, so this will be done in $O(n^2)$. Therefore, the final complexity of Algorithm 2 is also $O(n^4)$. □

Finally, as all linear extensions have the same probability of being selected, for a linear extension $(x_{(1)}, ..., x_{(p)}) \in \mathcal{L}(P)$, it follows that

$$P((x_{(1)}, ..., x_{(p)}) = \frac{1}{e(P)}.$$

This allows us to compute the number of linear extensions of $\mathcal{P}^2_*(X)$.

Theorem 7. *The number of linear extensions of $\mathcal{P}^2_*(X)$ is given by:*

$$e\left(\mathcal{P}^2_*(X)\right) = \binom{n}{2}! \prod_{i=1}^{n} \left(\frac{i(i + \binom{n}{2})}{i + \binom{n}{2} - \binom{n-i}{2}} \right).$$

Algorithm 2 RANDOM GENERATION OF 3-TOLERANT MEASURES

GENERATING A LINEAR EXTENSION

Apply Algorithm 1 to generate a linear extension $(x_{(1)}, ..., x_{(\binom{n}{2}+n)})$ of $\mathcal{P}_*^2(X)$.

GENERATING VALUES

Generate $\binom{n}{2} + n = \binom{n+1}{2}$ random values in $[0, 1]$.

Order the generated values in increasing order

$$v(1) \leq v(2) \leq ... \leq v(\binom{n+1}{2}).$$

GENERATING A 3-TOLERANT MEASURE

i=1

while $i \leq \binom{n+1}{2}$ **do**

Assign $\mu(x_{(i)}) = v(i), \quad i = i + 1.$

end

return μ.

Proof. It suffices to compute the probability of the linear extension starting with singletons (in any order) and then following by pairs (in any order) and apply the probabilities obtained in Theorem 6. □

Corollary 2. *Let X be a referential set of n elements. Then, the $(\binom{n}{2} + n)$-volume of $\mathcal{TOL}^3(X)$ is*

$$Vol(\mathcal{TOL}^3(X)) = \prod_{i=1}^{n} \left(\frac{i}{i + \binom{n}{2} - \binom{n-i}{2}} \right).$$

Proof. It suffices to apply Theorem 5. Hence, the volume is given by $\frac{e(P)}{[\binom{n}{2}+n]!}$ and thus,

$$\begin{aligned} Vol(\mathcal{TOL}_3(X)) &= \frac{\binom{n}{2}! \prod_{i=1}^{n} \left(\frac{i(i+\binom{n}{2})}{i+\binom{n}{2}-\binom{n-i}{2}} \right)}{\left[\binom{n}{2} + n\right]!} \\ &= \frac{\prod_{i=1}^{n} \left(\frac{i(i+\binom{n}{2})}{i+\binom{n}{2}-\binom{n-i}{2}} \right)}{\left[\binom{n}{2} + n\right] \times ... \times \left[\binom{n}{2} + 1\right]} \\ &= \prod_{i=1}^{n} \left(\frac{i}{i + \binom{n}{2} - \binom{n-i}{2}} \right). \end{aligned}$$

This finishes the proof. □

The first values of $e(\mathcal{P}^2_*(X))$ and $Vol(\mathcal{TOL}^3(X))$ are given in Table 3.

n	$e(\mathcal{P}^2_*(X))$	$Vol(\mathcal{TOL}^3(X)$
2	2	$\frac{2}{3!} = 0.333$
3	48	$\frac{48}{6!} = 0.066$
4	34 560	$\frac{34\,560}{10!} = 0.00952$
5	1 383 782 400	$\frac{1\,383\,782\,400}{15!} = 0.00106$

Table 3. First values of $e(\mathcal{P}^2_*(X))$ and $Vol(\mathcal{TOL}^3(X))$.

As the volume reduces very quickly, these values show us that generating points randomly in the $\left(\binom{n}{2} + n\right)$-hypercube until a point in $\mathcal{TOL}^3(X)$ is obtained is not an efficient way to generate random points.

4. Some properties of polytope $\mathcal{TOL}^3(X)$.

In this section we study some other properties of the polytope $\mathcal{TOL}^3(X)$. First, let us start computing the number of vertices of this polytope.

Corollary 3. *The number of vertices of* $\mathcal{TOL}^3(X)$ *when* $|X| = n$ *is*

$$\sum_{i=0}^{n} 2^{\binom{n-i}{2}} \binom{n}{i}.$$

Proof. Note that Theorem 1 establishes that the number of filters (and ideals) is indeed the number of vertices of the corresponding order polytope. Hence, the result follows from Proposition 1. □

Therefore, Table 1 provides the number of vertices of $\mathcal{TOL}^3(X)$ for the first values of $|X|$.

Next, k-dimensional faces are given in terms of a closed partition $\{B_\top, B_\perp, B_1, ..., B_k\}$ of $\hat{P}$ being connected and compatible. Hence, faces of $\mathcal{TOL}^3(X)$ can be found applying Theorem 2. In particular, the following holds.

Proposition 4. *Let us consider a finite referential* X *and consider* $\mathcal{TOL}^3(X)$. *Then, the number of facets is*

$$\frac{3n^2-n}{2}.$$

Proof. In this case, we have just to put together two elements of $\hat{P}$ in the same block in a way such that connectivity and compatibility are kept. There are three cases:

- A block containing a singleton i and a pair $\{i,j\}$. There are $n(n-1)$ possibilities.
- A singleton is included in $B_\perp$. There are n possibilities.
- A pair is included in $B_\top$. There are $\binom{n}{2}$ possibilities.

Summing up these quantities

$$\binom{n}{2}+n+n(n-1)=n\frac{3n-1}{2}=\frac{3n^2-n}{2},$$

and the result holds. □

Let us now deal with the k-dimensional faces. It can be proved as a consequence of Theorem 3 that the only possible 2-dimensional faces in an order polytope are squares and triangles. Next, the only possible 3-dimensional faces are triangular pyramids, triangular prisms, square pyramids and cubes. And for higher dimensions faces are just combinations of triangles and squares (2-dimensional cubes). Let us then study the form of hypercubes and simplices of $\mathcal{TOL}^3(X)$.

Proposition 5. *Let us consider $F_1,...,F_{2^r}$ a family of filters of $\mathcal{P}^2_*(X)$. Then, this family is the set of vertices of a r-dimensional cubical face of $\mathcal{TOL}^3(X)$ if and only if $\mathcal{P}^2_*(X)$ can be partitioned as $\{C_\top, C_\perp, C_1,...,C_r\}$ where C_i are connected $i=1,...,r$, $C_\top$ is a filter, $C_\perp$ is an ideal and for $\{x\}\in C_i$, then $\{x,y\}\notin C_j, j\in\{1,...,r\}\backslash i$ and such that each F_i can be written as*

$$F_i = C_\top \cup C_{i_1}\cup ...\cup C_{i_k}.$$

Proof. $\Rightarrow$) We already know that if $F_1,...,F_{2^r}$ determines a cubical face, then the corresponding partition $\mathfrak{B}=\{B_\top,B_\perp,B_1,...,B_r\}$ satisfies that $\{B_1,...,B_r\}$ is an antichain (Lemma 2).

As $\mathfrak{B}$ determines a face, it follows that B_i is connected, $i=1,...,r$. Moreover, F_i is constant on each block. Hence, each F_i can be written as

$$F_i = B_\top \cup B_{i_1}\cup ...\cup B_{i_k}.$$

As there are just 2^r possible combinations and we are considering 2^r filters, it follows that all combinations lead to a vertex. In particular, $B_\top\cup B_i$ is a filter. Hence, if $\{x\}\in B_i$, it follows that $\{x,y\}\in B_\perp\cup B_i$.

Finally, as $B_\top = F_i$ for some i, then $B_\top$ is a filter. And as $B_\top \cup B_1 \cup ... \cup B_r = F_j$ for some j, then $B_\perp$ is an ideal.

$\Leftarrow$) Remark that by the conditions on $C_i, i = 1, ..., r$, if $\{x\} \in C_i$, then $\{x, y\} \in C_i \cup C_\top$ ($\{x, y\} \notin C_\perp$ because it is an ideal). Thus, it follows that $C_1 \cup C_\top, ..., C_r \cup C_\top$ are filters. Hence,

$$F_i = C_\top \cup C_{i_1} \cup ... \cup C_{i_k}$$

is a filter for any possible combination because it is a union of filters. The partition that these filters determine on $\mathcal{P}^2_*(X)$ is $\{C_\top, C_\perp, C_1, ..., C_r\}$. Let us show that this partition is connected and compatible. Connectivity holds by hypothesis.

For compatibility, note that neither $C_i \succeq C_j$ nor $C_j \succeq C_i$ for $i, j = 1, ..., r$ by hypothesis.

If $C_\perp \succeq C_i$, this would imply that there exist $x, y \in X$ such that $\{x\} \in C_i, \{x, y\} \in C_\perp$ contradicting $C_\perp$ is an ideal.

Similarly, it can be seen that it is not possible $C_\top \preceq C_i$.

Then, $\{C_1, ..., C_r\}$ is an antichain and the result holds by Lemma 2. □

Next, let us study faces that are simplices. The following holds:

Proposition 6. *Consider $\mathcal{P}^2_*(X)$ and consider $\mathfrak{B} := \{B_\top, B_\perp, B_1, ..., B_r\}$ a partition of $\mathcal{P}^2_*(X)$. Then, $\mathfrak{B}$ determines a r-simplex face with vertices $F_0, ..., F_r$ if and only if $F_i := B_1 \cup ... \cup B_i \cup B_\top$, B_i is connected, $i = 1, ..., r$, and for all $i < j$, there exists $\{x\} \in B_j, \{x, y\} \in B_i$.*

Proof. $\Rightarrow$) As $\mathfrak{B}$ determines a face, it is a connected partition, so that B_i is connected, $i = 1, ..., r$.

Consider $F_0, ..., F_r$ the vertices of the simplex. Then, F_i and F_j are adjacent, for all i, j and hence, either $F_i \subset F_j$ or $F_j \subset F_i$. Consequently, these filters can be ranged as a chain $F_0 \subset F_1 \subset ... \subset F_r$.

On the other hand, F_i is constant (0 or 1) in each B_j. Hence, $B_j \subseteq F_i$ or $B_j \cap F_i = \emptyset,\ \forall i, j$.

Besides, as $F_i \subset F_{i+1}$, there exists F_j such that $B_i \subseteq F_j$ and j minimal. Then, $B_i \cap F_{j-1} = \emptyset, B_i \subseteq F_j \backslash F_{j-1} := C_j$. Consider $x \in C_j \setminus B_i$. It follows that $\{x\} \notin F_i,\ i < j$. Moreover, $\{x\} \in B_l$, for some $l \in \{\top, \perp, 1, ..., r\}$. If $l \neq i$, this implies that $B_i \cup B_l \subseteq F_j \subset F_{j+1} ...$ and $(B_i \cup B_l) \cap F_{j-1} = \emptyset$. These two facts contradict that $\mathfrak{B}$ is closed.

We then conclude that $B_i = C_j$ and hence, we can assume w.l.g. $C_i = B_i, i = 1, ..., r$. Finally, we also conclude that

$$F_i = B_1 \cup ... \cup B_i \cup B_\top, \quad i = 0, ..., r.$$

As $\mathfrak{B}$ determines a simplex, it follows that $\{B_1, ..., B_r\}$ is a chain by Lemma 2. Hence, either $B_i \preceq B_{i+1}$ or $B_i \succeq B_{i+1}$. Suppose $B_i \preceq B_{i+1}$. This implies

$$\exists x, y \in X,\ \{x\} \in B_i = C_i = F_i \setminus F_{i-1},\ \{x, y\} \in B_{i+1} = C_{i+1} = F_{i+1} \setminus F_i.$$

But then, $\{x\} \in F_i, \{x, y\} \notin F_i$, contradicting that F_i is a filter.

We conclude that $B_i \succeq B_{i+1}$ and this implies that $\exists x, y \in X, \{x\} \in B_{i+1}, \{x, y\} \in B_i$. As $\mathfrak{B}$ is compatible, $B_i \succeq B_j$ if $i < j$ and this implies that $\exists x, y \in X, \{x\} \in B_j, \{x, y\} \in B_i$.

$\Leftarrow$) Consider the filters $F_0, ..., F_r$ as defined in the theorem. Let us then show that they are the vertices of a face. As B_i is connected, it suffices to show the compatibility of $\mathfrak{B}$. Suppose $B_\perp \succeq B_i$. Then, $\exists x, y \in X, \{x\} \in B_i, \{x, y\} \in B_\perp$. But then, $\{x\} \in F_i \setminus F_{i-1}, \{x, y\} \notin F_i$, contradicting that F_i is a filter.

Similarly, we can show that $B_\top \not\preceq B_i$. As $B_i \succeq B_j, i < j$ by construction, it follows that $\mathfrak{B}$ is a connected and compatible partition that is a chain, so that the corresponding r-face is a simplex by Lemma 2 and its vertices are $F_0, ..., F_r$. $\square$

Given a polytope, its **adjacency graph** is defined as the graph with vertices the vertices of the polytope and where two vertices are joined by an edge if they are adjacent vertices in the polytope. Remember that the **distance** $d(\mu_1, \mu_2)$ between two vertices μ_1, μ_2 of a polytope is the number of edges of the shortest path between them in the adjacency graph. The **diameter** of the polytope is the maximum distance between two vertices. Let us study the diameter for $\mathcal{TOL}^3(X)$.

Proposition 7. *The diameter of $\mathcal{TOL}^3(X)$ is bounded by 6.*

Proof. We will see that any vertex of $\mathcal{TOL}^3(X)$ is at distance less or equal than 3 from the vertex whose filter is $\mathcal{P}^2_*(X)$. Let us consider a vertex and let us denote by F its corresponding filter. If $F \neq \mathcal{P}^2_*(X)$, then there exists a singleton $\{i_0\} \notin F$. Consider the filter

$$F_1 := F \cup \{\{i_0\}, \{i_0, j\} : j \neq i_0\}.$$

Hence, $F_1 \backslash F$ is connected, so that the corresponding vertices are adjacent. Let us now consider

$$F_2 := F_1 \setminus (\{\{j\} : \{j\} \in F_1\} \cup \{\{i_0, i\} : i \in X\}),$$

i.e. we remove all singletons and all pairs containing i_0. Hence, $F_1 \backslash F_2$ is connected. Consequently, the corresponding vertices are adjacent.

Finally, $\mathcal{P}^2_*(X) \backslash F_2$ is connected: consider $A \in \mathcal{P}^2_*(X) \backslash F_2$ and let us show that it can be connected to i_0. If $A = \{i, j\}$, then

$$\{i, j\} - \{i\} - \{i, i_0\} - \{i_0\}.$$

A similar path can be obtained for $A = \{i\}$. Hence, $d(F, \mathcal{P}_*^2(X)) \leq 3$ and the result holds. □

Proposition 8. *The diameter of $\mathcal{TOL}^3(X)$ is 6 if $|X| \geq 6$.*

Proof. By the previous proposition, it suffices to find two vertices at distance 6. Let us denote by $\mathcal{A}_2$ the set of pairs of $\mathcal{P}_*^2(X)$. Consider the vertices whose corresponding filters are

$$F_1 = \mathcal{A}_2\backslash\{\{1,2\},\{3,4\},\{5,6\}\}, \quad F_2 = \mathcal{A}_2\backslash\{\{1,3\},\{2,5\},\{4,6\}\}.$$

Consider F_1. Then, the set of filters adjacent to F_1 are (up to isomorphism):

- $\mathcal{A}_2\backslash\{\{1,2\},\{3,4\},\{5,6\},\{i,j\}\}$, i.e. another pair is removed from F_1.
- $\mathcal{A}_2\backslash\{\{3,4\},\{5,6\}\}$, i.e. a pair is added to F_1.
- $\{i\} \cup \mathcal{A}_2\backslash\{\{1,2\},\{3,4\},\{5,6\}\}$ with $i \neq 1,2,3,4,5,6$.
- $\{1\} \cup \mathcal{A}_2\backslash\{\{3,4\},\{5,6\}\}$.
- $\{1\} \cup \{2\} \cup \mathcal{A}_2\backslash\{\{3,4\},\{5,6\}\}$.

Now, the set of adjacent filters to one of these possibilities is:

- Two pairs are removed.
- One pair is removed and one is added.
- Two pairs are added.
- A singleton is added together with all pairs where this singleton appears and another pair is added/removed.
- Two singletons are added together with all pairs where these singletons appear.
- Two pairs, say $\{1,2\},\{3,4\}$ are added and $\{1\}$ and/or $\{2\}$ and $\{3\}$ and/or $\{4\}$ are added.
- First, a singleton is added together with all pairs where this singleton appears and next this singleton and some pairs containing it are removed.
- First, a couple of singletons say $\{1\}$, $\{2\}$ are added together with $\{1,2\}$ and next $\{1\}$ and/or $\{2\}$ are removed together with some pairs containing it/them.

On the other hand, the same can be done for F_2. As the pairs outside $\mathcal{A}_2$ are different for F_1 and F_2, it can be checked that none filter at two steps from F_1 is adjacent to a filter two steps from F_2. Hence, $d(F_1, F_2) \geq 6$. Indeed the distance is 6, for example considering the path

$$\mathcal{A}_2 \setminus \{\{1,2\},\{3,4\},\{5,6\}\} - \mathcal{A}_2 \setminus \{\{3,4\},\{5,6\}\} - \mathcal{A}_2 \setminus \{\{5,6\}\} - \mathcal{A}_2$$

$$-\mathcal{A}_2 \setminus \{\{1,3\}\} - \mathcal{A}_2 \setminus \{\{1,3\},\{2,5\}\} - \mathcal{A}_2 \setminus \{\{1,3\},\{2,5\},\{4,6\}\}.$$

□

For $|X| = 3, 4, 5$, it can be seen that the diameter of $\mathcal{TOL}^3(X)$ is 3, 4, 4, respectively.

To finish this section, we show that this polytope is combinatorial.

Definition 5.[26] A polytope is **combinatorial** if it satisfies the following conditions:

- All vertices are $\{0, 1\}$-valued.
- If two vertices x, y are not adjacent, then there are two other vertices u, v such that $x + y = u + v$.

Proposition 9. *The polytope $\mathcal{TOL}^3(X)$ is combinatorial.*

Proof. As $\mathcal{TOL}^3(X)$ is an order polytope, we know that vertices are $\{0, 1\}$-valued. Let us then study the second condition. Consider two filters F_1, F_2 such that the corresponding vertices are not adjacent. Assume w.l.g. that $F_2 \not\subset F_1$. We have two different cases:

- If $F_1 \not\subset F_2$, then consider $F_1 \cup F_2, F_1 \cap F_2 \in \mathcal{F}(\mathcal{P}^2_*(X))$. Hence, $F_1, F_2 \neq F_1 \cap F_2, F_1 \cup F_2$ and

$$I_{F_1}(A) + I_{F_2}(A) = I_{F_1 \cup F_2}(A) + I_{F_1 \cap F_2}(A), \forall A \in \mathcal{P}^2_*(X).$$

- Assume that $F_1 \subset F_2$ and $F_2 \setminus F_1$ is not connected. Let us denote by C one of the connected components of $F_2 \setminus F_1$. We consider $F_2 \setminus C$ and $F_1 \cup C$. Then,

$$I_{F_1}(A) + I_{F_2}(A) = I_{F_1 \cup C}(A) + I_{F_2 \setminus C}(A), \forall A \in \mathcal{P}^2_*(X).$$

  It suffices to show that $F_2 \setminus C$ and $F_1 \cup C$ are filters.
  Take $\{x\} \in F_2 \setminus C$ and suppose $\{x, y\} \notin F_2 \setminus C$. As $\{x\} \in F_2$, then $\{x, y\} \in F_2$. Hence $\{x, y\} \in C$, so that $\{x, y\} \notin F_1$. Hence, $\{x\} \notin F_1$ and thus $\{x\} \in C$ because C is a connected component of $F_2 \setminus F_1$, a contradiction.
  Similarly, take $\{x\} \in F_1 \cup C$. If $\{x, y\} \in F_1$, then $\{x, y\} \in F_1 \cup C$ and we are done. Otherwise, $\{x, y\} \notin F_1$, and thus $\{x\} \notin F_1$. Hence, $\{x\} \in C \subseteq F_2 \setminus F_1$. But then, $\{x, y\} \in F_2 \setminus F_1$ and hence $\{x, y\} \in C$ because C is a connected component of $F_2 \setminus F_1$.

Hence, the result holds. □

A graph is **Hamilton connected** if every pair of distinct nodes is joined by a Hamilton path. For combinatorial polytopes, the following can be shown:

Theorem 8.[26] *Let G be the graph of a combinatorial polytope. Then, G is either a hypercube or else is Hamilton connected.*

As $\mathcal{TOL}^3(X)$ is not an hypercube except for $|X| = 2$, we conclude that it is Hamilton connected. A Hamiltonian path between $\mathcal{P}^2_*(X)$ and $\emptyset$ for the graph of $\mathcal{TOL}^3(X)$ when $|X| = 3$ can be seen in Figure 3.

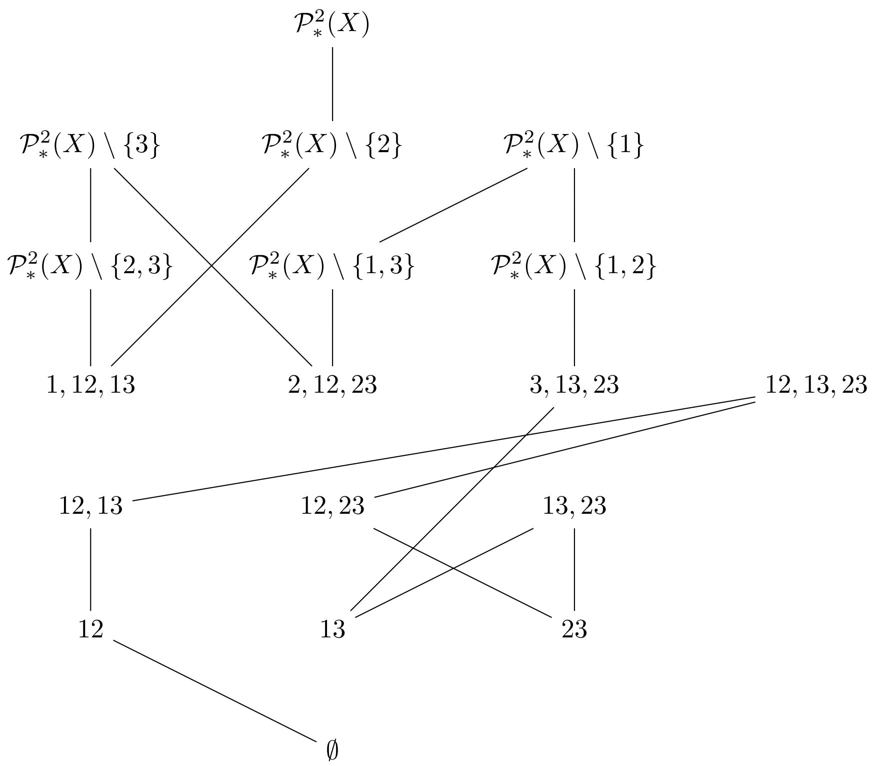

Fig. 3. Hamiltonian path between $\mathcal{P}^2_*(X)$ and $\emptyset$ in the adjacency graph of $\mathcal{TOL}^3(X)$.

5. Related subfamilies

The results developed in the previous sections can be straightforwardly extended to some other families of fuzzy measures. In this section we consider three other cases.

5.1. *3-intolerant measures*

Similarly to the case of k-tolerant measures, we can consider the case in which an object is rejectable if it is not satisfactory for at least $n-k+1$ criteria. To translate this into the language of fuzzy measures, the family of k-intolerant measures arises.

Definition 6.[21] A fuzzy measure μ is **k-intolerant** if $\mu(A) = 0$ when $|A| \leq n-k$ and there exists A such that $|A| = n-k+1$ and $\mu(A) > 0$.

Let us denote by $\mathcal{INTOL}^k(X)$ the set of fuzzy measures on X being k'-intolerant, with $k' \leq k$. Remark that for $\mu \in \mathcal{INTOL}^k(X)$, we just need to define the values of $\mu(A), |A| > n-k+1$. Hence, the same as for $\mathcal{TOL}^k(X)$, the polytope $\mathcal{INTOL}^k(X)$ can be seen as an order polytope whose corresponding poset is given by

$$\overline{\mathcal{P}_*^k(X)} := \{A \subseteq X : |A| \geq n-k+1, A \neq X\}.$$

It can be seen that the dual measure of a k-tolerant measure is a k-intolerant measure.[21] Hence, we can apply the results for generating a measure in $\mathcal{INTOL}^3(X)$ just applying the algorithm developed in Section 3 to generate a random measure in $\mathcal{TOL}^3(X)$ and then consider the corresponding dual measure. Moreover, the posets $\mathcal{P}_*^k(X)$ and $\overline{\mathcal{P}_*^k(X)}$ are dually isomorphic via the map

$$\begin{array}{rcl} \Phi : \mathcal{P}_*^k(X) & \rightarrow & \overline{\mathcal{P}_*^k(X)} \\ A & \mapsto & X \setminus A \end{array}.$$

Hence, the combinatorial structure of $\mathcal{TOL}^k(X)$ and $\mathcal{INTOL}^k(X)$ is the same and both polytopes have the same number of vertices, edges, diameter, ... Indeed, all the results for $\mathcal{TOL}^3(X)$ can be translated to $\mathcal{INTOL}^3(X)$ just changing the term filter (elements attaining value 1) for $\mathcal{P}_*^2(X)$ by the term ideal (elements attaining value 0) in $\overline{\mathcal{P}_*^2(X)}$.

5.2. *3-interactive measures*

Let us now treat the case of 3-interactive fuzzy measures. In many subfamilies of fuzzy measures, what is aimed is a reduction in the number of coefficients needed to define a measure because the goal is to reduce the number of coefficients that the decision maker should assign. However, if these values are not determined by an expert and they have to be obtained with the help of a sample, we have to take into account in the optimization problem that these values should satisfy the monotonicity constraints. In many subfamilies, the number of constraints dealing with monotonicity (that grows exponentially, indeed $n2^{n-1}$) does not reduce[4] even if there is a reduction in the number of coefficients.

The subfamily of k-interactive measures is specifically designed to deal with this problem. Indeed, for k-interactive measures, both the number of coefficients and the number of constraints needed for defining a fuzzy measure are reduced. This subfamily allows interactions for big cardinalities of the referential set, but these interactions are determined by the values of the fuzzy measure for small subsets. Besides, the corresponding Choquet integral of a k-interactive fuzzy measure has a nice and simple form. For more details about k-interactive fuzzy measures, see.[4]

Definition 7.[4] A fuzzy measure μ is **k-interactive** for $k < n$ if for some chosen $C \in [0, 1]$, it follows

$$\mu(A) = C + \frac{|A| - k - 1}{n - k - 1}(1 - C), \, \forall A, |A| > k.$$

Hence, for a k-interactive measure, the value for $\mu(A)$ is fixed for cardinalities greater than k. In particular, k-tolerant measures arise for $C = 1$. We will denote by $\mathcal{INT}^k_C(X)$ the set of fuzzy measures being at most k-interactive with fixed constant C. Thus, for fixed C, in order to determine a k-interactive measure, we only need to know the values of $\mu(A), 1 \leq |A| \leq k$ and these values satisfy monotonicity and lay in $[0, C]$. Consequently, the set $\mathcal{INT}^k_C(X)$ is the uniform scaling of $\mathcal{TOL}^k(X)$ by C for cardinality A s.t. $|A| \leq k$, and we can translate all the results derived in the previous sections for $\mathcal{TOL}^3(X)$ for the case $\mathcal{INT}^3_C(X)$.

For example, in order to generate a 3-interactive fuzzy measure in $\mathcal{INT}^3_C(X)$ in a random way, it suffices to generate a 3-tolerant measure μ following the algorithm developed in Section 3 and take the 3-interactive measure given by

$$\nu(A) = C\mu(A), \forall A, |A| \leq k, \nu(A) = C + \frac{|A| - k - 1}{n - k - 1}(1 - C), \, \forall A, |A| > k.$$

See also[2] for an alternative way to generate k-interactive measures based on order polytopes.

5.3. *2-truncated measures*

In many practical applications of fuzzy measures, some subsets B do not arise and thus, it makes no sense to consider the value $\mu(B)$. This is especially important in Game Theory, where X denotes the set of players and some coalitions may fail to form. It can be shown that in this case, the corresponding set of measures defined on a subset $\mathcal{S}$ of $\mathcal{P}(X)$ define an order polytope, no matter the structure of $\mathcal{S}$.[24]

Definition 8. We say that a fuzzy measure is k-**truncated** with respect to cardinalities $r_1, ..., r_k$ if the only subsets where μ is defined are those with cardinalities $r_1, ..., r_k$.

We denote the set of truncated fuzzy measures w.r.t. cardinalities $r_1, ..., r_k$ by $\mathcal{TFM}^k_{\{r_1,...,r_k\}}(X)$. This subsection is devoted to the case of 2-truncated fuzzy measures over sets of cardinalities r_1 and r_2, namely $\mathcal{TFM}^2_{\{r_1,r_2\}}(X)$. Especially we will study the case $r_1 = 1$. Let us denote by $\mathcal{P}^2_{r_1,r_2}(X)$ the Boolean poset restricted to the sets of size r_1 and r_2. It is obvious that $\mathcal{TFM}^2_{\{r_1,r_2\}}(X)$ is an order polytope whose associated poset is $\mathcal{P}^2_{r_1,r_2}(X)$. Here we are going to treat the problem of sampling uniformly capacities of $\mathcal{TFM}^2_{\{1,r_2\}}(X)$ by sampling linear extensions of $\mathcal{P}^2_{1,r_2}(X)$.

For this, we can adapt the proofs of Section 3, so that the following holds for $\mathcal{TFM}^2_{\{1,r_2\}}(X)$ and $\mathcal{P}^2_{1,r_2}(X)$.

Lemma 6. *Let F be a filter of $\mathcal{P}^2_{1,r_2}(X)$. If x, y are two singletons in F, then $F\backslash\{x\} \cong F\backslash\{y\}$. Similarly, if $A_1, A_2 \in \mathcal{MIN}(F), |A_1| = r_2 = |A_2|$, then $F\backslash\{A_1\} \cong F \setminus \{A_2\}$.*

Lemma 7. *Let F_1 and F_2 be two filters of $\mathcal{P}^2_{1,r_2}(X)$. Then, $F_1 \cong F_2 \Leftrightarrow F_1$ and F_2 have the same number of singletons and r_2-subsets.*

Let us denote by $F(t_1, t_2)$ a general filter having t_1 singletons and t_2 r_2-subsets.

Corollary 4. *The number of non-isomorphic filters of $\mathcal{P}^2_{1,r_2}(X)$ is $\binom{n+1}{r_2+1} + n + 1$.*

Theorem 9. *Let $F(t_1, t_2)$ be a filter of $\mathcal{P}^2_{1,r_2}(X)$ and let us denote by $I(t_1, t_2) := \mathcal{P}^2_{1,r_2}(X) \setminus F(t_1, t_2)$ its associated ideal. Let x be a singleton of $\mathcal{MIN}(F(t_1, t_2))$ and A a r_2-subset of $\mathcal{MIN}(F(t_1, t_2))$. Then:*

$$P(x|I(t_1,t_2)) = \frac{t_1 + \binom{n}{r_2} - \binom{n-t_1}{r_2}}{t_1(t_1+t_2)}, \qquad P(A|I(t_1,t_2)) = \frac{1}{t_1+t_2}.$$

With the last probabilities we can sample linear extensions of $\mathcal{P}^2_{1,r_2}(X)$ and therefore we can sample fuzzy measures in $\mathcal{TFM}^n_{\{1,r_2\}}(X)$. Finally, we can use the last result to get the number of linear extensions of $\mathcal{P}^2_{1,r_2}(X)$.

Theorem 10. *The number of linear extensions of $\mathcal{P}^2_{1,r_2}(X)$ and the volume of $\mathcal{TFM}^2_{1,r_2}(X)$ are given by:*

$$e\left(\mathcal{P}^2_{1,r_2}(X)\right) = \binom{n}{r_2}! \prod_{i=1}^{n} \frac{i\left(i + \binom{n}{r_2}\right)}{i + \binom{n}{r_2} - \binom{n-i}{r_2}}.$$

$$Vol(\mathcal{TFM}^2_{1,r_2}(X)) = \prod_{i=1}^{n} \frac{i}{i + \binom{n}{r_2} - \binom{n-i}{r_2}}.$$

The first values of $e(\mathcal{P}^2_{1,r_2}(X))$ are given in Table 4.

We finally remark that this can be applied to $\mathcal{TFM}^2_{n-r_2-1,n-1}(X)$ by duality the same way as for $\mathcal{TOL}^3(X)$ and $\mathcal{INTOL}^3(X)$.

6. Conclusions

In this paper we have studied the polytope of 3-tolerant measures. We have derived a procedure that allows a fast way to generate fuzzy measures inside this subfamily. Next, we have obtained several properties of the combinatorial structure of this polytope, e.g. the diameter. We have also characterized cubical faces and faces being a simplex. Finally, we have seen that these results can be also extended to other subfamilies of fuzzy measures, as 3-intolerant measures, 3-interactive measures or 2-truncated measures.

$n \setminus r_2$	2	3	4	5
2	2			
3	48	6		
4	34 560	720	24	
5	1 383 782 400	746 496 000	17 280	120

Table 4. First values of $e(B_2^n(1, r_2))$.

Next step should be to extend the results for $\mathcal{TOL}^k(X), k > 2$. However, this does not seem to be an easy problem and more research is needed. In particular, there is not an obvious way to extend Theorem 6. Similar problems arise for the combinatorial properties developed in Section 4. Related to this problem, another approach based on Markov chains has been recently presented in.[2]

Acknowledgements

This paper has been supported by the Ministry of Economy and Competitiveness of Spain under Grant PGC2018-095194-B-100.

1. I. Bárány and G. Füredi. Computing the volume is difficult. *Discrete Comput. Geom.*, 2(4):319–326, 1987.
2. G. Beliakov, F. J. Cabrerizo, E. Herrera-Viedma, and J.Z. Wu. Random generation of k-interactive capacities. *Fuzzy Sets and Systems*, 430: 48-55, 2022.
3. G. Beliakov, S. James, and J.Z. Wu. *Dicrete Fuzzy Measures: Computational aspects.* Springer, 2019.
4. G. Beliakov and J.Z. Wu. Learning fuzzy measures from data: simplifications and optimisation strategies. *Information Sciences*, 494:100–113, 2019.
5. G. Beliakov and J. Z. Wu. Random generation of capacities and its application in comprehensive decision aiding. *Information Sciences*, vol. 577:424–435, 2021.
6. G. Brightwell and P. Winkler. Counting linear extensions. *Order*, 8(3):225–242, 1991.
7. G. Brightwell and P. Winkler. Counting linear extensions is # P-complete. In *STOC '91: Proceedings of the twenty-third annual ACM symposium on Theory of computing*, pages 175–181, New York, NY, USA, 1991. ACM.
8. G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, (5):131–295, 1953.
9. E. F. Combarro and P. Miranda. Identification of fuzzy measures from sample data with genetic algorithms. *Computers and Operations Research*, 33(10):3046–3066, 2006.
10. E. F. Combarro and P. Miranda. On the polytope of non-additive measures. *Fuzzy Sets and Systems*, 159(16):2145–2162, 2008.
11. E. F. Combarro and P. Miranda. Adjacency on the order polytope with applications to the theory of fuzzy measures. *Fuzzy Sets and Systems*, 180:384–398, 2010.

12. B. A. Davey and H. A. Priestley. *Introduction to lattices and order.* Cambridge University Press, 2002.
13. D. Denneberg. *Non-additive measures and integral.* Kluwer Academic, Dordrecht (The Netherlands), 1994.
14. L. Devroye. *Non-uniform random variate generation.* Springer-Verlag, New York, 1986.
15. M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 5:587–607, 1997.
16. M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, (92):167–189, 1997.
17. M. Grabisch. *Set functions, games and capacities in Decision Making*, volume 46 of *Theory and Decision Library.* Springer, 2016.
18. A. Karzanov and L. Khachiyan. On the conductance of order Markov chains. *Order*, 8(1):7–15, 1995.
19. J. Leydold and W. Hörmann. A Sweep-Plane Algorithm for Generating Random Tuples in Simple Polytopes. *J. Math. Comp.*, (67):1617–1635, 1998.
20. Karel De Loof, Hans De Meyer, and Bernard De Baets. Exploiting the lattice of ideals representation of a poset. *Fundam. Inf.*, 71(2,3):309–321, 2006.
21. J.-L. Marichal. k-intolerant capacities and Choquet integral. *European Journal of Operational Research*, 177:1453–1468, 2007.
22. Jiri Matousek. *Lectures on Discrete Geometry.* Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.
23. P. Miranda and E.F. Combarro. On the poytopes of belief and plausibility functions. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 15(6):1068–1081, 2010.
24. P. Miranda and P. García-Segador. Order cones: A tool for deriving k-dimensional faces of cones of subfamilies of monotone games. *Annals of Operational Research*, 295(1):117–137, 2020.
25. P. Miranda, M. Grabisch, and P. Gil. p-symmetric fuzzy measures. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10 (Suppl.):105–123, 2002.
26. D. Naddef and W. R. Pulleyblank. Hamiltonicity and Combinatorial Polyhedra. *Journal of Combinatorial Theory, Series B*, 31:297–312, 1981.
27. R. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.
28. M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, 1974.
29. G. Ziegler. *Lectures on Polytopes.* Springer-Verlag, 1995.