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On the Approximate Controllability for Second Order Nonlinear Parabolic Boundary Value Problems.

In this communication we develop and improve some of the results of [4] on the approximate controllability of several semilinear parabolic boundary value problems where the nonlinear term appears either at the second order parabolic equation or at the flux boundary condition. We also distinguish the cases where the control function acts on the interior *of the parabolic set $Q := \Omega \times (0,T)$ from the one in which the control acts* on the boundary $\Sigma := \partial\Omega \times (0,T)$. *Most of our results will concern to control problems with* final observation *i.e. our goal is to prove that the set $\{y(T,\cdot : v)\}$ generated by the value of solutions at time T is dense in $L^2(\Omega)$ when v runs the set of controls. Nevertheless we also consider a control problem with a* boundary observation. *In that case we shall prove that if $\Sigma_1 \subset \Sigma$ then the set $\{y(\cdot,\cdot : v)|_{\Sigma_1}\}$ generated by the trace of solutions on Σ_1 is a dense subset of $L^2(\Sigma_1)$ when v runs the set of controls.*

1. Cancelation Method.

Given $v \in L^p(Q)$, $y_0 \in L^p(\Omega)$ and f a real continuous function, we denote by $y(v)$ to the solution of

$$(\mathcal{P}_D)\begin{cases} y_t - \Delta y + f(y) = v & \text{in } Q = \Omega \times (0,T) \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T) \\ y(0) = y_0 & \text{on } \Omega. \end{cases}$$

Theorem 1. *(Linear case) Let $h \in L^p(Q)$, $a \in L^\infty(Q)$, $Y_0 \in L^p(\Omega)$ and $Y(\cdot : v)$ the solution of*

$$\begin{cases} Y_t - \Delta Y + aY = h + v & \textit{in } Q \\ Y = 0 & \textit{on } \Sigma \\ Y(0) = Y_0 & \textit{on } \Omega. \end{cases}$$

Then, if $\mathcal{U}$ is dense in $L^p_+(Q)$, the set $F := \{Y(T : v) : \ v \in \mathcal{U}\}$ is dense in $Y(T : 0) + L^p_+(\Omega)$.

Proof. It uses the geometric version of the Hahn-Banach theorem.

Theorem 2. *(Nonlinear case) Let f be a nondecreasing continuous real function. If $y_0 \in L^\infty(\Omega)$ and $\mathcal{U}$ is dense in $L^p_+(Q)$ then the set $F = \{y(T : v)$ solution of $(\mathcal{P}_D)$; $v \in \mathcal{U}\}$ is dense in $y(T : 0) + L^p_+(\Omega)$.*

Proof. As $y_0 \in L^\infty(\Omega)$, $y(\cdot : 0) \in L^\infty(Q)$ and $h(\cdot) := -f(y(\cdot : 0)) \in L^\infty(Q)$. Then, applying Theorem 1, there exists $w_\varepsilon \in L^\infty_+(Q)$ such that $\| Y(T : w_\varepsilon) - y_d \|_{L^p(\Omega)} < \varepsilon$. Besides, $f(Y(\omega_\varepsilon)) \in L^p(Q)$. Now, given $\delta > 0$, let $\tilde{y}$ be the unique function satisfying

$$(\mathcal{P}^*_D)\begin{cases} \tilde{y}_t - \Delta\tilde{y} + f(\tilde{y} + Y(\omega_\varepsilon)) = f(Y(\omega_\varepsilon)) + \delta & \text{in } Q \\ \tilde{y} = 0 & \text{on } \Sigma \\ \tilde{y}(0) = 0 & \text{on } \Omega. \end{cases}$$

Then, if we define $y = \tilde{y} + Y(\omega_\varepsilon)$, we easily check that y is solution of $(\mathcal{P}_D)$ with

$$v_\varepsilon = w_\varepsilon + f(Y(\omega_\varepsilon)) - f(y(\cdot : 0)) + \delta \in L^p(Q).$$

Moreover, $v_\varepsilon \geq 0$ and $Y(\cdot : \omega_\varepsilon) \geq Y(\cdot : 0) = y(\cdot : 0)$. Using the density of $\mathcal{U}$ and the continuous dependence on the data in problem $(\mathcal{P}^*_D)$, we can choose $v \in \mathcal{U}$ such that $\| v - v_\varepsilon \|_{L^p(Q)} \leq \varepsilon$. Finally applying Hölder and Young inequalities, we conclude (for $\delta > 0$ small enough) that $\| \tilde{y}(T) \|_{L^p(\Omega)} \leq C_1\varepsilon$ which proves the result.

Remark . *The cancelation method also applies to the case of Neumann boundary controls (see [1]).*

2. Approximate Controllability Via Kakutani Fixed Point Theorem.

Some approximate controllability results for nonlinear parabolic problems can be obtained by using Kakutani fixed point theorem instead the previous cancelation method. Here we merely state the results. We send the interested reader to [1] for proofs and further remarks. Some related methods are given in [5], [3] and [2].

Observation in time T.

Consider the following problem

$$(\mathcal{P}_1)\begin{cases} y_t - \Delta y + f(y) + A(x,t)\beta(y) \ni h & \text{in } Q \\ \dfrac{\partial y}{\partial \nu} = v\chi_{\mathcal{O}} & \text{on } \Sigma \\ y(0) = y_0 & \text{on } \Omega, \end{cases}$$

where $y_0 \in L^2(\Omega)$, $A(x,t) \in L^\infty(Q)$, $\beta(\cdot)$ is a bounded maximal monotone graph of $\mathbb{R}^2$ with $D(\beta) = \mathbb{R}$ and $\mathcal{O}$ is a subset of Σ with nonempty interior.

Theorem 4. *Let f be a continuous function such that there exists $f'(s_0)$ for some $s_0 \in \mathbb{R}$ and $|f(s)| \leq c_1 + c_2|s|$ assumed $|s| > M$. Then the set $F := \{y(T,v) : v \in L^\infty(\mathcal{O})\}$ is dense in $L^2(\Omega)$.*

Boundary observation. Now we consider the problem

$$(\mathcal{P}_2)\begin{cases} y_t - \Delta y + A(x,t)\beta(y) \ni h & \text{in } Q \\ \dfrac{\partial y}{\partial \nu} + f(y) = 0 & \text{on } \Sigma_1 \\ \dfrac{\partial y}{\partial \nu} = v & \text{on } \Sigma_2 \\ y(x,0) = y_0(x) & \text{on } \Omega, \end{cases}$$

where $\Sigma_2 = \Sigma \backslash \Sigma_1$ has nonempty interior set.

Theorem 5. *Let f be a continuous real function such that $|f(r)| \leq C(1+|r|)$ for some constant $C > 0$. Then the set $F := \{y(v)|_{\Sigma_1} : v \in L^2(\Sigma_2)\}$ is dense in $L^2(\Sigma_1)$.*

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3. References

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