

Some Results About the Approximate Controllability Property for Quasilinear Diffusion Equations

Jesus Ildefonso DIAZ and Angel Manuel RAMOS

J. I. D.: Departamento de Matemática Aplicada, Universidad Complutense de Madrid,
28040 Madrid, Spain.

A. M. R.: Departamento de Informática y Automática, Universidad Complutense de Madrid,
28040 Madrid, Spain.

Abstract. We study the approximate controllability property for some quasilinear diffusion equations of the type $y_t - \Delta\varphi(y) = u\chi_\omega$, on $\Omega \times (0, T)$, where Ω is a bounded open set of \mathbb{R}^N and $\omega \subset \Omega$. First we show some negative results for the case $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$, by means of an obstruction phenomenon. In a second part, we obtain a positive answer to this property on the space $H^{-1-\gamma}(\Omega)$, for any $\gamma > 0$, for a class of functions φ essentially linear at infinity, by using a higher order vanishing viscosity argument.

Quelques resultats sur la propriété de la contrôlabilité approchée pour les équations de diffusion quasi-linéaires

Résumé. On étudie la propriété de la contrôlabilité approchée de quelques équations de diffusion quasi-linéaires de type $y_t(x, t) - \Delta\varphi(y(x, t)) = u(x, t)\chi_\omega$, où $(x, t) \in \Omega \times (0, T)$ et $\omega \subset \Omega$. D'abord on montre quelques résultats négatifs pour les fonctions $\varphi(s) = |s|^{m-1}s$, $0 < m < 1$ (qui sont strictement sous-linéaires à l'infini), par un phénomène d'obstruction. Finalement, on donne une réponse positive dans les espaces $H^{-1-\gamma}(\Omega)$, $\gamma > 0$ arbitraire, pour une famille de fonctions qui sont essentiellement linéaires à l'infini, en utilisant un argument de viscosité d'ordre supérieur.

Versión francesa abreviada

On étudie la contrôlabilité approchée de l'équation de diffusion quasi-linéaire

$$\mathcal{P}(u) \begin{cases} y_t(x, t) - \Delta\varphi(y(x, t)) = h(x, t) + u(x, t)\chi_\omega, & (x, t) \in Q := \Omega \times (0, T), \\ \varphi(y(x, t)) = 0, & (x, t) \in \Sigma := \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x), & x \in \Omega, \end{cases}$$

où Ω est un ouvert borné régulier de \mathbb{R}^N , ω est un sous-ensemble ouvert de Ω , $h \in L^2(0, T : H^{-1}(\Omega))$, $y_0 \in L^2(\Omega)$ sont des données et $u \in L^2(\omega \times (0, T))$ est le contrôle. La fonction φ est supposée continue, non décroissante et sous-linéaire à l'infini (i.e. $|\varphi(s)| \leq C(1 + |s|)$ pour $|s| > M$). L'existence et l'unicité d'une solution $y \in \mathcal{C}([0, T] : H^{-1}(\Omega))$ sont bien connues ([2]). Etant donné $\gamma > 0$, on veut savoir si pour chaque état

desiré $y_d \in H^{-1-\gamma}(\Omega)$, $T > 0$ et $\delta > 0$ il existe $u \in L^2(\omega \times (0, T))$ tel que la solution y de $\mathcal{P}(u)$ verifie

$$(1) \quad \|y(T) - y_d\|_{H^{-1-\gamma}(\Omega)} \leq \delta.$$

On commence par prouver (Théorème 1) qu'un *phénomène d'obstruction* apparait dans le cas des fonctions estrictement sous-linéaires à l'infini, comme par exemple

$$(2) \quad \varphi(s) = |s|^{m-1}s, \quad 0 < m < 1.$$

Bien que la classe restante de fonctions sous-linéaires est assez étroite, on montre (Théorème 2) la contrôlabilité approchée dans $H^{-1-\gamma}(\Omega)$ pour des fonctions φ qui sont essentiellement linéaires à l'infini. Cette classe de fonctions inclut celle qui correspond au problème de Stefan à deux phases. Ce resultat est montré via le problème de viscosité artificiel d'ordre supérieur $y_t + \varepsilon \Delta^2 y - \Delta \varphi(y) = h + u \chi_\omega$.

THÉORÈME 1. – Soit $y(\cdot; u) \in \mathcal{C}([0, T] : L^1(\Omega))$ solution du problème $\mathcal{P}(u)$, avec φ donnée pour (2), $y_0 \in L^2(\Omega)$, $h \equiv 0$ et $u \in L^1(\omega \times (0, T))$. Alors, on peut choisir un état $y_d \in L^1(\Omega)$ tel que $\|y(T; u) - y_d\|_{L^1(\Omega)} > \delta$, quelque soit le contrôl $u \in L^1(\omega \times (0, T))$, avec $\delta > 0$ assez petit.

La démonstration est liée au suivant resultat:

LEMME 1. – ([9]) Soient $m \in (0, 1)$, $R > 0$ et $y, \hat{y} \in \mathcal{C}([0, T] : L^1(B_R(x_0)))$ satisfaisant l'équation

$$(3) \quad y_t - \Delta(|y|^{m-1}y) = 0$$

dans $\mathcal{D}'(B_{2R}(x_0) \times (0, T))$. Supposons $y \geq \hat{y}$. Alors, pour chaque $t, s \in [0, T]$, il existe $C = C(N, m)$ tel que

$$(4) \quad \int_{B_R(x_0)} |y(t) - \hat{y}(t)| \leq C \left[\int_{B_{2R}(x_0)} (|y(s) - \hat{y}(s)| + |t - s|^\alpha R^{-\beta}) \right],$$

où $\alpha = 1/(1 - m)$ et $\beta = 2/(1 - m) - N$.

REMARQUE 1. – Grâce au Principe de Comparaison on peut aussi montrer, dans le cas de φ donnée pour (2), l'existence de deux fonctions U_- et U_+ telles que $U_-(x, t) \leq y(x, t; u) \leq U_+(x, t)$ pour tout $t \in [0, T]$ et p.p. $x \in \Omega \setminus \omega$ et pour tout $u \in L^2(\omega \times (0, T))$. Si $m \in (1/2, 1)$ ces inegalités impliquent aussi une obstruction dans $H^{-2}(\Omega)$ et alors dans tous les espaces $X \subset H^{-2}(\Omega)$ avec inclusion continue (voir Proposition 2).

THÉORÈME 2. – Soit φ une fonction continue non-décroissante telle que $\varphi \in C^1(\mathbb{R} \setminus [-M_1, M_1])$ pour quelque constante $M_1 \geq 0$ et $\varphi(0) = 0$. Supposons qu'il existe deux constantes positives k et C_1 telles que

$$(5) \quad |\varphi'(s) - k| \leq \frac{C_1}{|s|} \quad \forall |s| > M_1,$$

$$(6) \quad |\varphi(s) - ks| \leq C_2 \quad \forall s \in \mathbb{R}.$$

Alors, si $\varphi'(s) \geq c > 0$ p.p. $s \in \mathbb{R}$ ou $h \in L^2(Q)$, le problème $\mathcal{P}(u)$ vérifie la propriété de la contrôlabilité approchée dans $H^{-1-\gamma}(\Omega)$, pour tout $\gamma > 0$.

Pour prouver ce théorème on passe à la limite, $\varepsilon \rightarrow 0$, sur les problèmes d'ordre supérieur

$$\mathcal{P}_\varepsilon(u) \begin{cases} y_t(x, t) + \varepsilon \Delta^2 y(x, t) - \Delta \varphi(y(x, t)) = h(x, t) + u(x, t) \chi_\omega, & (x, t) \in Q, \\ y(x, t) = \Delta y(x, t) = 0 & (x, t) \in \Sigma, \\ y(x, 0) = y_0(x), & x \in \Omega. \end{cases}$$

THÉORÈME 3. – On suppose φ vérifiant les hypothèses du Théorème 2. Alors, étant donne $\gamma > 0$ arbitraire et $y_d \in H^{-1-\gamma}(\Omega)$, il existe une constante positive K , qui dépend de k, C_1, C_2, M_1 mais qui est indépendant de ε et une suite de contrôls $v_\varepsilon \in L^2(\omega \times (0, T))$ vérifiant

$$(7) \quad \|v_\varepsilon\|_{L^2(\omega \times (0, T))} \leq K \quad \forall \varepsilon > 0,$$

telles que les solutions $y(\cdot; v_\varepsilon)$ de $\mathcal{P}_\varepsilon(v_\varepsilon)$ vérifient

$$(8) \quad \|y(T; v_\varepsilon) - y_d\|_{H^{-1-\gamma}(\Omega)} < \delta.$$

1. Introduction.

We study the approximate controllability of problem $\mathcal{P}(u)$, where Ω is a bounded open subset of \mathbb{R}^N , $T > 0$, ω is a nonempty open subset of Ω , $h \in L^2(0, T; H^{-1}(\Omega))$ and $y_0 \in L^2(\Omega)$ are prescribed data and $v \in L^2(\omega \times (0, T))$ represents the searched output control. The function φ is assumed to be continuous, nondecreasing and *sublinear at infinity* (i.e. $|\varphi(s)| \leq C(1 + |s|)$ for $|s| > M$). The existence and uniqueness of a solution $y \in \mathcal{C}([0, T]; H^{-1}(\Omega))$ are well-known ([2]). Given $\gamma > 0$, for any desired state $y_d \in H^{-1-\gamma}(\Omega)$, any fixed time T and any $\delta > 0$ we want to find a control $u \in L^2(\omega \times (0, T))$ such that (1) holds. We start by proving that (in contrast with the case of semilinear equations; see [8] and [5]) an *Obstruction Phenomenon* (implying the impossibility of the approximate controllability) arises in the case of φ being strictly sublinear ($\varphi(s) = |s|^{m-1}s$, $0 < m < 1$). Although the remaining range of sublinear functions is quite narrow, we shall show that the approximate controllability holds for functions φ which are *essentially linear at infinity*. This class of functions includes the one associated to the two-phase Stefan Problem. The result is obtained via the vanishing viscosity higher order problem $y_t + \varepsilon \Delta^2 y - \Delta \varphi(y) = h + u \chi_\omega$.

2. Obstruction phenomenon when the nonlinearity is strictly sublinear.

THEOREM 1. – *Let $y(\cdot; u) \in \mathcal{C}([0, T]; L^1(\Omega))$ be a solution of $\mathcal{P}(u)$ with φ given by (2), $y_0 \in L^1(\Omega)$, $h \equiv 0$ and $u \in L^1(\omega \times (0, T))$. Then we can choose $y_d \in L^1(\Omega)$ such that $\|y(T; u) - y_d\|_{L^1(\Omega)} > \delta$ for any $u \in L^1(\omega \times (0, T))$ and any $\delta > 0$ small enough.*

The proof is based on the following lemma

LEMMA 1. – ([9]) *Let $m \in (0, 1)$ and $R > 0$. Let $y, \hat{y} \in \mathcal{C}([0, T]; L^1(B_R(x_0)))$ be two functions satisfying equation (3) in $\mathcal{D}'(B_{2R}(x_0) \times (0, T))$. Assume that $y \geq \hat{y}$. Then, for any $t, s \in [0, T]$, there exists $C = C(N, m)$ such that inequality (9) holds, where $\alpha = 1/(1 - m)$ and $\beta = 2/(1 - m) - N$.*

Proof of Theorem 1. – Let $x_0 \in \Omega \setminus \omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset \Omega \setminus \omega$. Let $y_0^+ := \sup(y_0, 0)$, $y_0^- := \sup(-y_0, 0)$. Define analogously u^+ and u^- . Let Y_+ (resp. Y_-) be the (unique) solution of problem $\mathcal{P}(u^+)$ with initial datum y_0^+ (resp. $\mathcal{P}(u^-)$ with initial datum y_0^-). Then, by the comparison principle (see [10])

$$-Y_-(x, t) \leq y(x, t) \leq Y_+(x, t) \quad \text{and} \quad Y_+(x, t) \quad (\text{resp. } Y_-(x, t)) \geq 0$$

for any $t \in [0, T]$ and a.e. $x \in \Omega$. Then the function Y_+ (resp. Y_-) and $\hat{y} \equiv 0$ satisfy (3) in $\mathcal{D}'(B_{2R}(x_0) \times (0, T))$ and therefore, by (4),

$$\int_{B_R(x_0)} Y_+(x, t) dx \leq C \left[\int_{B_{2R}(x_0)} (y_0^+(x) + t^\alpha R^{-\beta}) dx \right]$$

for any $t \in [0, T]$. In consequence

$$(9) \quad \int_{B_R(x_0)} |y(x, t)| dx \leq C \left[\int_{B_{2R}(x_0)} (|y_0(x)| + t^\alpha R^{-\beta}) dx \right]$$

for any $t \in [0, T]$. It is clear that (9) implies an obstruction for the $L^1(\Omega)$ -norm of $y(t; u)$. ■

We have obtained the non-controllability result in $L^1(\Omega)$ by means of an *energy obstruction phenomenon* but the following result shows that a *pointwise obstruction phenomenon* also arises when φ is given by (2).

PROPOSITION 1. – *Let $y(\cdot; u)$ be a solution of $\mathcal{P}(u)$ with φ given by (2). Then, for a large class of functions $y_0 \in L^2(\Omega)$ (those satisfying (13)), there exist two functions $\underline{U}, \bar{U} \in L^\infty(0, T; L_{loc}^\infty(\Omega \setminus \omega))$ such that, for any $u \in L^2(\omega \times (0, T))$, $\underline{U}(x, t) \leq y(x, t; u) \leq \bar{U}(x, t)$ for any $t \in [0, T]$, a.e. $x \in \Omega \setminus \omega$.*

Proof. – For any $\lambda > 0$ we take $p := 1/m$ (therefore $p > 1$) and we define the function $Y_\lambda^+(x)$ (resp. $Y_\lambda^-(x)$) as the (unique) solution of

$$(10) \quad \begin{cases} -\Delta Y_\lambda^+ + \lambda |Y_\lambda^+|^{p-1} Y_\lambda^+ = 0 & \text{in } \Omega \setminus \omega, \\ Y_\lambda^+ = 0 & \text{on } \partial\Omega, \\ Y_\lambda^+ = \infty \quad (\text{resp. } Y_\lambda^- = -\infty) & \text{on } \partial\omega \end{cases}$$

(see, e.g., [1]). Then it is easy to prove that, for any constant $C > 0$, the function

$$(11) \quad \bar{U}(x, t) := (Y_\lambda^+(x))^p [(1-m)\lambda t + C^{1-m}]^{\frac{1}{1-m}} \quad (\text{resp. } \underline{U}(x, t) := -|Y_\lambda^-(x)|^p [(1-m)\lambda t + C^{1-m}]^{\frac{1}{1-m}})$$

is a solution of

$$(12) \quad \begin{cases} \bar{U}_t - \Delta(|\bar{U}|^{m-1}\bar{U}) = 0 & \text{in } \mathcal{D}'(\Omega \setminus \omega \times (0, T)) \\ |\bar{U}|^{m-1}\bar{U} = 0 & \text{on } \Sigma \\ \bar{U} = \infty \quad (\text{resp. } \underline{U} = -\infty) & \text{on } \partial\omega \times (0, T) \\ \bar{U}(0, x) = C Y_\lambda^+(x) & \text{in } \Omega \setminus \omega. \end{cases}$$

Thus, if we assume that

$$(13) \quad \begin{cases} \text{there exist } C > 0 \text{ and } \lambda > 0 \text{ such that} \\ -C|Y_\lambda^-(x)|^p \leq y_0(x) \leq C(Y_\lambda^+(x))^p \quad \text{a.e. } x \in \Omega \setminus \omega, \end{cases}$$

the comparison principle leads to the pointwise obstruction estimate $\underline{U}(x, t) \leq y(x, t; u) \leq \bar{U}(x, t)$ for any $t \in [0, T]$, a.e. $x \in \Omega \setminus \omega$ and any $u \in L^2(\omega \times (0, T))$. ■

We point out that the uniqueness of solutions of (12) with arbitrary initial data may fail (in contrast with the case of non singular solutions of $\mathcal{P}(u)$ or singular solutions of semilinear equations). This is the case if, for instance, $y_0 \equiv 0$ (for any $\lambda > 0$ the function $U_\lambda(x, t) := [(1-m)\lambda t]^{1/(1-m)} (Y_\lambda^+(x))^p$ is a solution of (12) with zero initial value).

It is well-known ([1]) that $Y_\lambda^-(x)$, $Y_\lambda^+(x)$ behave as $C_\lambda d(x, \partial\omega)^{-2/(p-1)}$, $C_\lambda > 0$, when $x \in \Omega \setminus \omega$ is near $\partial\omega$. Then, if $p \in (1, 2)$ (i.e. $m \in (1/2, 1)$), we have that $z(x, t) = (-\Delta)^{-1}y(x, t)$ (with homogeneous Dirichlet conditions on $\partial\Omega$) satisfies $|z(x, t)| \leq K(t)d(x, \partial\omega)^{-2(2-p)/(p-1)}$ for some $K(t) > 0$ and for $x \in \Omega \setminus \omega$ near $\partial\omega$. So we get

PROPOSITION 2. – *Let $y \in \mathcal{C}([0, T] : H^{-1}(\Omega))$ be the solution of $\mathcal{P}(u)$ with φ given by (2) and $m \in (1/2, 1)$, $y_0 \in L^2(\Omega)$ satisfying (13) and $h \equiv 0$. Then we can choose $y_d \in H^{-2}(\Omega)$ such that, for any $u \in L^2(\omega \times (0, T))$, $\|y(T; u) - y_d\|_{H^{-2}(\Omega)} > \delta$ for any $\delta > 0$ small enough.*

REMARK 1. – Notice that Proposition 2 implies the non-approximate controllability in any Banach space $X \subset H^{-2}(\Omega)$ with continuous embedding.

3. An approximate controllability result when the nonlinearity is essentially linear at infinity.

THEOREM 2. – *Let φ be a continuous nondecreasing function such that $\varphi \in \mathcal{C}^1(\mathbb{R} \setminus [-M_1, M_1])$ for some constant $M_1 \geq 0$ and $\varphi(0) = 0$. Assume that there exists two positive constants k and C_1 such that (5) and (6) hold. Then, if $\varphi'(s) \geq c > 0$ a.e. $s \in \mathbb{R}$ or $h \in L^2(Q)$, problem $\mathcal{P}(u)$ satisfies the approximate controllability property on $H^{-1-\gamma}(\Omega)$ for any $\gamma > 0$.*

In order to prove Theorem 2 we take $\varepsilon > 0$ arbitrary and we study the approximate controllability for the vanishing viscosity higher order problem $\mathcal{P}(u)$.

ABOUT THE CONTROLLABILITY PROPERTY FOR QUASILINEAR EQUATIONS

THEOREM 3. – *If we assume φ like in Theorem 2, $\gamma > 0$ and $y_d \in H^{-1-\gamma}(\Omega)$, then there exists a positive constant K , depending on k, C_1, C_2, M_1 but independent of ε and a sequence of controls $v_\varepsilon \in L^2(\omega \times (0, T))$ satisfying (7) such that the associated solutions $y(\cdot; v_\varepsilon)$ satisfy (8).*

Sketch of proof of Theorem 3. – Since assumption (5) clearly implies that $\varphi'(s) \rightarrow k$ as $|s| \rightarrow \infty$, it is natural to define the function $\varphi_0(s) := \varphi(s) - ks$, (so that $\varphi'_0(s) \rightarrow 0$ as $|s| \rightarrow \infty$). Then, we linearize function φ_0 which (by convenience) will be done near a point $s_\varepsilon \in \mathbb{R}$ chosen depending on ε in a suitable way. Some elementary Calculus techniques lead to

LEMMA 2. – *Let $\varphi \in C^0(\mathbb{R})$ (non necessarily nondecreasing) satisfying (5). Given $\varepsilon > 0$ there exists $s_\varepsilon \in \mathbb{R}$ such that the function*

$$(14) \quad g_\varepsilon(s) := \frac{\varphi_0(s) - \varphi_0(s_\varepsilon)}{s - s_\varepsilon}$$

satisfies $g_\varepsilon \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ and

$$(15) \quad \|g_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \sqrt{\varepsilon}.$$

If in addition φ satisfies (6), then there exists a positive constant K_2 , depending on C_1, C_2 and M_1 but independent of ε , such that

$$(16) \quad |g_\varepsilon(s)s_\varepsilon| \leq K_2, \quad \text{for any } \varepsilon > 0 \text{ and any } s \in \mathbb{R}.$$

End of the sketch of proof of Theorem 3. – Returning to our linearizing process, since $\varphi_0(s) = \varphi_0(s_\varepsilon) + g_\varepsilon(s)s - g_\varepsilon(s)s_\varepsilon$, we consider the approximate controllability for a linear problem obtained by replacing the term $\varphi(y)$ by $ky + g_\varepsilon(z)y + \varphi_0(s_\varepsilon) - g_\varepsilon(z)s_\varepsilon$, where z is an arbitrary function in $L^2(Q)$. Notice that when $z = y$ this expression coincides with $\varphi(y)$ and that if we denote $h_\varepsilon(z) := \Delta(\varphi_0(s_\varepsilon) - g_\varepsilon(z(t, z))s_\varepsilon)$, then $h_\varepsilon(z) \in L^\infty(0, T : H^{-2}(\Omega))$ for all $z \in L^2(Q)$ and for all $\varepsilon > 0$. Now, we consider the approximate controllability property corresponding to the linear problem

$$(17) \quad \begin{cases} y_t + \varepsilon \Delta^2 y - k \Delta y - \Delta((g_\varepsilon(z)y) = h + h_\varepsilon(z) + u_\varepsilon \chi_\omega & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Then, in a way similar to the proof of Theorem 1 of [6] but taking into account now Lemma 2, we deduce from the assumption $\gamma > 0$ that, given a desired state $y_d \in H^{-1-\gamma}(\Omega)$ and $\delta > 0$, there exists a constant K , depending on k, C_1, C_2, M_1 but independent of ε , such that we can find solutions of (17) satisfying (1) with

$$(18) \quad \|u_\varepsilon\|_{L^2(Q)} \leq K.$$

Furthermore, if we define the operator $\Lambda_\varepsilon : L^2(Q) \rightarrow \mathcal{P}(L^2(Q))$ by $\Lambda_\varepsilon(z) := \{y_\varepsilon \text{ satisfying (17), (1), with a control } u_\varepsilon \text{ satisfying (18)}\}$, then, by Kakutani's fixed point Theorem, we deduce that for all $\varepsilon > 0$ there exists a fixed point y_ε of Λ_ε and y_ε is the solution of $\mathcal{P}(u)$ for some control u satisfying (18). ■

Proof of Theorem 2. - First step. Assume additionally that $\varphi \in C^1(\mathbb{R})$. For any $\varepsilon > 0$, let v_ε and y_ε be the functions given in Theorem 3. Assume $\varphi'(s) \geq c > 0$ for all $s \in \mathbb{R}$ (resp. $h \in L^2(Q)$). Since the equation of $\mathcal{P}_\varepsilon(u)$ holds on $L^2(0, T : (H^2(\Omega) \cap H_0^1(\Omega))')$, multiplying by $y_\varepsilon \in L^2(0, T : H^2(\Omega) \cap H_0^1(\Omega))$ and applying Young (resp. Gronwall) inequality we obtain, thanks to (7), that there exists a constant $C > 0$ independent of ε such that

$$(19) \quad \|y_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} + \int_Q \varphi'(y_\varepsilon) |\nabla y_\varepsilon|^2 dx dt \leq C.$$

Therefore, from (19) we obtain that y_ε is uniformly bounded in $L^\infty(0, T : L^2(\Omega))$ and by the equation of $\mathcal{P}_\varepsilon(u)$, $(y_\varepsilon)_t$ is uniformly bounded in $L^\infty(0, T : H^{-4}(\Omega))$. Then, since $L^2(\Omega) \subset H^{-1}(\Omega) \subset H^{-4}(\Omega)$ with compact embeddings, we have (see [12]) that y_ε is relatively compact in $\mathcal{C}([0, T] : H^{-1}(\Omega))$. Further, from

(19) and the boundedness of function φ' (notice that $\varphi' \in L^\infty(\mathbb{R})$ by (5)), we deduce that there exists a constant $K > 0$ independent of ε such that

$$\int_0^T \|\nabla\varphi(y_\varepsilon)\|_{L^2(\Omega)}^2 dt = \int_Q \varphi'(y_\varepsilon(x, t)) \varphi'(y_\varepsilon(x, t)) |\nabla(y_\varepsilon(x, t))|^2 dx dt < K.$$

Thus, there exist $y \in L^\infty(0, T : L^2(\Omega))$ and $\zeta \in L^2(0, T : H_0^1(\Omega))$ (recall that $\varphi(0) = 0$) such that $y_\varepsilon \rightarrow y$ strongly in $\mathcal{C}([0, T] : H^{-1}(\Omega))$ and $\varphi(y_\varepsilon) \rightharpoonup \zeta$ weakly in $L^2(0, T : H_0^1(\Omega))$. But the operator $Au := -\Delta\varphi(u)$, $D(A) := \{u \in H^{-1}(\Omega) \cap L^1(\Omega) : \varphi(u) \in H_0^1(\Omega)\}$ is a maximal monotone operator on the space $H^{-1}(\Omega)$ ([2]). Thus, the extension operator \mathcal{A} of A on $L^2(0, T : H^{-1}(\Omega))$ is also a maximal monotone operator ([3], Example 2.33). Finally, as any maximal monotone operator is strongly-weakly closed ([3], Proposition 2.5), we obtain that $\zeta = \varphi(y)$ in $L^2(0, T : H_0^1(\Omega))$. Moreover, from estimate (7) we have that $v_\varepsilon \rightharpoonup v$ weakly in $L^2(\omega \times (0, T))$, with

$$(20) \quad \|v\|_{L^2(\omega \times (0, T))} \leq K.$$

Then we deduce that $y \in \mathcal{C}([0, T] : H^{-1}(\Omega))$ is solution of $\mathcal{P}(v)$ and that $\|y(\cdot, T) - y_d\|_{H^{-1-\gamma}(\Omega)} \leq \delta$.

Second step. Let φ as in Theorem 2. We approximate φ by $\varphi_n \in \mathcal{C}^1(\mathbb{R})$, φ_n nondecreasing, satisfying (5) and (6) with the same constants k , C_1 , C_2 and M_1 that the ones for φ . Then the controls v_n build as in step 1 are uniformly bounded (see (20) and so the conclusion comes from well-known results expressing the convergence $y_n \rightarrow y$ in $\mathcal{C}([0, T] : H^{-1}(\Omega))$ when $\varphi_n \rightarrow \varphi$ (see e.g. [4]).

■

REMARK 2. – In [7] we have applied this viscosity method for the semilinear equation $y_t - \Delta y + \varphi(y) = h + u\chi_\omega$ by assuming merely that φ is sublinear at infinity and differentiable at a point. It is also shown in [7] that the approximate controllability in $H^{-1}(\Omega)$ for problem $\mathcal{P}(u)$ is reduced to check a convergence result independent of the control process.

References

- [1] **Bandle, C. and Markus, M., 1992.** “Large” solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, *Journal d’Analyse Mathématique*, 58, pp. 9-24.
- [2] **Brézis, H., 1971.** Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In *Nonlinear Functional Analysis* (E. Zantonello ed.), Academic Press, New York, pp. 101-156.
- [3] **Brézis, H., 1973.** *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam.
- [4] **Damlamian, A., 1977.** Some results on the multi-phase Stefan problem, *Comm. Part. Diff. Eq.*, 2, pp. 1017-1044.
- [5] **Díaz, J.I. and Ramos, A.M., 1995.** Positive and negative approximate controllability results for semilinear parabolic equations, *Rev. Real Acad. de Ciencias*, Tomo LXXXIX, 1-2, pp. 11-30.
- [6] **Díaz, J.I. and Ramos, A.M., 1997.** On the Approximate Controllability for Higher Order Parabolic Nonlinear Equations of Cahn-Hilliard Type. To appear in *Proceedings of the International Conference on Control and Estimation of Distributed Parameter Systems*, Vorau (Austria).
- [7] **Díaz, J.I. and Ramos, A.M., 1997.** Un método de viscosidad para la controlabilidad aproximada de ciertas ecuaciones parabólicas cuasilineales. To appear in a volume in honor of A. Valle. Universidad de Sevilla.
- [8] **Fabre, C., Puel, J.P. and Zuazua, E., 1995.** Approximate controllability of the semilinear heat equation, *Proceedings of the Royal Society of Edinburgh*, 125A, pp. 31-61.
- [9] **Herrero, M.A. and Pierre, M., 1985.** The Cauchy Problem for $u_t = \Delta u^m$ when $0 < m < 1$, *Trans. Amer. Math. Soc.*, 291, pp. 145-158.
- [10] **Kalashnikov, A.S., 1987.** Some problems of the qualitative theory of nonlinear degenerate second-order parabolic equations, *Russ. Math. Survs.*, 42, pp. 169-222.
- [11] **Lions, J.L., 1990.** Remarques sur la contrôlabilité approchée. In *Proceedings of Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos*, Univ. de Malaga, pp. 77-88.
- [12] **Simon, J., 1987.** Compact Sets in the Space $L^p(0, T; B)$. *Annali di Mat. Pura ed Appl.*, Serie 4, 146, pp. 65-96.