Approximate Controllability and Obstruction Phenomena for Quasilinear Diffusion Equations

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1. INTRODUCTION.

The approximate controllability for parabolic problems has received an intensive study in the last three decades. References to the pioneering works devoted to linear equations can be found in the book of Lions (1968) and in the survey of Russell (1978). For numerical aspects see Carthel, Glowinski and Lions (1994), Glowinski and Lions (1994), (1995). The study of this property for nonlinear parabolic equations seems to have its origins in the work of Henry (1978). Since then, many other results are today available in the literature (see some references in Díaz (1995a), (1995b)) but, to the best of our knowledge, always restricted to the case of semilinear parabolic equations in which the presence of a dominating linear term allows to arrive to a positive conclusion.

In this paper we start a series of works devoted to purely quasilinear parabolic equations, i.e., without assuming the presence of a dominating linear term in the equation. To fix ideas, we shall consider the question of the approximate controllability for the, so called, *nonlinear diffusion equation*

$$
\left\{
\begin{array}{ll}
y_t - \Delta\varphi(y) = h + v\chi_\omega & \text{in } Q := \Omega\times(0,T), \\
\varphi(y) = 0 & \text{on } \Sigma := \partial\Omega\times(0,T), \\
y(0) = y_0 & \text{in } \Omega,
\end{array}
\right. \tag{1}
$$

where Ω is a bounded open subset of $I\!R^N$ of class C^4, $T>0$, ω is a nonempty open subset of Ω, φ is a continuous nondecreasing real function, $h\in L^2(0,T:H^{-1}(\Omega))$ and $y_0\in L^2(\Omega)$ are prescribed data and v represents the searched output control answering to the approximate controllability property; i.e. such that $\| y(t;v)-y_d \|_{L^2(\Omega)}\leq\delta$ for a given $\delta>0$ and for some *desired state* $y_d\in L^2(\Omega)$ (here $y(t;v)$ denotes the solution of (1) associated to the control v). In the rest of the paper we always assume $\omega\subset\Omega$ but $\omega\not\equiv\Omega$ (the approximate controllability when $\omega\equiv\Omega$ is a consequence of the results of Díaz and Fursikov (1994)). Before continuing, we recall that the class of equations (1) arises in many important physical settings (see, e.g. the surveys Peletier (1981), Díaz (1986), Kalashnikov (1987) and Vázquez (1992)).

This paper is devoted to the case in which φ is assumed to be *sublinear at infinity.* i.e. such that

$$
|\varphi(s)|\leq C(1+|s|) \qquad \text{for } |s|>M, \tag{2}
$$

for some $M>0$ (the superlinear case will be considered in a next work). We recall that this type of conditions are sufficient and, in some sense, *necessary* in order to have the approximate controllability of semilinear parabolic equations of the type

$$
y_t + (-\Delta)^m y + \varphi(y) = h + v\chi_\omega \tag{3}
$$

(see Díaz and Ramos (1997b) for $m\geq 1$ and its references on the case $m=1$). More precisely, if for instance

$$
\varphi(s) = |s|^{m-1}s, \qquad s\in I\!R \tag{4}
$$

and φ is superlinear (i.e. $m>1$) then an *obstruction phenomenon* occurs for the solutions of the Cauchy-Dirichlet problem associated to (3) and thus the approximate controllability fails for a general desired state y_d (see Díaz (1991), (1994), Díaz and Ramos (1997a) for $m=1$ and Díaz and Ramos (1997b) for $m>1$). In contrast with that, we shall prove in Section 2 that an obstruction phenomenon occurs for solutions of the nonlinear diffusion equation (1) when φ is a *strictly sublinear* function as, for instance, φ given by (4) with $m\in(0,1)$. Therefore, again, the approximate controllability fails in this situation if y_d is suitably chosen. Nevertheless, we shall prove, in Section 3, that although the remaining range of sublinear functions φ (satisfying (2)) which are not strictly sublinear is quite narrow, the approximate controllability holds for a certain class of functions φ which are *essentially linear* at infinity (see assumptions (13) and (14) bellow). This class of functions includes the one associated to some type of *two phase Stefan problem* ($\varphi(s)=ks$ for $s<0$, $\varphi(s)=0$ in $[0,L]$ and $\varphi(s)=ks$ for $s>L$, for some positive constants k and L). The result is obtained through the application of the main theorem of Díaz and Ramos (1997b) to the vanishing viscosity higher order problem

$$
\left\{
\begin{array}{ll}
y_t + \varepsilon\Delta^2 y - \Delta\varphi(y) = h + v\chi_\omega & \text{in } Q, \\
\dfrac{\partial^j y}{\partial\nu^j} = 0 \quad , \quad j=0,1 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } \Omega
\end{array}
\right. \tag{5}
$$

(with $\varepsilon > 0$ arbitrary) and posterior passing to the limit $\varepsilon \to 0$. This vanishing viscosity argument seems to lead to approximate controllability results for a very large class on nonlinear parabolic equations even in non divergence form as

$$y_t - \mathcal{F}(t, x, y, \nabla y, D^2 y) = v\chi_\omega.$$

2. OBSTRUCTION PHENOMENON WHEN φ IS STRICTLY SUBLINEAR.

In this section we shall prove that when φ is *strictly sublinear* at infinity as, for instance, when φ is given by (2) with $m \in (0, 1)$, then an *obstruction phenomenon* arises and therefore problem (1) does not satisfy, in general, the approximate controllability property (in contrast with semilinear parabolic problems). Several proofs of this fact are possible. We start with an energy argument.

Theorem 1 *Let $m \in (0, 1)$ and $y_0 \in L^2(\Omega)$. Let $y(t; u) \in \mathcal{C}([0, T]; L^2(\Omega))$ with $|y|^{m-1}y \in L^2(0, T; H_0^1(\Omega))$ be a function satisfying*

$$\mathcal{P}(u, y_0) \begin{cases} y_t - \Delta(|y|^{m-1}y) = u\chi\omega & \text{in } \mathcal{D}'(Q) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

with external control $u \in L^2(\omega \times (0, T))$. Then we can choose $y_d \in L^2(\Omega)$ such that $\| y(T; u) - y_d \|_{L^2(\Omega)} > \varepsilon$ for any $u \in L^2(\omega \times (0, T))$ and any $\varepsilon > 0$ small enough.

The main ingredient of the proof is the following technical result due to Herrero and Pierre (1985) (see their Lemma 3.1 and following Remark).

Lemma 1 (Herrero and Pierre (1985)). *Let $m \in (0, 1)$, $R > 0$ and $y, \widehat{y} \in \mathcal{C}([0, T]; L^1(B_R(x_0)))$ satisfying the equation*

$$y_t - \Delta(|y|^{m-1}y) = 0 \quad \text{in } \mathcal{D}'((0, T) \times B_{2R}(x_0)). \tag{6}$$

Assume that $y \geq \widehat{y}$. Then, for any $t,\ s \in [0, T]$, there exists $C = C(N, m)$ such that

$$\int_{B_R(x_0)} |y(t) - \widehat{y}(t)| \leq C \left[\int_{B_{2R}(x_0)} (|y(s) - \widehat{y}(s)| + |t - s|^\alpha R^{-\gamma}) \right], \tag{7}$$

where $\alpha = 1/(1 - m)$ and $\gamma = 2/(1 - m) - N$.

Proof of Theorem 1. Let $x_0 \in \Omega \backslash \omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset \Omega \backslash \omega$. Let $y_0^+ := \sup(y_0, 0)$, $y_0^- := sup(-y_0, 0)$. Define analogously u^+ and u^-. Let Y_+ (resp. Y_-) be the (unique) solution of problem $\mathcal{P}(u^+, y_0^+)$ (resp. $\mathcal{P}(u^-, y_0^-)$) (see, for instance, Brézis (1971)). Then, by the comparison principle (see references in Kalashnikov (1987))

$$-Y_-(t, x) \leq y(t, x) \leq Y_+(t, x) \qquad \text{and} \qquad Y_+(t, x) \;\; (\text{resp. } Y_-(t, x)) \geq 0$$

for any $t \in [0, T]$ and a.e. $x \in \Omega$. Then the function Y_+ (resp. Y_-) and $\widehat{y} \equiv 0$ satisfy (6) in $\mathcal{D}'((0, T) \times B_{2R}(x_0))$ and therefore, by (7),

$$\int_{B_R(x_0)} Y_+(t, x) dx \leq C \left[\int_{B_{2R}(x_0)} (y_0^+(x) + t^\alpha R^{-\gamma}) dx \right]$$

for any $t \in [0,T]$. Then

$$\int_{B_R(x_0)} |y(t,x)|dx \leq C \left[\int_{B_{2R}(x_0)} (|y_0(x)| + t^{\alpha} R^{-\gamma}) dx \right] \tag{8}$$

for any $t \in [0,T]$. It is clear that (8) implies an obstruction for the $L^2(\Omega)$-norm of $y(t;u)$ (independent of u) and that the conclusion holds by choosing $y_d \in L^2(\Omega)$ with

$$\int_{B_{2R}(x_0)} |y_d(x)|dx >> \int_{B_{2R}(x_0)} (|y_0(x)| + T^{\alpha} R^{-\gamma}) dx. \qquad \blacksquare$$

Remark 1 We point out that a *pointwise obstruction phenomenon* also arises when $m \in (0,1)$. It is a consequence of the existence of a (unique) function $Y^+_{\lambda,\infty}(x)$ (resp. $Y^-_{\lambda,\infty}(x)$) satisfying

$$\begin{cases} -\Delta Y^+_{\lambda,\infty} + \lambda |Y^+_{\lambda,\infty}|^{p-1} Y^+_{\lambda,\infty} = 0 & \text{in } \Omega \backslash \omega \\ Y^+_{\lambda,\infty} = 0 & \text{on } \partial\Omega \\ Y^+_{\lambda,\infty} = \infty \qquad (\text{resp. } Y^-_{\lambda,\infty} = -\infty) & \text{on } \partial\omega, \end{cases} \tag{9}$$

for any prescribed $\lambda > 0$ and $p > 1$ (see e.g. Bandle and Markus (1992)). Assume now that

$$\begin{cases} \text{there exist } C > 0 \text{ and } \lambda > 0 \text{ such that} \\ C Y^-_{\lambda,\infty}(x) \leq y_0(x) \leq C Y^+_{\lambda,\infty}(x) \quad a.e.\ x \in \Omega \backslash \omega. \end{cases} \tag{10}$$

Then it is possible to construct $U^+(t,x)$ (resp. $U^-(t,x)$) satisfying

$$\begin{cases} U^+_t - \Delta(|U^+|^{m-1} U^+) = 0 & \text{in } \mathcal{D}'(\Omega \backslash \omega \times (0,T)) \\ U^+ = 0 & \text{on } \Sigma \\ U^+ = \infty \quad (\text{resp. } U^- = -\infty) & \text{on } \partial\omega \times (0,T) \\ U^+(0,x) = y_0(x) & \text{in } \Omega \backslash \omega. \end{cases} \tag{11}$$

The main idea is to use the supersolution

$$\overline{U}(t,x) := Y^+_{\lambda,\infty}(x)(m-1) \left[\lambda t + C^{1-m} \right]^{\frac{1}{1-m}}, \tag{12}$$

where $Y^+_{\lambda,\infty}(x)$ is the solution of (9) with $p := 1/m$. Then the comparison principle leads to the pointwise obstruction estimate $U_-(t,x) \leq y(t,x;u) \leq U_+(t,x)$ for any $t \in [0,T]$, a.e. $x \in \Omega \backslash \omega$ and any solutions U^+ (resp. U^-) of (11). We point out that the uniqueness of solutions U^+ (resp. U^-) of (11) may fail (in contrast with the case of non singular solutions or semilinear equations). This is the case if, for instance, $y_0 \equiv 0$ (for any $\lambda > 0$ the functions $U_\lambda(t,x) := (m-1)(\lambda t)^{1/(1-m)} Y^+_{\lambda,\infty}(x)$ is a solution of (11) with zero initial value).

3. AN APPROXIMATE CONTROLLABILITY RESULT WHEN φ IS ESSENTIALLY LINEAR AT INFINITY.

The main result of this section is the following:

Theorem 2 *Let φ be a continuous nondecreasing function with $\varphi(0) = 0$. Assume that there exists $k \geq 0$ such that*

$$\left\{ \begin{array}{l} \varphi \in \mathcal{C}^1(I\!R \backslash [-M_1, M_1]) \text{ and } |\varphi'(s) - k| \leq \dfrac{C_1}{|s|} \quad \text{if } |s| > M_1, \\ \text{for some positive constants } C_1 \text{ and } M_1 \end{array} \right. \tag{13}$$

and

$$|\varphi(s) - ks| \leq C_2 \qquad \forall\, s \in I\!R. \tag{14}$$

Then the approximate controllability property holds for problem (1), i.e., given $y_d \in L^2(\Omega)$ and $\delta > 0$ there exists $v \in L^2(0,T; L^2(\omega))$ such that $\| y(T; v) - y_d \|_{L^2(\Omega)} < \delta$.

Remark 2 Notice that assumptions (13) and (14) are not fulfilled when φ is given by (2) with $m \in (0, 1)$.

As mentioned at the Introduction, the proof of Theorem 2 will be obtained through the study of the approximate controllability for the evanescent viscosity higher order problem (5).

Theorem 3 *Assume $\varphi \in \mathcal{C}^0(I\!R)$ (non necessarily nondecreasing) satisfying (2). Let $y_d \in L^2(\Omega)$ and $\delta > 0$. Then, for any $\varepsilon > 0$ there exists a control $v_\varepsilon \in L^\infty((0,T) \times \omega))$ such that if $y(t; v)$ is the corresponding solution of (5) we have*

$$\| y(T; v_\varepsilon) - y_d \|_{L^2(\Omega)} < \delta. \tag{15}$$

If in addition φ satisfies (13) and (14), then there exists a positive constant K, depending on k, C_1, C_2 and M_1 but independent of ε, such that the above controls v_ε can be taken satisfying

$$\| v_\varepsilon \|_{L^\infty((0,T)\times\omega)} \leq K, \qquad \text{for any } \varepsilon > 0. \tag{16}$$

The proof of the first part of Theorem 3 is an special formulation of the main result (Theorem 1) of Díaz and Ramos (1997b). The second part reproduces some of the steps of the proof of Theorem 1 of Díaz and Ramos (1997b) that here will be merely sketched but putting emphasis on the new arguments needed to arrive to the conclusion. The first step consists in proving the approximate controllability for a linearized problem (a posterior fixed point argument will extend the conclusion to the nonlinear problem). Since assumption (13) clearly implies that $\varphi'(s) \to k$ as $|s| \to \infty$, it is natural to define the function

$$\varphi_0(s) := \varphi(s) - ks, \qquad s \in I\!R \tag{17}$$

(so that $\varphi_0'(s) \to 0$ as $|s| \to \infty$). Then, it suffices to linearize function φ_0 which (by convenience) will be done near a point $s_\varepsilon \in I\!R$ depending on ε in a suitable way as shows the following result (that can be proved by elementary techniques of calculus)

Lemma 2 *Let $\varphi \in \mathcal{C}^0(I\!R)$ (non necessarily nondecreasing) satisfying (13). Given $\varepsilon > 0$ there exists $s_\varepsilon \in I\!R$ such that the function*

$$g_\varepsilon(s) := \frac{\varphi_0(s) - \varphi_0(s_\varepsilon)}{s - s_\varepsilon} \tag{18}$$

satisfies $g_\varepsilon \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$ *and*

$$\| g_\varepsilon \|_{L^\infty(I\!R)} \leq \sqrt{\varepsilon}. \tag{19}$$

If in addition φ *satisfies (14), then there exists a positive constant* K_2*, depending on* C_1*,* C_2 *and* M_1 *but independent of* ε*, such that*

$$|g_\varepsilon(s)s_\varepsilon| \leq K_2, \qquad \textit{for any } \varepsilon > 0 \textit{ and any } s \in I\!R. \tag{20}$$

Now we return to our linearizing process. Since $\varphi_0(s) = \varphi_0(s_\varepsilon) + g_\varepsilon(s)s - g_\varepsilon(s)s_\varepsilon$, we shall start by considering the approximate controllability for a linear problem obtained by replacing the term $\varphi(y)$ by $ky + g_\varepsilon(z)y + \varphi_0(s_\varepsilon) - g_\varepsilon(z)s_\varepsilon$, where z is an arbitrary function in $L^2(Q)$. Notice that when $z = y$ this expression coincides with $\varphi(y)$ and that if we denote $h_\varepsilon(z) := \Delta\left(\varphi_0\,(s_\varepsilon) - g_\varepsilon(z(t,z))s_\varepsilon\right)$, then $h_\varepsilon(z) \in L^\infty(0,T;H^{-2}(\Omega))$ for all $z \in L^2(Q)$ and for all $\varepsilon > 0$. Now, we consider the approximate controllability property corresponding to the linear problem

$$\left\{\begin{array}{ll} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta\left((g_\varepsilon(z)y\right) = h + h_\varepsilon(z) + u_\varepsilon\chi_\omega & \text{in } Q, \\ \dfrac{\partial^j y}{\partial\nu^j} = 0 \quad , \quad j = 0,1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{array}\right. \tag{21}$$

The existence and uniqueness of a solution $y \in L^2(0,T;H_0^2(\Omega))$, with $y_t \in L^2(0,T;H^{-2}(\Omega))$ was given in Proposition 4 of Díaz and Ramos (1997b).

Before stating an approximate controllability result for this problem, following Lions (1990), Fabre-Puel-Zuazua (1992) (1995) and Díaz-Ramos (1994), (1997a), we consider $\delta > 0$ and $y_d \in L^2(\Omega)$ and we introduce the functional $J_\varepsilon = J_\varepsilon(\cdot;z,y_d) : L^2(\Omega) \to I\!R$ defined by

$$J_\varepsilon(p^0;z,y_d) = J_\varepsilon(p^0) = \frac{1}{2}\left(\int_{\omega\times(0,T)} |p(t,x)|dxdt\right)^2 + \delta \| p^0 \|_{L^2(\Omega)} - \int_\Omega y_d p^0 dx \tag{22}$$

where $p(t,x)$ is the solution of the backward problem

$$\left\{\begin{array}{ll} -p_t + \varepsilon\Delta^2 p - k\Delta p - (g_\varepsilon(z))\Delta p = 0 & \text{in } Q, \\ \dfrac{\partial^j p}{\partial\nu^j} = 0 \quad , \quad j = 0,1 & \text{on } \Sigma, \\ p(T) = p^0 & \text{in } \Omega, \end{array}\right. \tag{23}$$

for any $p^0 \in L^2(\Omega)$ given. The existence and uniqueness of a solution $p \in L^2(0,T;H_0^2(\Omega))$, with $p_t \in L^2(0,T;H^{-2}(\Omega))$ was given in Proposition 1 of Díaz and Ramos (1997b). Moreover, some easy modifications of the arguments given in Fabre, Puel and Zuazua (1992), (1995) and the Unique Continuation property (see Saut and Scheurer (1987)) allow to show that the functional $J_\varepsilon(\cdot;z,y_d)$ is continuous, strictly convex on $L^2(\Omega)$ and satisfies

$$\liminf_{\|p^0\|_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(p^0;z,y_d)}{\| p^0 \|_{L^2(\Omega)}} \geq \delta. \tag{24}$$

Then $J_\varepsilon(\cdot;z,y_d)$ attains its minimum at a unique point $\widehat{p}^0_\varepsilon$ in $L^2(\Omega)$. Furthermore, $\widehat{p}^0_\varepsilon = 0$ iff $\| y_d \|_{L^2(\Omega)} \leq \delta$.

Concerning the approximate controllability of problem (21) we have

Theorem 4 *Let $z \in L^2(Q)$. Assume g_ε satisfying (19) and (20). Let $\| y_d - y(T;z,0) \|_{L^2(\Omega)} > \delta$ and let $\widehat{p}_\varepsilon$ be the solution of (23) corresponding to $\widehat{p}(T) = \widehat{p}_\varepsilon^0$, with $\widehat{p}_\varepsilon^0$ minimum of $J_\varepsilon(\cdot; z, y_d - y(T;z,0))$, where in general $y(t;z,u)$ denotes the solution of (21) corresponding to the control u. Then there exists $\widehat{q}_\varepsilon \in sgn(\widehat{p}_\varepsilon)\chi_\omega$ such that the solution y_ε of*

$$\begin{cases} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta\left((g_\varepsilon(z))y\right) = h + h_\varepsilon(z) + \| \widehat{p}_\varepsilon \|_{L^1((0,T)\times\omega)} \widehat{q}_\varepsilon \chi_\omega & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad j = 0,1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\| y_\varepsilon(T) - y_d \|_{L^2(\Omega)} \leq \delta. \tag{25}$$

Moreover, if $\| y_d - y(T;z,0) \|_{L^2(\Omega)} \leq \delta$, then property (25) holds for the control $v_\varepsilon \equiv 0$. Finally, if φ satisfies (13) and (14), there exists a positive constant K, depending on k, C_1, C_2 and M_1 but independent of ε, such that the above functions $\widehat{p}_\varepsilon$ satisfy

$$\| \widehat{p}_\varepsilon \|_{\mathcal{C}([0,T];L^2(\Omega))} \leq K, \quad \textit{for any } \varepsilon > 0 \textit{ and any } z \in L^2(Q). \tag{26}$$

Remark 3 Theorem 4 solves the approximate controllability problem for (21) with control $u_\varepsilon := \| \widehat{p}_\varepsilon \|_{L^1((0,T)\times\omega)} \widehat{q}_\varepsilon$. Therefore

$$\| u_\varepsilon \|_{L^\infty(Q)} \leq K. \tag{27}$$

Proof of Theorem 4. We put $y_\varepsilon = L_\varepsilon + Y_\varepsilon$, where $L_\varepsilon = L_\varepsilon(z)$ satisfies

$$\begin{cases} L_t + \varepsilon\Delta^2 L - k\Delta L - \Delta\left((g_\varepsilon(z))L\right) = h + h_\varepsilon(z) & \text{in } Q, \\ \dfrac{\partial^j L}{\partial \nu^j} = 0 \qquad j = 0,1 & \text{on } \Sigma, \\ L(0) = y_0 & \text{in } \Omega \end{cases} \tag{28}$$

and $Y_\varepsilon = Y_\varepsilon(z)$ is taken associated to the approximate controllability problem

$$\begin{cases} Y_t + \varepsilon\Delta^2 Y - k\Delta Y - \Delta\left((g_\varepsilon(z))Y\right) = u_\varepsilon(z)\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j Y}{\partial \nu^j} = 0 \qquad j = 0,1 & \text{on } \Sigma, \\ Y(0) = 0 & \text{in } \Omega, \end{cases}$$

with desired state $y_d - L_\varepsilon(T)$, i.e. such that $\| Y_\varepsilon(T) - (y_d - L\varepsilon(T)) \| \leq \delta$. Assuming (2), by Theorem 2 of Díaz and Ramos (1997b), there exists $\widehat{q}_\varepsilon \in sign\ (\widehat{p}_\varepsilon)\chi_\omega$, with $\widehat{p}_\varepsilon$ solution of (23) of initial value $\mathcal{M}_\varepsilon(z, y_d - L_\varepsilon(T))$, where $\mathcal{M}_\varepsilon : L^2(Q) \times L^2(\Omega) \longrightarrow L^2(\Omega)$ with $\mathcal{M}(z, y_d) = \widehat{p}_\varepsilon^0$ (it can be shown that, if K is a compact subset of $L^2(\Omega)$, then, for any fixed $\varepsilon > 0$, $\mathcal{M}_\varepsilon(L^2(Q) \times K)$ is a bounded subset of $L^2(\Omega)$), such that $u_\varepsilon(z) := \| \widehat{p}_\varepsilon \|_{L^1((0,T)\times\omega)} \widehat{q}_\varepsilon$ leads to $\| Y(T) - \widehat{y}_d \|_{L^2(\Omega)} = \delta$, where $\widehat{y}_d := y_d - L_\varepsilon(T)$ (in the case $\| \widehat{y}_d \|_{L^2(\Omega)} \leq \delta$ it suffices to take $u_\varepsilon \equiv 0$). For the proof of (26) we have

Lemma 3 *Assume (19) and (20). Let $z \in L^2(Q)$. Let $p_0 \in L^2(\Omega)$ be given. Then, if p_ε is the solution of (23), we have*

$$\| p_\varepsilon \|_{\mathcal{C}([0,T];L^2(\Omega))} \leq e^T \| p^0 \|_{L^2(\Omega)} \qquad \textit{for any } \varepsilon > 0 \textit{ and any } z \in L^2(Q). \tag{29}$$

Proof. If we "multiply" in (23) by p_ε, for any $t \in (0,T]$ we obtain

$$\frac{1}{2} \parallel p_\varepsilon(t) \parallel^2_{L^2(\Omega)} + \varepsilon \parallel \Delta p_\varepsilon \parallel^2_{L^2((t,T)\times\Omega)} + k \parallel \nabla p_\varepsilon \parallel^2_{L^2((t,T)\times\Omega)} \leq$$

$$\frac{1}{2} \parallel p_\varepsilon(T) \parallel^2_{L^2(\Omega)} + \parallel g_\varepsilon(z(t,x)) \parallel_{L^\infty(Q)} \parallel \Delta p_\varepsilon \parallel_{L^2((t,T)\times\Omega)} \parallel p_\varepsilon \parallel_{L^2((t,T)\times\Omega)} .$$

Then, if we apply Young's inequality, we have that

$$\frac{1}{2} \parallel p_\varepsilon(t) \parallel^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \parallel \Delta p_\varepsilon \parallel^2_{L^2((t,T)\times\Omega)} \leq \frac{1}{2} \parallel p_\varepsilon(T) \parallel^2_{L^2(\Omega)} + \frac{1}{2} \parallel p_\varepsilon \parallel^2_{L^2((t,T)\times\Omega)} .$$

Then we obtain that

$$\parallel p_\varepsilon(t) \parallel^2_{L^2(\Omega)} \leq \parallel p_\varepsilon(T) \parallel^2_{L^2(\Omega)} + \int_t^T \parallel p_\varepsilon \parallel^2_{L^2(\Omega)} .$$

Applying Gronwall's inequality, we deduce the following inequality leading to (29)

$$\parallel p_\varepsilon(t) \parallel^2_{L^2(\Omega)} \leq \parallel p_\varepsilon(T) \parallel^2_{L^2(\Omega)} e^{T-t} \qquad \forall\, t \in [0,T].$$

■

Completion of proof of Theorem 4. From (20) we deduce that there exists a constant K_3, depending on C_1, C_2 and M_1 but independent of ε, such that $\parallel L_\varepsilon(z) \parallel_{\mathcal{C}([0,T];L^2(\Omega))} \leq K_3$ for any $\varepsilon > 0$ and any $z \in L^2(Q)$. Moreover, Lemma 3 implies that for any $\varepsilon > 0$ and $z \in L^2(Q)$

$$J_\varepsilon(p^0; z, y_d) \leq \frac{1}{2}|\omega| e^{2T} T^2 \parallel p^0 \parallel^4_{L^2(\Omega)} + \parallel p^0 \parallel_{L^2(\Omega)} - \int_\Omega y_d p^0 dx.$$

Thus, there exists a constant K_4, depending on C_1, C_2 and M_1 but independent of ε, such that, if $\widehat{p}^0_\varepsilon$ is the minimum of $J_\varepsilon(\cdot; z, y_d - L_\varepsilon(T))$, we have $\parallel \widehat{p}^0_\varepsilon \parallel_{L^2(\Omega)} \leq K_4$ for any $\varepsilon > 0$ and any $z \in L^2(Q)$. Lemma 3 implies (26) with $K = e^T K_4$.

■

Proof of Theorem 3. The first part was proved in Theorem 1 of Díaz and Ramos (1997b) by applying Kakutani's fixed point theorem to the operator $\Lambda_\varepsilon : L^2(Q) \to \mathcal{P}(L^2(Q))$ defined by $\Lambda_\varepsilon(z) := \{y_\varepsilon$ satisfying (21), (25), with a control u_ε satisfying (27)$\}$, where the constant K of (27) depends on ε. Finally, if φ satisfies (13) and (14), then Proposition 2 shows that (26) holds, which leads to (16) with $K = e^T K_4$.

■

Proof of Theorem 2. *First step.* Assume additionally that $\varphi \in \mathcal{C}^1(I\!R)$. For any $\varepsilon > 0$, let v_ε and y_ε be the functions given in Theorem 3. Since the equation of (5) holds on $L^2(0,T;H^{-2}(\Omega))$, multiplying by $y_\varepsilon \in L^2(0,T;H^2_0(\Omega))$ and applying Young and Gronwall inequalities we obtain, from the uniform estimate (16), that there exists a constant $C > 0$ independent of ε such that

$$\parallel y_\varepsilon \parallel_{\mathcal{C}([0,T];L^2(\Omega))} + \int_Q \varphi'(y_\varepsilon)|\nabla(y_\varepsilon)|^2 dxdt \leq C. \tag{30}$$

Therefore, from (30) we obtain that y_ε is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and by the equation of (5), $(y_\varepsilon)_t$ is uniformly bounded in $L^\infty(0,T;H^{-4}(\Omega))$. Then, since $L^2(\Omega) \subset H^{-1}(\Omega) \subset H^{-4}(\Omega)$ with compact imbeddings, we have (see Aubin (1963)) that y_ε is relatively compact in $L^\infty(0,T;H^{-1}(\Omega))$. Further, from (30) and the boundedness of function φ' (notice that $\varphi' \in L^\infty(I\!R)$ by (13)), we deduce that there exists a constant $K > 0$ independent of ε such that

$$\int_0^T \| \nabla\varphi(y_\varepsilon) \|^2_{L^2(\Omega)} \, dt = \int_Q \varphi'(y_\varepsilon(x,t)) \, \varphi'(y_\varepsilon(x,t)) |\nabla(y_\varepsilon(x,t))|^2 dxdt < K.$$

Thus, there exist $y \in L^\infty(0,T;L^2(\Omega))$ and $\zeta \in L^2(0,T;H^1_0(\Omega))$ such that $y_\varepsilon \to y$ strongly in $L^2(0,T;H^{-1}(\Omega))$ and $\varphi(y_\varepsilon) \rightharpoonup \zeta$ weakly in $L^2(0,T;H^1_0(\Omega))$. But the operator $Au := -\Delta\varphi(u)$, $D(A) := \{u \in H^{-1}(\Omega) : \varphi(u) \in H^1_0(\Omega)\}$ is a maximal monotone operator on the space $H^{-1}(\Omega)$ (see Brézis (1971)). Thus, the extension operator $\mathcal{A}$ of A is also a maximal monotone operator on $L^2(0,T;H^{-1}(\Omega))$ (see Brézis (1973), Example 2.33). Finally, as any maximal monotone operator is strongly-weakly closed (see Brézis (1973), Proposition 2.5), we obtain that $\zeta = \varphi(y)$ in $L^2(0,T;H^1_0(\Omega))$. Moreover, from estimate (16) we have that $v_\varepsilon \rightharpoonup v$ $*$-weakly in $L^\infty((0,T)\times\omega)$, with

$$\| v \|_{L^\infty((0,T)\times\omega)} \leq K. \tag{31}$$

Then we deduce that $y \in \mathcal{C}([0,T];H^{-1}(\Omega))$ is solution of (1). Further, since $\| y_\varepsilon(T) \|_{L^2(\Omega)}$ is uniformly bounded and $y_\varepsilon(T) \to y(T)$ strongly in $H^{-1}(\Omega)$, we deduce that $y_\varepsilon(T) \rightharpoonup y(T)$ in the weak topology of $L^2(\Omega)$, which implies that

$$\| y(T) - y_d \|_{L^2(\Omega)} \leq \liminf_{\varepsilon\to 0} \| y_\varepsilon(T) - y_d \|_{L^2(\Omega)} \leq \delta.$$

Second step. Let φ as in the statement of Theorem 2. It is clear that we can approximate φ by $\varphi_n \in \mathcal{C}^1(I\!R)$, φ_n nondecreasing, satisfying (13) and (14) with the same constants k, C_1, C_2 and M_1 that the ones for φ. Then the respective controls v_n build as in step 1 are uniformly bounded and therefore the conclusion comes as an easy modification of the well-known result expressing the continuous dependence on φ of solutions of (1) (see e.g. Benilan and Crandall (1981)).

■

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