On the Approximate Controllability for Higher Order Parabolic Nonlinear Equations of Cahn-Hilliard Type

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Abstract. We prove the approximate controllability property for some higher order parabolic nonlinear equations of Cahn-Hilliard type when the nonlinearity is of sublinear type at infinity. We also give a counterexample showing that this property may fail when the nonlinearity is of superlinear type.

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1 Introduction.

Let Ω be a bounded open subset of $I\!R^N$ of class C^{2m}, $T > 0$, ω a nonempty open subset of Ω, f a continuous real function and $k \in I\!N$ such that $0 \leq 2k \leq m$. The main goal of this work is the study of the approximate controllability of the following semilinear equation with Dirichlet boundary conditions:

$$(1) \qquad \begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k f(y) = h + v\chi_\omega & \text{in } Q := \Omega \times (0,T), \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

where v is a suitable output control, χ_ω is the characteristic function of ω, ν is the unit outward normal vector, $h \in L^2(0,T : H^{-m}(\Omega))$ and $y_0 \in L^2(\Omega)$. Due to the term χ_ω the controls are assumed supported on the set $\mathcal{O} := \omega \times (0,T)$. Problems as (1), sometimes known as Cahn-Hilliard problems, appear, with $m = 2$, in the study of phase separation in cooling binary solutions and in other contexts generating spatial pattern formation (see [6], [8] and the references cited therein).

We recall that problem (1) satisfies the approximate controllability property, at time T with states space X and controls space Y, if the set

$$\{\ y(T, \cdot : v) :\ v \in Y,\ y \text{ solution of (1)}\}$$

is dense in X.

The main goal of this paper is to extend the approximate controllability results on second order problems, $m = 1$ and $k = 0$ (see e.g. [9], [10] and [7]) to the case of higher order equations for which the maximum principle does not hold, in general. Our first result gives a positive answer when f is assumed to be sublinear at the infinity:

Theorem 1 *Assume that f satisfies the following conditions: there exist some positive constants c_1 and c_2 such that*

$$(2) \qquad |f(s)| \leq c_1 + c_2|s| \quad \textit{for all } s \in I\!R$$

and

$$(3) \qquad \textit{there exists } f'(s_0) \textit{ for some } s_0 \in I\!R.$$

Then problem (1) satisfies the approximate controllability property at time T with states space $X = L^2(\Omega)$ and controls space $Y = L^2(\mathcal{O})$.

In contrast to the above result, we shall prove that when f is superlinear the approximate controllability property does not hold in general, as explained in Section 4. Therefore if, for instance, $f(s) = |s|^{p-1}s$ Theorem 1 gives a positive approximate controllability result for $0 < p \leq 1$. The results of section 6 provide a negative approximate controllability answer when $1 < p < \infty$. The similar alternative was obtained in Díaz-Ramos [7] for second order parabolic semilinear problems.

We remark that the existence of solutions in the class

$$y \in L^2(0,T;H_0^m(\Omega)) \cap C([0,T];L^2(\Omega)),\ f(y) \in L^2(Q),\ \Delta^k f(y) \in L^2(0,T;H^{-m}(\Omega)),$$

is also obtained as a by-product of Theorem 1 for a suitable subclass of controls. The uniqueness of solutions can be easily proved if, for instance, f is nondecreasing or Lipschitz continuous. Those uniqueness results are not needed in our arguments.

2 Approximate Controllability for an Associated Linear problem.

In order to prove Theorem 1 we follows the same scheme of proof than in [9], [10] and [7]. We define the function

$$g(s) = \frac{f(s) - f(s_0)}{s - s_0}.$$

From assumptions (2) and (3) we have that $g \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$. The conclusion will be derived from a fixed point argument. As $f(s) = f(s_0) + g(s)s - g(s)s_0$, we shall start by considering the approximate controllability for a linear problem obtained by replacing the term $f(y)$ by

$$g(z)y + f(s_0) - g(z)s_0,$$

where z is an arbitrary function in $L^2(Q)$. Notice that when $z = y$ this expression coincides with $f(y)$ and that if we denote $g(z(t,x)) := a(t,x)$ and

$$h(a) := -(-\Delta)^k f(s_0) + (-\Delta)^k (a(t,x)s_0), \tag{4}$$

then $a \in L^\infty(Q)$ and $h(a) \in L^\infty(0,T;H^{-2k}(\Omega))$. More in general, given $a \in L^\infty(Q)$ and $h(a)$ defined by (4), we consider the approximate controllability property corresponding to the linear problem

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k (a(t,x)y) = h + h(a) + u\chi_\omega & \text{in } Q := \Omega \times (0,T), \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\cdots,m-1 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{5}$$

Before stating an approximate controllability result for this problem, following Lions [14] and Fabre-Puel-Zuazua [9], [10], we consider $\varepsilon > 0$ and $y_d \in L^2(\Omega)$ and we introduce the functional $J = J(\cdot; a, y_d) : L^2(\Omega) \to I\!R$ defined by

$$J(\varphi_0; a, y_d) = J(\varphi^0) = \frac{1}{2}\left(\int_{\mathcal{O}} |\varphi(t,x)| dxdt\right)^2 + \varepsilon \parallel \varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega y_d\, \varphi^0, dx \tag{6}$$

where $\varphi(t,x)$ is the solution of the backward problem

$$\begin{cases} -\varphi_t + (-\Delta)^m \varphi + a(t,x)\Delta^k \varphi = 0 & \text{in } Q := \Omega \times (0,T), \\ \dfrac{\partial^j \varphi}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\cdots,m-1 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ \varphi(T) = \varphi^0 & \text{in } \Omega. \end{cases} \tag{7}$$

To study the above backward problem we introduce the space

$$W := \{y \in L^2(0,T;H_0^m(\Omega)),\ y_t \in L^2(0,T;H^{-m}(\Omega))\}.$$

The following result will be used later

Proposition 1 *Given $h \in L^2(0,T;H^{-m}(\Omega))$ and $y_0 \in L^2(\Omega)$, there exists a unique function $y \in W$ satisfying*

$$(8) \qquad \begin{cases} y_t + (-\Delta)^m y + a(t,x)\Delta^k y = h & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\cdots,m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Furthermore, we have the estimate

$$(9) \qquad \|y\|_{L^2(0,T;H_0^m(\Omega))} + \|y_t\|_{L^2(0,T;H^{-m}(\Omega))} \le C\left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right),$$

where the constant C depends only on $M := \| a \|_{L^\infty(Q)}$ (provided that Ω, T and m are kept fixed). Moreover, if $h \in L^2(Q)$, the solution y also satisfies that

$$(10) \qquad y \in L^2(\delta,T;H^{2m}(\Omega)) \quad \text{and} \quad y_t \in L^2((\delta,T)\times\Omega) \quad \text{for all } \delta \in (0,T).$$

Proof. For all $n \in I\!N$ we define y^{n+1} as the solution of the following iterative problem

$$\begin{cases} y_t^{n+1} + (-\Delta)^m y^{n+1} = h - a(t,x)\Delta^k y^n & \text{in } Q, \\ \dfrac{\partial^j y^{n+1}}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\cdots,m-1 & \text{on } \Sigma, \\ y^{n+1}(0) = y_0 & \text{in } \Omega, \end{cases}$$

where $y^0(t) := 0$ for all $t \in [0,T]$. The existence of a solution $y^n \in W$ can be found, for instance, in Theorem 3.4.1 of Lions-Magenes [15]. Thus, for all $n \in I\!N\backslash\{0,1\}$, $y^{n+1} - y^n$ satisfies

$$(11) \qquad \begin{cases} (y^{n+1}-y^n)_t + (-\Delta)^m (y^{n+1}-y^n) = -a(t,x)\Delta^k (y^n - y^{n-1}) & \text{in } Q, \\ \dfrac{\partial^j (y^{n+1}-y^n)}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\cdots,m-1 & \text{on } \Sigma, \\ (y^{n+1}-y^n)(0) = 0 & \text{in } \Omega \end{cases}$$

and therefore

$$y^{n+1} - y^n \in H^{1,2m}(Q) := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^{2m}(\Omega))$$

and

$$\| y^{n+1} - y^n \|_{H^{1,2m}(Q)} \le c_1 \| a\Delta^k (y^n - y^{n-1}) \|_{L^2(Q)}$$

(see, for instance, Theorem 4.6.1 of Lions-Magenes [16]). Then, since

$$H^{1,2m}(Q) \subset \mathcal{C}([0,T];H^m(\Omega))$$

with continuous imbedding (see, for instance, Theorems 1.3.1 and 1.9.6 of Lions-Magenes [15]), there exists $c_2 = c_2(T)$ such that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];H_0^m(\Omega))} \le c_2 \| a\Delta^k (y^n - y^{n-1}) \|_{L^2(Q)} .$$

Further, it is clear that we can choose $C_2 = C_2(T)$ such that for all $t \in [0,T]$

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,t];H_0^m(\Omega))} \le C_2 \| a\Delta^k (y^n - y^{n-1}) \|_{L^2((0,t)\times\Omega)} .$$

Hence,

$$\| (y^{n+1} - y^n)(t) \|^2_{H_0^m(\Omega)} \le (C_2 M)^2 \int_0^t \| \Delta^k (y^n - y^{n-1})(\tau) \|^2_{L^2(\Omega)} \, d\tau, \quad \text{for all } t \in [0,T]$$

and therefore, by using the Poincaré inequality, there exists a constant K, independent of M, such that

$$\| (y^{n+1} - y^n)(t) \|^2_{H_0^m(\Omega)} \le (KM)^2 \int_0^t \| (y^n - y^{n-1})(\tau) \|^2_{H_0^m(\Omega)} \, d\tau, \quad \text{for all } t \in [0,T].$$

Then, for every $t \in [0,T]$ we deduce that

$$\| (y^{n+1} - y^n)(t) \|^2_{H_0^m(\Omega)} \le (K^2 M^2)^{n-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \| (y^2 - y^1)(\tau_n) \|^2_{H_0^m(\Omega)} \, d\tau_n \cdots d\tau_1$$

$$\leq (K^2M^2)^{n-1}\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];H_0^m(\Omega))}\, d\tau_n\cdots d\tau_1$$

$$\leq (K^2M^2)^{n-1}\frac{t^{n-1}}{(n-1)!} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];H_0^m(\Omega))}$$

$$\leq \frac{(K^2M^2T)^{n-1}}{(n-1)!} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];H_0^m(\Omega))},$$

which implies that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];H_0^m(\Omega))} \to 0 \quad \text{as } n\to\infty$$

and therefore, by (11), we deduce that

$$\| (y^{n+1} - y^n)_t \|_{L^2(0,T;H^{-m}(\Omega))} \to 0 \quad \text{as } n\to\infty.$$

Then, there exists $y \in W$ such that

$$y_n \to y \quad \text{in } W \quad \text{as } n\to\infty.$$

In order to prove that y satisfies (8) we point out that

$$\Delta^m y^n \to \Delta^m y \quad \text{in } L^2(0,T;H^{-m}(\Omega)) \text{ as } n\to\infty,$$

$$\Delta^k y^n \to \Delta^k y \quad \text{in } L^2(\Omega) \text{ as } n\to\infty,$$

and

$$y_t^n \to y_t \quad \text{in } L^2(0,T;H^{-m}(\Omega)) \text{ as } n\to\infty.$$

this implies (passing to the limit) that y is the solution of (8). In order to prove (9), we "multiply" in (8) by y. Then it is easy to see that

$$\|y\|_{L^2(0,T;H_0^m(\Omega))} + \|y_t\|_{L^2(0,T;H^{-m}(\Omega))} \leq C\left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)} + \|y\|_{L^2(Q)}\right). \tag{12}$$

Furthermore,

$$\| y(t) \|^2_{L^2(\Omega)} \leq \left(\| y(0) \|^2_{L^2(\Omega)} + c_2 \| h \|^2_{L^2(0,T;H^{-m}(\Omega))}\right) + c_3\int_0^t \| y(s) \|^2_{L^2(\Omega)}\, ds.$$

Then, applying Gronwall's inequality (see, for instance, Lemma 4 of Haraux [11]), we deduce that

$$\| y(t) \|^2_{L^2(\Omega)} \leq \left(\| y(0) \|^2_{L^2(\Omega)} + c_2 \| h \|^2_{L^2(0,T;H^{-m}(\Omega))}\right) e^{c_3 t} \qquad \forall\, t\in[0,T].$$

From here, we obtain that

$$\| y \|_{L^2(Q)} \leq c_4\left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right)$$

which implies, together with (12), inequality (9). Now, thanks to (9) and the linearity of Problem (8), we deduce the uniqueness of solution.

Finally, if $h \in L^2(Q)$, since $y(\delta) \in H_0^m(\Omega)$ for all $\delta \in (0,T)$, taking $y(\delta)$ as initial datum and applying Theorem 4.6.1 of [16], we get (10).

■

As usual in Controllability Theory we shall use a *unique continuation* property for solutions of the *dual problem* (in our case Problem (7)).

Lemma 1 *Let ω be a nonempty open subset of Ω. Assume that*

$$\varphi \in L^2(0,T;H_0^m(\Omega)) \,\cap\, C([0,T];L^2(\Omega))$$

satisfies (7) and that $\varphi \equiv 0$ in $\mathcal{O} = \omega\times(0,T)$. Then $\varphi\equiv 0$ in Q.

Proof. From Proposition 1 (applied with backward time) we deduce that $\varphi \in L^2(0,T-\delta;H^{2m}(\Omega))$ for all $\delta \in (0,T)$. Then Lemma 1 follows from Theorem 3.2 of Saut-Scheurer [17].

■

The following two results are easy adaptations (by using Lemma 1) of the similar ones given in [9], [10] for second order parabolic problems.

Proposition 2 *The functional $J(\cdot;a,y_d)$ is continuous, strictly convex on $L^2(\Omega)$ and verifies*

$$(13)\qquad \liminf_{\|\varphi^0\|_{L^2(\Omega)}\to\infty} \frac{J(\varphi^0;a,y_d)}{\|\varphi^0\|_{L^2(\Omega)}} \geq \varepsilon.$$

Further $J(\cdot;a,y_d)$ attains its minimum at a unique point $\widehat{\varphi}^0$ in $L^2(\Omega)$ and

$$(14)\qquad \widehat{\varphi}^0 = 0 \quad \Leftrightarrow \quad \| y_d \|_{L^2(\Omega)} \leq \varepsilon.$$

Proposition 3 *Let $\mathcal{M}$ be the mapping*

$$\begin{array}{rccc} \mathcal{M}: & L^\infty(Q)\times L^2(\Omega) & \rightarrow & L^2(\Omega) \\ & (a(t,x),y_d) & \longrightarrow & \widehat{\varphi}^0. \end{array}$$

If B is a bounded subset of $L^\infty(Q)$ and K is a compact subset of $L^2(\Omega)$, then $\mathcal{M}(B\times K)$ is a bounded subset of $L^2(\Omega)$.

In order to characterize the duality of problem (7), we recall that given a convex and proper function $V : X \to I\!R \cup \{+\infty\}$ on the Banach space X, it is said that a element p_0 of V' belongs to the set $\partial V(x_0)$ (subdifferential of V at $x_0 \in X$) if

$$V(x_0) - V(x) \leq (p_0, x_0 - x) \quad \forall\, x \in X.$$

It is well known that that if V is Gateaux differentiable its differential coincides with its subdifferential and that x_0 minimizes V over X (or over a convex subset of X) if and only if $0 \in \partial V(x_0)$. Finally, if V is a lower semicontinuous function, then $p_0 \in \partial V(x_0)$ if and only if

$$(p_0, x) \leq \lim_{h\to 0^+} \frac{V(x_0+hx)-V(x_0)}{h} (< +\infty) \quad \forall\, x \in X.$$

(See, for instance, Aubin-Ekeland [3]). Coming back to the functional J we have:

Lemma 2 *For every $\varphi^0 \in L^2(\Omega)$ ($\varphi^0 \neq 0$), if φ is the solution of (7) satisfying $\varphi(T) = \varphi^0$, we have that*

$$\partial J(\varphi^0;a,y_d) = \{\xi \in L^2(\Omega),\ \exists\ v \in sgn(\varphi)\chi_{\mathcal{O}} \text{ satisfying}$$

$$\begin{aligned} \int_\Omega \xi(x)\theta^0(x)dx &= \left(\int_{\mathcal{O}} |\varphi(t,x)| d\Sigma\right)\left(\int_{\mathcal{O}} v(t,x)\theta(t,x)d\Sigma\right) \\ &\quad + \varepsilon \int_\Omega \frac{\varphi^0(x)}{\|\varphi^0\|_{L^2(\Omega)}}\theta^0(x)dx - \int_\Omega y_d(x)\theta^0(x)dx \ \forall \theta^0 \in L^2(\Omega)\}, \end{aligned}$$

where θ is the solution of (7) satisfying $\theta(T) = \theta^0$.

Proof. It is an easy modification of Proposition 2.4 of [10].

Let us prove the approximate controllability property for an special version of the linear problem given in (5).

Theorem 2 *If $\| y_d \|_{L^2(\Omega)} > \varepsilon$ and $\widehat{\varphi}$ is the solution of (7) corresponding to $\widehat{\varphi}(T) = \widehat{\varphi}^0$, with $\widehat{\varphi}^0$ minimum of $J(\cdot;a,y_d)$. Then there exists $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$ such that the solution of*

$$(15)\qquad \left\{ \begin{array}{ll} y_t + (-\Delta)^m y + (-\Delta^k)(a(t,x)y) = \| \widehat{\varphi} \|_{L^1(\mathcal{O})} v\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad (j = 0 \cdots (m-1)) & \text{on } \Sigma, \\ y(0) = 0 & \text{in } \Omega, \end{array} \right.$$

satisfies

$$y(T) = y_d - \varepsilon \frac{\widehat{\varphi}^0}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}},$$

and then $\| y(T) - y_d \|_{L^2(\Omega)} = \varepsilon$.

Remark 1 In the case $\| y_d \|_{L^2(\Omega)} \leq \varepsilon$, if we use the null control, we obtain $y = 0$ and therefore $\| y(T) - y_d \|_{L^2(\Omega)} \leq \varepsilon$.

First of all we prove the existence and uniqueness to problem given by (5).

Proposition 4 *Assumed $y_0 \in L^2(\Omega)$, $h \in L^2(0,T;H^{-m}(\Omega))$ and $a(t,x) \in L^\infty(Q)$, there exists a unique function $y \in W$ satisfying*

$$(16) \qquad \begin{cases} y_t + (-\Delta)^m y + \Delta^k (a(t,x)y) = h & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Moreover, we have the estimate

$$(17) \qquad \|y\|_{L^2(0,T;H_0^m(\Omega))} + \|y_t\|_{L^2(0,T;H^{-m}(\Omega)} \leq C \left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)} \right),$$

where the constant C depends only on M (provided that Ω, T and m are kept fixed).

Proof. For all $n \in I\!N$ we define again y^{n+1} as the solution of the iterative problem

$$\begin{cases} y_t^{n+1} + (-\Delta)^m y^{n+1} = h - \Delta^k (a(t,x)y^n) & \text{in } Q, \\ \dfrac{\partial^j y^{n+1}}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma, \\ y^{n+1}(0) = y_0 & \text{in } \Omega, \end{cases}$$

where $y^0(t) := 0$ for all $t \in [0,T]$. The existence of a solution $y^n \in W$ can be found, for instance, in Theorem 3.4.1 of Lions-Magenes [15]. Thus, for all $n \in I\!N \backslash \{0,1\}$, $y^{n+1} - y^n$ is solution of

$$(18) \qquad \begin{cases} (y^{n+1} - y^n)_t + (-\Delta)^m (y^{n+1} - y^n) = -\Delta^k [a(t,x)(y^n - y^{n-1})] & \text{in } Q, \\ \dfrac{\partial^j (y^{n+1} - y^n)}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma, \\ (y^{n+1} - y^n)(0) = 0 & \text{in } \Omega \end{cases}$$

and therefore (see again Theorem 3.4.1 of Lions-Magenes [15]) $y^{n+1} - y^n \in W$ and

$$(19) \qquad \| y^{n+1} - y^n \|_W \leq c_1 \| a(y^n - y^{n-1}) \|_{L^2(Q)} .$$

Then, since $W \subset \mathcal{C}([0,T]; L^2(\Omega))$ with continuous imbedding (see, for instance, [12] or [15]), we have that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];L^2(\Omega))} \leq c_2 \| a(y^n - y^{n-1}) \|_{L^2(Q)} .$$

Further, as in the proof of Proposition 1, we can choose $C_2 = C_2(T)$ such that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,t];L^2(\Omega))} \leq C_2 \| a(y^n - y^{n-1}) \|_{L^2((0,t)\times\Omega)}, \quad \text{for all } t \in [0,T].$$

Hence,

$$\| (y^{n+1} - y^n)(t) \|_{L^2(\Omega)}^2 \leq (C_2 M)^2 \int_0^t \| (y^n - y^{n-1})(\tau) \|_{L^2(\Omega)}^2 \, d\tau, \quad \text{for all } t \in [0,T]$$

Then, for every $t \in [0,T]$ we deduce that

$$\begin{aligned} \| (y^{n+1} - y^n)(t) \|_{L^2(\Omega)}^2 &\leq (C_2^2 M^2)^{n-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \| (y^2 - y^1)(\tau_n) \|_{L^2(\Omega)}^2 \, d\tau_n \cdots d\tau_1 \\ &\leq (C_2^2 M^2)^{n-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \| y^2 - y^1 \|_{\mathcal{C}([0,T];L^2(\Omega))}^2 \, d\tau_n \cdots d\tau_1 \\ &\leq (C_2^2 M^2)^{n-1} \frac{t^{n-1}}{(n-1)!} \| y^2 - y^1 \|_{\mathcal{C}([0,T];L^2(\Omega))}^2 \\ &\leq \frac{(C_2^2 M^2 T)^{n-1}}{(n-1)!} \| y^2 - y^1 \|_{\mathcal{C}([0,T];L^2(\Omega))}^2, \end{aligned}$$

which implies that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];L^2(\Omega))} \to 0 \quad \text{as} \quad n \to \infty$$

and therefore, by (19), we deduce that

$$\| y^{n+1} - y^n \|_W \to 0 \quad \text{as} \quad n \to \infty.$$

Then, there exists $y \in W$ such that

$$y_n \to y \quad \text{in } W \quad \text{as } n \to \infty.$$

The end of the proof is similar to the end of the proof of Proposition 1. ■

Proof of Theorem 2. Using the subdifferentiability of $J(.;a,y_d)$ at $\widehat{\varphi}^0$ ($\neq 0$ by (14)), we know that

$$0 \in \partial J(\widehat{\varphi}^0),$$

which is equivalent, from Lemma 2, to the existence of $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$, such that

$$(20) \qquad -\| \widehat{\varphi} \|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}} v(x,t)\theta(x,t)dxdt \right) = \frac{\varepsilon}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}} \int_\Omega \widehat{\varphi}^0(x)\theta^0(x)dx - \int_\Omega y_d(x)\theta^0(x)dx.$$

On the other hand, as $y \in W$, if we "multiply" by θ in (15) we obtain, by (7), that

$$(21) \qquad \int_\Omega y(T,x)\theta^0(x)dxdt = \| \widehat{\varphi} \|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}} v(x,t)\theta(x,t)dxdt \right)$$

Then, from (20) and (21), we obtain

$$\int_\Omega y(T,x)\theta^0(x)dxdt = \int_\Omega (y_d(x) - \varepsilon \frac{\widehat{\varphi}^0(x)}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}})\theta^0(x)dxdt \quad \forall\ \theta^0 \in L^2(\Omega)$$

and we conclude that $y(T) = y_d - \varepsilon \dfrac{\widehat{\varphi}^0}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}}$. ■

Now we are ready to prove a linear version of Theorem 1 for problem (5)

Corollary 1 *Let $\| y_d \|_{L^2(\Omega)} > \varepsilon$ and $\widehat{\varphi}$ the solution of (7) corresponding to $\widehat{\varphi}(T) = \widehat{\varphi}^0$, with $\widehat{\varphi}^0$ minimum of $J(\cdot;a,y_d - y(T;a,0))$, where in general $y(t;a,u)$ denotes the solution of (5) corresponding to the control u. Then there exists $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$ such that the solution of*

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta^k)(a(t,x)y) = h + h(a) + \| \widehat{\varphi} \|_{L^1(\mathcal{O})} v\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad (j = 0 \cdots (m-1)) & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\| y(T) - y_d \|_{L^2(\Omega)} \leq \varepsilon.$$

Proof. We put $y = L + Y$, where $L = L(a)$ satisfies

$$(22) \qquad \begin{cases} L_t + (-\Delta)^m L + (-\Delta^k)(a(t,x)L) = h + h(a) & \text{in } Q, \\ \dfrac{\partial^j L}{\partial \nu^j} = 0 \quad (j = 0 \cdots (m-1)) & \text{on } \Sigma, \\ L(0) = y_0 & \text{in } \Omega \end{cases}$$

and $Y = Y(a)$ is taken associated to the approximate controllability problem

$$\begin{cases} Y_t + (-\Delta)^m Y + (-\Delta^k)(a(t,x)Y) = u(a)\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j Y}{\partial \nu^j} = 0 \quad (j = 0 \cdots (m-1)) & \text{on } \Sigma, \\ Y(0) = 0 & \text{in } \Omega, \end{cases}$$

with desired state $y_d - L(T)$, i.e. such that $\| Y(T) - (y_d - L(T)) \| \leq \varepsilon$. Notice that the existence of such a control $u(a)$ is consequence of Theorem 2. In particular, if $\| y_d - L(T) \| \leq \varepsilon$, we can take $u(a) \equiv 0$ and if $\| y_d - L(T) \| > \varepsilon$, then we take $u(a) = \| \widehat{\varphi}(a) \|_{L^1(\Omega)} v(a)$, where $v(a) \in sgn(\widehat{\varphi}(a))\chi_{\mathcal{O}}$ and $\widehat{\varphi}(a)$ is the solution of (7) with initial value $\mathcal{M}(\ (a(x,t), y_d - L(T))\ )$ defined in Proposition 3. It is obvious that such function y and such control $u(a)$ lead to the conclusion.

■

3 Controllability for the nonlinear problem.

As mentioned before, we shall use a fixed point argument to prove Theorem 1. In fact we shall deal with multivalued operators. Let us recall a well-known result: the Kakutani's fixed point Theorem. The usual continuity assumption in other fixed pont theorems is replaced here by the following notion:

Definition 1 *Let X, Y two Banach spaces and, $\Lambda : X \to \mathcal{P}(Y)$ a multivalued function. We say that Λ is upper hemicontinuous at $x_0 \in X$, if for every $p \in Y'$, the function*

$$x \to \sigma(\Lambda(x), p) = \sup_{y \in \Lambda(x)} < p, y >_{Y' \times Y}$$

is upper semicontinuous at x_0. We say that the multivalued function is upper hemicontinuous on a subset K of X, if it satisfies this properties for every point of K.

Theorem 3 (Kakutani's fixed point Theorem). *Let $K \subset X$ be a convex and compact subset and $\Lambda : K \to K$ an upper hemicontinuous application with convex, closed and nonempty values. Then, there exists a fixed point x_0, of Λ.*

For a proof see, for instance, Aubin [2].

Proof of Theorem 1. We fix $y_d \in L^2(\Omega)$ and $\varepsilon > 0$. By using Corollary 1, for each $z \in L^2(Q)$ and $\varepsilon > 0$ it is possible to find two functions $\varphi(z) \in L^1(Q)$ and $v(z) \in sgn(\varphi(z))\chi_{\mathcal{O}}$ such that the solution $y = y^z$ of

$$\text{(23)} \qquad \begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k (g(z)y) = h + h(g(z)) + u\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0\ ,\ j = 0, 1, \cdots m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

(where $u = u(z) = |\varphi(z)|_{L^1(\mathcal{O})} v(z))$ satisfies

$$\text{(24)} \qquad |y(T) - y_d|_{L^2(\Omega)} \leq \varepsilon.$$

Here $\varphi(z)$ is the solution of (7) with initial value $M(\ (g(z), y_d - L(z;T))\ )$ (see Proposition 3) and $a(t,x) = g(z)$, where is $L(z;T)$ the solution of (22), with $a = g(z)$, at time T .

Lemma 3 *The set*

$$\{y_d - L(z;T),\ z \in L^2(Q)\},$$

is relatively compact in $L^2(\Omega)$.

Proof of Lemma 3. Applying Proposition 4 it is easy to see that the set of solutions $L(z)$ of

$$\text{(25)} \qquad \begin{cases} L_t + (-\Delta)^m L + (-\Delta)^k (g(z)y) = h + h(g(z)) & \text{in } Q, \\ \dfrac{\partial^j L}{\partial \nu^j} = 0\ ,\ j = 0, 1, \cdots m-1 & \text{on } \Sigma, \\ L(0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfy

$$\text{(26)} \qquad \| L(z) \|_W \leq K(1 + \| y_0 \|_{L^2(\Omega)} + \| h \|_{L^2(0,T;H^{-m}(\Omega))})\ \ \forall\ z \in L^2(Q)$$

with $K > 0$ independent of z. Recall that $\| g(z) \|_{L^\infty(Q)} \leq M$ with M independent of z. Now, let $L(z_n)$ be a sequence of solutions (25) with $z_n \in L^2(Q)$. We must prove that there exists a subsequence (that we rewrite as $L(z_n)$), such that

$$\| L(z_n;T) - L(z_{n+1};T) \|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

By a compactness result due to Aubin [1], we know that

$$W \subset L^2(0,T;H^{m-1}(\Omega)) \ \text{ with compact imbedding.}$$

Therefore, by (26), we can suppose that

$$\| L(z_n) - L(z_{n+1}) \|_{L^2(0,T;H^{m-1}(\Omega))} \to 0 \quad \text{as } n \to \infty.$$

Further, it is easy to prove that $L(z_n) - L(z_{n+1})$ satisfies

$$\| L(z_n;T) - L(z_{n+1};T) \|^2_{L^2(\Omega)}$$

$$\leq -\int_0^T < D^k \left(g(z_n)L(z_n) - g(z_{n+1})L(z_{n+1})\right), D^k \left(L(z_n) - L(z_{n+1})\right) >_{H^{-k}(\Omega)\times H_0^k(\Omega)} dt$$

$$+ \int_0^T < D^k \left(g(z_n)s_0 - g(z_{n+1})s_0\right), D^k \left(L(z_n) - L(z_{n+1})\right) >_{H^{-k}(\Omega)\times H_0^k(\Omega)} dt.$$

Then, by (26), since $k \leq m-1$ (notice that $k=0$ if $m=1$),

$$\| L(z_n;T) - L(z_{n+1};T) \|^2_{L^2(\Omega)} \leq \widetilde{K} \| L(z_n) - L(z_{n+1}) \|^2_{L^2(0,T;H^{m-1}(\Omega))} \to 0 \quad \text{as } n \to \infty$$

and the proof ends.

■

Completion of Proof of Theorem 1. From Lemma 3, we obtain that $y_d - L(z;T)$ belongs to a compact set for all $z \in L^2(Q)$ and so, by using Propositions 3 and 1, we obtain that

(27)
$$\{\| \varphi(z) \|_{L^1(\mathcal{O})} v(z), \ z \in L^2(Q)\} \quad \text{is bounded in } L^\infty(Q)$$

Thus

(28)
$$K_1 = \sup_{z \in L^2(Q)} \| \varphi(z) \|_{L^1(\mathcal{O})} < \infty.$$

Obviously, $u = u(z)$ satisfies

(29)
$$\| u \|_{L^2(Q)} \leq K_2.$$

Therefore, if we define the operator

$$\Lambda : L^2(Q) \to \mathcal{P}(L^2(Q))$$

by

$$\Lambda(z) = \{y \text{ satisfies (23), (24) for some } u \text{ satisfying (29) }\},$$

we have seen that for each $z \in L^2(Q)$, $\Lambda(z) \neq \emptyset$. In order to apply Kakutani's fixed point theorem, we have to check that the next properties hold:

(i) There exists a compact subset U of $L^2(Q)$, such that for every $z \in L^2(Q)$, $\Lambda(z) \subset U$.

(ii) For every $z \in L^2(Q)$, $\Lambda(z)$ is a convex, compact and nonempty subset of $L^2(Q)$.

(iii) Λ is upper hemicontinuous.

The proof of these properties is as follows:

(i) From Proposition 4 we know that, there exists a bounded subset U of W such that for every $z \in L^2(Q)$, $\Lambda(z) \subset U$. Now, to see that we can choose U compact we shall prove that the set

$$\mathcal{Y} = \{y \text{ satisfying (23) for some } z \in L^2(Q) \text{ and } u \text{ satisfying (29)}\}$$

is a relatively compact subset of $L^2(Q)$. But this is easy to prove by using that

(30)
$$W \subset L^2(Q) \text{ with compact imbedding}$$

(see Lions [12] or Simon [18]).

(ii) We have already seen that for every $z \in L^2(Q)$, $\Lambda(z)$ is a nonempty subset of $L^2(Q)$. Further $\Lambda(z)$ is obviously convex, because $B(y_d, \varepsilon)$ and $\{u \in L^2(Q) :$ satisfying (29)$\}$ are convex sets. Then, we have to see that $\Lambda(z)$ is a compact subset of $L^2(Q)$. In (i) we have proved that $\Lambda(z) \subset U$ with U compact. Let $(y^n)_n$ be a sequence of elements of $\Lambda(z)$ which converges in $L^2(Q)$ to $y \in U$. We have to prove that $y \in \Lambda(z)$. We know that there exist $u^n \in L^2(Q)$ satisfying (29) such that

$$\text{(31)} \qquad \begin{cases} y^n_t + (-\Delta)^m y^n + (-\Delta)^k(g(z)y^n) = h + h(g(z)) + u^n\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y^n}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots, m-1 & \text{on } \Sigma, \\ y^n(0) = y_0 & \text{in } \Omega, \\ |y^n(T) - y_d|_2 \leq \varepsilon. \end{cases}$$

Now, by using that the controls u^n are uniformly bounded, we deduce that $u^n \to u$ in the weak topology of $L^2(Q)$ and u satisfies (29) (see Proposition III.5 of Brezis [5]). Then, using (31) and Proposition 4 we can see that $(y^n)_n$ converges to y in the weak topology of W (and so, by (30), strongly in $L^2(Q)$). Therefore, passing to the limit in (31) we obtain

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k(g(z)y) = h + h(g(z)) + u\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots, m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Further, $v^n = y - y^n$ is solution of

$$\text{(32)} \qquad \begin{cases} v^n_t + (-\Delta)^m v^n + (-\Delta)^k(g(z)v^n) = (u - u^n)\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j v^n}{\partial \nu^j} = 0 \ , \ j = 0, 1, ..., m-1 & \text{on } \Sigma, \\ v^n(0) = 0 & \text{in } \Omega \end{cases}$$

and satisfies $v^n \in W$ (see Proposition 4). Further, if we "multiply" in (32) by v^n and integrate, we obtain that

$$\| v^n(T) \|^2_{L^2(\Omega)} \leq k \int_Q (u - u^n)\chi_{\mathcal{O}} v^n dxdt \to 0 \quad \text{as } n \to \infty.$$

Thus $y^n(T)$ converges to $y(T)$ in the strong topology of $L^2(\Omega)$ and $\| y(T) - y_d \|_2 \leq \varepsilon$. This prove that $y \in \Lambda(z)$ and concludes the proof of (ii).

(iii) We must prove that for every $z_0 \in L^2(Q)$

$$\limsup_{z_n \overset{L^2(Q)}{\longrightarrow} z_0} \sigma(\Lambda(z_n), k) \leq \sigma(\Lambda(z_0), k), \ \forall \ k \in L^2(Q).$$

We have seen in (ii) that $\Lambda(z)$ is a compact set, which implies that for every $n \in \mathbb{N}$ there exists $y^n \in \Lambda(z_n)$ such that

$$\sigma(\Lambda(z_n), k) = \int_Q k(x,t) y^n(x,t) dxdt.$$

Now, by (i), $(y^n)_n \subset U$ (compact set of $L^2(Q)$). Then, there exists $y \in L^2(Q)$ such that (after extracting a subsequence) $y^n \to y$ in $L^2(Q)$. We shall prove that $y \in \Lambda(z_0)$. We know that there exist $u^n \in L^2(Q)$ satisfying (29) such that

$$\text{(33)} \qquad \begin{cases} y^n_t + (-\Delta)^m y^n + (-\Delta)^k(g(z_n)y^n) = h + h(z_n) + u^n\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y^n}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots, m-1 & \text{on } \Sigma, \\ y^n(0) = y_0 & \text{in } \Omega, \\ |y^n(T) - y_d|_2 \leq \varepsilon. \end{cases}$$

Then there exists $u \in L^2(Q)$ satisfying (29) such that $u^n \to u$ in the weak topology of $L^2(\mathcal{O})$. On the other hand, by using the smoothing effect of the parabolic linear equation (in a similar way to the proof of (ii)) and that $g \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$, we deduce that y satisfies (23) and (24) with $z = z_0$ for some $u \in L^2(Q)$ satisfying (29), which implies that $y \in \Lambda(z_0)$. Then, for every $k \in L^2(Q)$,

$$\sigma(\Lambda(z_n), k) = \int_Q k(x,t) y^n(x,t) dxdt \to \int_Q k(x,t) y(x,t) dxdt$$

$$\leq \sup_{\overline{y} \in \Lambda(z_0)} \int_Q k(x,t) \overline{y}(x,t) dxdt = \sigma(\Lambda(z_0), k),$$

which proves that Λ is upper hemicontinuous and conclude the proof of (iii).

Finally, if we restrict Λ to $K = conv(U)$ (the convex envelope of U), which is a compact set of $L^2(Q)$, it satisfies the assumptions of Kakutani's fixed point theorem. Then, Λ has a fixed point $y \in K$. Further, by construction, there exists a control $u \in L^2(Q)$ satisfying (29) such that

$$(34) \qquad \left\{ \begin{array}{ll} y_t + (-\Delta)^m y + (-\Delta)^k (f(y)) = h + u\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \\ |y(T) - y_d|_2 \leq \varepsilon. & \end{array} \right.$$

Therefore, y is the solution that we were looking for.

■

Remark 2 Several generalizations seem possible. For instance, the equation of (1) could be replaced by other ones with a more general nonlinearity

$$y_t + (-\Delta)^m y + \sum_{i=0}^{k} (-\Delta)^i f_i(y) = h + v\chi_\omega$$

or a more general lower order differential operator

$$y_t + (-\Delta)^m y + L(f(y)) = h + v\chi_\omega,$$

with L suitable linear partial differential operator of degree lower than $2m$. The key point in those generalizations is that the unique continuation result of Lemma 1, for the associated dual problem, remains true thanks to Theorem 3.2 of Saut-Scheurer [17] and the rest of arguments of the proof of Theorem 1 apply.

4 Non-controllability for superlinear problems.

In this section we assume $k = 0$. We shall prove a result of non-controllability for a superlinear nonlinear term with $\overline{\omega} \subset \Omega$.

Theorem 4 *Let $p > 1$ and let $y(t; u) = y \in L^2(0, T; H^m(\Omega)) \cap \mathcal{C}([0, T]; L^2(\Omega))$ a function satisfying*

$$\left\{ \begin{array}{ll} y_t + (-\Delta)^m y + |y|^{p-1} y = u\chi_\omega & \textit{in } Q, \\ y(0) = y_0 & \textit{in } \Omega, \end{array} \right.$$

associated to any "natural" boundary condition and with control $u \in L^2(Q)$. Then we can choose $y_d \in L^2(\Omega)$ and $\varepsilon > 0$ such that

$$(35) \qquad \| \, y(T; u) - y_d \, \|_{L^2(\Omega)} > \varepsilon \quad \textit{for any} \quad u \in L^2(Q).$$

In order to prove Theorem 4 we introduce, previously, some auxiliar functions. Given $R > 0$ we define, on $I\!R^N$, the functions

$$\xi_R(x) = (R^2 - |x|^2)/R \ \text{ if } \ |x| < R, \ \ \xi_R(x) = 0 \ \text{ if } |x| \geq R$$

and

$$(36) \qquad d_R(x) = R - |x| \ \text{ if } \ |x| < R, \ \ d_R(x) = 0 \ \text{ if } |x| \geq R.$$

It is clear that

$$d_R(x) \le \xi_R(x) \le 2d_R(x) \tag{37}$$

for all $x \in I\!R^N$.

The following result was proved in Bernis [4].

Proposition 5 *Let $s \ge 2m$ and $R > 0$. Then, for each $\varepsilon > 0$ there exist a constant C depending only on N, m, s and ε (thus independent of R) such that the following inequality holds for all $y \in H^m_{loc}(I\!R^N)$:*

$$((-\Delta)^m y, \xi_R^s y)_{H^{-m}_{loc}(I\!R^N)\times H^m_c(I\!R^N)} \ge (1-\varepsilon)\int_{I\!R^N} \xi_R^s |D^m y|^2 dx - C\int_{I\!R^N} \xi_R^{s-2m} y^2 dx.$$

Remark 3 *Since $s \ge 2m$, $\xi_R^s \in W_c^{2m,\infty}(I\!R^N)$. Hence $\xi_R^s \in \mathcal{C}_c^m(I\!R^N)$ (see e.g. Corollary IX.13 of [5]) and $\xi_R^s u \in H_c^m(I\!R^N)$ (see e.g. Note IX.4 of [5]).*

Corollary 2 *Let $s \ge 2m$ and $R > 0$ such that $\overline{B_R} \subset \Omega$. Then, for each $\varepsilon > 0$ there exist a constant C depending only on N, m, s and ε (thus independent of R) such that the following inequality holds for all $y \in H^m(\Omega)$:*

$$((-\Delta)^m y, \xi_R^s y)_{H^{-m}(\Omega)\times H_0^m(\Omega)} \ge (1-\varepsilon)\int_{\Omega} \xi_R^s |D^m y|^2 dx - C\int_{\Omega} \xi_R^{s-2m} y^2 dx.$$

Proof. Let $\overline{y} \in H^m(\Omega)$ such that $\overline{y} = y$ in Ω (such $\overline{y}$ exists by standar results: see, e.g., Chapter IX of Brezis [5]). Then, by Proposition 5, the inequality holds for $\overline{y}$, but as $\overline{B_R} \subset \Omega$ we obtain the result. ∎

Theorem 5 *Let $p > 1$, $r = p+1$, $y_0 \in L^2(\Omega)$ and $u \in L^{r'}(Q)$. Then any solution $y \in L^r(Q) \cap L^2(0,T;H^m(\Omega))$ of*

$$\begin{cases} y_t + (-\Delta)^m y + |y|^{p-1}y = u & \text{in } \mathcal{D}'(Q), \\ y(0) = y_0 & \text{on } \Omega, \end{cases} \tag{38}$$

with any "natural" boundary condition, satisfies the local estimate

$$\sup_{0<t<T}\int_{B_R} y(x,t)^2 dx + \int_{B_R\times(0,T)} (|D^m y|^2 + |y|^r) dxdt$$

$$\le K\left(1 + \int_{B_{R_1}\times(0,T)} |u|^{r'} dxdt + \int_{B_{R_1}} y_0^2 dx\right)$$

if $\overline{B_{R_1}} \subset \Omega$ and $0 < R \le R_1$. Moreover, the constant K depends only on N, m, p, R, R_1 and T.

Proof of Theorem 5. We take $X_r = L^r(Q) \cap L^2(0,T;H_0^m(\Omega))$. Then the equation of (38) is satisfied in $X_r' = L^{r'}(Q) + L^2(0,T;H^{-m}(\Omega))$. Then, if $s \ge 2m$, we can multiply (38) by $\xi_R^s y$ with the duality product $(\cdot,\cdot)_{X_r'\times X_r}$ and we obtain

$$\frac{1}{2}\int_{B_R} \xi_R^s y(x,T)^2 dx + ((-\Delta)^m y, \xi_R^s y)_{L^2(0,T;H^{-m}(\Omega))\times L^2(0,T;H_0^m(\Omega))} + (|y|^{p-1}y, \xi_R^s y)_{L^{r'}(Q)\times L^r(Q)}$$

$$= \frac{1}{2}\int_{B_R} \xi_R^s y_0(x)^2 dx + (u, \xi_R^s y)_{L^{r'}(Q)\times L^r(Q)}.$$

Now, from Corollary 2 it follows that

$$\begin{aligned} &\frac{1}{2}\int_{B_R} \xi_R^s y(x,T)^2 dx + \int_{B_R\times(0,T)} \xi_R^s (|D^m y|^2 + |y|^r) dxdt \\ &\le C\int_{B_R} \xi_R^s y_0(x)^2 dx + C\int_{B_R\times(0,T)} \xi_R^{s-2m} y^2 dxdt + C\int_{B_R\times(0,T)} \xi_R^s uy dxdt. \end{aligned} \tag{39}$$

By (36) and (37) we can replace in (39) $\xi_R(x)$ by $R - |x|$ (modifying the constants). Further, writing $s - 2m = 2s/r + (s(r-2)/r) - 2m$, we can apply Hölder's or Young's inequality with exponents $q = r/2$ and $q' = r/r-2$ and we obtain

$$\int_{B_R\times(0,T)} (R-|x|)^{s-2m} y^2 dxdt$$

$$\leq \varepsilon \int_{B_R\times(0,T)} (R-|x|)^s |y|^r dxdt + K(\varepsilon,q) \int_{B_R\times(0,T)} (R-|x|)^{s-\gamma} dxdt$$

with

$$K(\varepsilon,q) = \frac{1}{q'(q\varepsilon)^{q'/q}} \quad \text{and} \quad \gamma = \frac{2mr}{r-2} \quad .$$

Hence, if we choose $s > \gamma - 1$, the last integral is finite and equal to $\widetilde{C}R^{s+N-\gamma}$. On the other hand, we can apply again Young's inequality and we have

$$\int_{B_R\times(0,T)} (R-|x|)^s uydxdt \leq \varepsilon \int_{B_R\times(0,T)} (R-|x|)^s |y|^r dxdt + k(\varepsilon,r) \int_{B_R\times(0,T)} (R-|x|)^s |u|^{r'} dxdt.$$

Thus, by changing the constants, we deduce that

$$\frac{1}{2}\int_{B_R} (R-|x|)^s y(x,T)^2 dx + \int_{B_R\times(0,T)} (R-|x|)^s (|D^m y|^2 + |y|^r) dxdt$$

$$\leq C\left(\int_{B_R} (R-|x|)^s y_0(x)^2 dx + R^{s+N-\gamma} + \int_{B_R\times(0,T)} (R-|x|)^s |u|^{r'} dxdt \right).$$

Finally, by replacing R by R_1 and by taking into account that $R_1 - |x| \geq R_1 - R$ and $R_1 - |x| \leq R_1$ if $|x| \leq R$ we deduce the result with

$$K = \max\left\{ C(\frac{R_1}{R_1 - R})^s,\ \frac{CR_1^{s+N-\gamma}}{(R_1-R)^s} \right\}. \quad \blacksquare$$

Proof of Theorem 4. It is a trivial consequence of Theorem 5 since, if R_1 satisfies $\overline{B_{R_1}} \subset \Omega\backslash\omega$, then

$$\| y(u;T) \|^2_{L^2(\Omega)} \leq K(1+ \| y_0 \|^2_{L^2(\Omega)}) \quad \forall u \in L^{r'}(Q).$$

Therefore, taking y_d with $\| y_d \|_{L^2(\Omega)}$ large enough, we obtain (35) for $\varepsilon > 0$ small enough. $\blacksquare$

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