Pointwise Control of the Burgers Equation and Related Nash Equilibrium Problems. A Computational Approach[1] [2]

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Abstract. This article is concerned with the numerical solution of multi-objective control problems associated to non-linear partial differential equations and more precisely to the Burgers equation. For this kind of problems we look for the Nash equilibrium, which is the solution to a non-cooperative game. To compute the solution of the problem, we use a combination of finite-difference methods for the time discretization, finite element methods for the space discretization, and a quasi-Newton algorithm à la BFGS for the iterative solution of the discrete control problem. Finally, we apply the above methodology to the solution of several tests problems. To be able to compare our results with existing results in the literature, we discuss first a simple single-objective control problem, already investigated by other authors. Finally, we discuss the multi-objective case.

Key Words. Burgers Equation, pointwise control, Nash equilibria, adjoint systems, Dirac measures, quasi-Newton algorithms, single objective control problems, multi-objective control problems.

1 Introduction.

This article is a continuation of a previous article (See Ref. 1). In the above reference we proved the existence and uniqueness of a Nash equilibrium for the control of linear partial differential equations of the parabolic type and developed an algorithm to approximate the control solution; numerical experiments validated our methodology. In Ref. 1 we fully used the fact that the *state* equation was linear. One of our long term goals is to study Nash equilibria associated to control problems for non linear partial differential equations, such as the Navier-Stokes equations. Since this problem is quite difficult both from a mathematical and a numerical point of view, it seems reasonable to first investigate a simpler model problem. Here we consider the viscous Burgers equation, since it retains many of the interesting features of the Navier-Stokes equations and can be used for the modeling of *weak shock waves* when the

[1]To appear in Journal of Optimization Theory and Applications

[2]The authors are very grateful to M. Berggren, J.I. Díaz and J.L. Lions for helpful comments and suggestions. The support of the Universidad Complutense de Madrid, of Dassault Aviation, of the Higher Education Texas Coordinating Board and of the DGES Spanish Project PB96-0583 is also acknowledged.

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flow of interest is a perturbation of a uniform sonic gas flow (see, e.g. Ref. 2). The viscous Burgers equation is

$$y_t - \nu y_{xx} + y y_x = f \qquad \text{in } Q = (0,1) \times (0,T),$$

where T ($0 < T < \infty$) is an horizon time, $\nu > 0$ is a *viscosity* parameter and f is a density of *external forces.*

We look for M *controls* $v_m(t)$ *forcing* the solution at the points $a_m \in (0,1)$, $m = 1, \cdots, M$. We complete the equation with *initial* and *boundary conditions* (in order to be able to do meaningful comparisons we employ the boundary conditions used in Refs. 3, 4 and 5); we obtain then the following *state system*

$$\begin{cases} y_t - \nu y_{xx} + y y_x = f + \sum_{m=1}^{M} v_m \delta(x - a_m) & \text{in } Q, \\ y_x(0,t) = 0, \ y(1,t) = 0 & \text{in } (0,T), \\ y(0) = y_0 & \text{in } (0,1), \end{cases} \tag{1}$$

where $x \to \delta(x - a_m)$ denotes the *Dirac measure* at a_m. A *variational* formulation of the above state system is provided by

$$\begin{cases} y(t) \in L^2(0,T;V_0) \cap H^1(0,T;V_0'), \text{ such that } \forall z \in L^2(0,T;V_0) \text{ we have} \\ \\ \displaystyle\int_0^T < y_t, z >_{V_0' \times V_0} dt + \nu \int_0^T (y_x, z_x) dt + \int_0^T (y y_x, z) dt = \int_0^T (f, z) dt + \sum_{m=1}^{M} \int_0^T v_m z(a_m) dt, \\ \\ y(0) = y_0, \end{cases}$$

where $V_0 = \{z \in H^1(0,1) : z(1) = 0\}$ and $(\cdot,\cdot)$ denotes the scalar product in $L^2(0,1)$ defined by $(y,z) = \displaystyle\int_0^1 yz dx$.

In Section 2 we discuss a single-objective control problem (with a unique functional J to be minimized) and we compare our results with those obtained in Refs. 3, 4 and 5.

Finally, in Section 3 we develop an algorithm, based on the solution methods for single-objective control problems treated in Section 2, to approximate the Nash equilibria associated to a multi-objective control problem.

2 A Single-Objective Control Problem.

2.1 Problem Formulation.

Before introducing the Nash equilibrium problem, and in order to be able to compare our results with existing results in the literature, we discuss a simple control problem which has been already investigated by other authors (see, e.g., Refs. 3, 4 and 5).

Let us consider $\omega_d, \omega_T \subset (0,1)$ and the target functions $y_d \in L^2(\omega_d \times (0,T))$ and $y_T \in L^2(\omega_T)$. We define the control space as $\mathcal{U} = L^2(0,T; \mathbb{R}^M)$. The goal is to find a control $u = \{u_m\}_{m=1}^M$ so that y is close to y_d in $\omega_d \times (0,T)$ and $y(T)$ is close to y_T in ω_T at a minimal cost for the control. To do this, we define the *cost function* J by

$$J(v) = \frac{\alpha}{2} \parallel v \parallel_{\mathcal{U}}^2 + \frac{k}{2} \parallel y - y_d \parallel_{L^2(\omega_d \times (0,T))}^2 + \frac{l}{2} \parallel y(T) - y_T \parallel_{L^2(\omega_T)}^2,$$

where $v = \{v_m\}_{m=1}^M$, and where $\alpha > 0$, $k, l \geq 0$ and $k + l > 0$. The control problem is then

$$(\mathcal{CP}) \begin{cases} \text{Find } u \in \mathcal{U}, \text{ such that} \\ \\ J(u) \leq J(v), \quad \forall v \in \mathcal{U}. \end{cases}$$

A common way to solve this problem is to solve the problem

$$J'(u) = 0,$$

where J' denotes the differential of J. Now, it is a well-known result (see, e.g., Refs. 3, 4 and 5) that

$$J'(v) = \{\alpha v_m + p(a_m)\}_{m=1}^M,$$

i.e.,

$$(J'(v), w) = \sum_{m=1}^M \int_0^T (\alpha v_m + p(a_m)) w_m dt,$$

where p is the solution of the adjoint system

$$\begin{cases} -p_t - \nu p_{xx} - y p_x = k(y(v) - y_d)\chi_{\omega_d} & \text{in } Q, \\ \\ y(0,t)p(0,t) + \nu p_x(0,t) = 0, \ p(1,t) = 0 & \text{in } (0,T), \\ \\ p(T) = l(y(T;v) - y_T) & \text{in } (0,1). \end{cases}$$

2.2 Time Discretization.

We consider the time discretization step Δt, defined by $\Delta t = T/N$, where N is a positive integer. Then, if $t^n = n\Delta t$, we have $0 < t^1 < t^2 < \cdots < t^N = T$. We approximate then problem $(\mathcal{CP})$ by the following finite-dimensional minimization problem:

$$(\mathcal{CP})^{\Delta t} \begin{cases} \text{Find } u^{\Delta t} = \{u_m^n\}_{m=1\cdots M}^{n=1\cdots N} \in \mathcal{U}^{\Delta t}, \text{ such that} \\ \\ J^{\Delta t}(u) \leq J^{\Delta t}(v), \ \forall v = \{v_m^n\}_{m=1\cdots M}^{n=1\cdots N} \in \mathcal{U}^{\Delta t}, \end{cases}$$

with the *discrete control space* $\mathcal{U}^{\Delta t} = I\!R^{MN}$ and

$$J^{\Delta t}(v) = \frac{\Delta t}{2} \sum_{n=1}^N \sum_{m=1}^M |v_m^n|^2 + \frac{k\Delta t}{2} \sum_{n=1}^N \parallel y^n - y_d(n\Delta t) \parallel_{L^2(\omega_d)}^2$$

$$+ \frac{l}{2} \left((1-\theta) \parallel y^{N-1} - y_T \parallel_{L^2(\omega_T)}^2 + \theta \parallel y^N - y_T \parallel_{L^2(\omega_T)}^2 \right),$$

where $\theta \in (0,1]$ and $\{y^n\}_{n=1}^N$ is defined from the solution of the following *second order accurate time discretization* scheme of the Burgers equation (1):

$$y^0 = y_0,$$

$$\begin{cases} \dfrac{y^1 - y^0}{\Delta t} - \nu \dfrac{\partial^2}{\partial x^2}(\dfrac{2}{3}y^1 + \dfrac{1}{3}y^0) + y^0 \dfrac{\partial y^0}{\partial x} = f^1 + \dfrac{2}{3}\sum_{m=1}^{M} v_m^1 \delta(x - a_m) \quad \text{in } (0,1), \\ \dfrac{\partial y^1}{\partial x}(0) = 0, \quad y^1(1) = 0, \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} \dfrac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} - \nu \dfrac{\partial^2}{\partial x^2} y^n + (2y^{n-1} - y^{n-2}) \dfrac{\partial}{\partial x}(2y^{n-1} - y^{n-2}) = f^n + \sum_{m=1}^{M} v_m^n \delta(x - a_m), \\ \dfrac{\partial y^n}{\partial x}(0) = 0, \quad y^n(1) = 0. \end{cases}$$

2.3 Full discretization.

We consider the space discretization step h, defined by $h = 1/I$, where I is a positive integer. Then, if $x_i = (i-1)h$, we have $0 = x_1 < x_2 < \cdots < x_I < x_{I+1} = 1$. We approximate V_0 by

$$V_{0h} = \{z \in \mathcal{C}^0[0,1] : z(1) = 0, z|_{(x_i, x_{i+1})} \in P_1, i = 1, \cdots, I\},$$

where P_1 is the space of the polynomials of degree ≤ 1. We define a_h and b_h by

$$a_h(y,z) = \int_0^1 y_x z_x dx, \qquad b_h(w,y,z) = \int_0^1 w y_x z dx.$$

We approximate then problem $(\mathcal{CP})$ by the following finite-dimensional minimization problem:

$$(\mathcal{CP})_h^{\Delta t} \begin{cases} \text{Find } u_h^{\Delta t} = \{u_m^n\}_{m=1\cdots M}^{n=1\cdots N} \in \mathcal{U}^{\Delta t}, \text{ such that} \\ J_h^{\Delta t}(u_h^{\Delta t}) \leq J_h^{\Delta t}(v), \ \forall v = \{v_m^n\}_{m=1\cdots M}^{n=1\cdots N} \in \mathcal{U}^{\Delta t}; \end{cases}$$

with

$$J_h^{\Delta t}(v) = \frac{\Delta t}{2}\sum_{n=1}^{N}\sum_{m=1}^{M} |v_m^n|^2 + \frac{k\Delta t}{2}\sum_{n=1}^{N} \| y_h^n - y_d(n\Delta t) \|_{L^2(\omega_d)}^2 + \frac{l}{2}\Big((1-\theta) \| y_h^{N-1} - y_T \|_{L^2(\omega_T)}^2 + \theta \| y_h^N - y_T \|_{L^2(\omega_T)}^2\Big),$$

where $\theta \in (0,1]$ and $\{y_h^n\}_{n=1}^N$ is defined from the solution of the following full discretization of the Burgers equation (1):

$$\begin{cases} y_h^0 \in V_{0h}, \\ (y_h^0, z) = (y_0, z), \quad \forall z \in V_{0h}; \end{cases}$$

$$\begin{cases} y_h^1 \in V_{0h}, \\ \left(\dfrac{y_h^1 - y_h^0}{\Delta t}, z\right) + \nu a_h(\dfrac{2}{3}y_h^1 + \dfrac{1}{3}y_h^0, z) + b_h(y_h^0, y_h^0, z) = (f^1, z) + \dfrac{2}{3}\sum_{m=1}^{M} v_m^1 z(a_m), \qquad \forall z \in V_{0h}; \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} y_h^n \in V_{0h}, \\ \left(\dfrac{\frac{3}{2}y_h^n - 2y_h^{n-1} + \frac{1}{2}y_h^{n-2}}{\Delta t}, z\right) + \nu a_h(y_h^n, z) + b_h(2y_h^{n-1} - y_h^{n-2}, 2y_h^{n-1} - y_h^{n-2}, z) \\ \qquad\qquad\qquad\qquad\qquad\qquad = (f^n, z) + \displaystyle\sum_{m=1}^{M} v_m^n z(a_m), \qquad \forall z \in V_{0h}. \end{cases}$$

As for the continuous case, to solve problem $(\mathcal{CP})_h^{\Delta t}$, we look for the solution $u_h^{\Delta t}$ of

$$\frac{\partial J_h^{\Delta t}}{\partial v}(u_h^{\Delta t}) = 0.$$

Computing $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$ is more complicated than in the continuous case but, following the same approach, we can show that

$$< \frac{\partial J_h^{\Delta t}}{\partial v}(v), w >= \Delta t \sum_{n=1}^{N} \sum_{m=1}^{M} (\alpha v_m^n + p_h^n(a_m)) w_m^n,$$

where $\{p_h^n\}_{n=1}^{N+2}$ is the solution of

$$\begin{cases} p_h^{N+2} \in V_{0h}, \\ (p_h^{N+2}, z) = -8l(1-\theta)\displaystyle\int_{\omega_T} (y_h^{N-1} - y_T) z dx - 2l\theta \int_{\omega_T} (y_h^N - y_T) z dx, \quad \forall z \in V_{0h}; \end{cases}$$

$$\begin{cases} p_h^{N+1} \in V_{0h}, \\ (p_h^{N+1}, z) = -2l(1-\theta)\displaystyle\int_{\omega_T} (y_h^{N-1} - y_T) z dx, \quad \forall z \in V_{0h}; \end{cases}$$

$$\begin{cases} p_h^N \in V_{0h}, \\ \left(\dfrac{\frac{3}{2}p_h^N - 2p_h^{N+1} + \frac{1}{2}p_h^{N+2}}{\Delta t}, z\right) + \nu a_h(p_h^N, z) = k\displaystyle\int_{\omega_d} (y_h^N - y_d(N\Delta t)) z dx, \quad \forall z \in V_{0h}; \end{cases}$$

$$\begin{cases} p_h^{N-1} \in V_{0h}, \\ \left(\dfrac{\frac{3}{2}p_h^{N-1} - 2p_h^N + \frac{1}{2}p_h^{N+1}}{\Delta t}, z\right) + \nu a_h(p_h^{N-1}, z) + b_h(2y_h^{N-1} - y_h^{N-2}, z, 2p_h^N) \\ \qquad\qquad + b_h(z, 2y_h^{N-1} - y_h^{N-2}, 2p_h^N) = k\displaystyle\int_{\omega_d} (y_h^{N-1} - y_d((N-1)\Delta t)) z dx, \quad \forall z \in V_{0h}; \end{cases}$$

and for $n = N-2, \cdots, 1$,

$$\begin{cases} p_h^n \in V_{0h}, \\ \left(\dfrac{\frac{3}{2}p_h^n - 2p_h^{n+1} + \frac{1}{2}p_h^{n+2}}{\Delta t}, z\right) + \nu a_h(p_h^n, z) + b_h(2y_h^n - y_h^{n-1}, z, 2p_h^{n+1}) + b_h(2y_h^{n+1} - y_h^n, z, -p_h^{n+2}) \\ \quad + b_h(z, 2y_h^n - y_h^{n-1}, 2p_h^{n+1}) + b_h(z, 2y_h^{n+1} - y_h^n, -p_h^{n+2}) = k\displaystyle\int_{\omega_d}(y_h^n - y_d(n\Delta t))z dx, \quad \forall z \in V_{0h}. \end{cases}$$

Now, once we know how to compute $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$ we use a *quasi-Newton method* à la BFGS (see, e.g., Ref. 6 for BFGS algorithms and their implementations) to compute the solution of the fully discrete control problem $(\mathcal{CP})_h^{\Delta t}$.

2.4 Numerical experiments for the single-objective control problem.

We consider the test problem defined as follows: $T = 1$, $I = 128$, $N = 256$, $\nu = 10^{-2}$, $k = 0$, $l = 8$,

$$f(x,t) = \begin{cases} 1 & \text{if } (x,t) \in (0, 1/2) \times (0, T), \\ 2(1-x) & \text{if } (x,t) \in [1/2, 1) \times (0, T), \end{cases}$$

$y_0 = 0$ and $y_T(x) = 1 - x^3$. This test problem has already been addressed in Refs. 3, 4 and 5 (In Ref. 3 the authors use $I = N = 60$).

Remark 2.1 This is a controllability type problem, since $k = 0$ and $l \neq 0$. We point out that the Burgers equation, with pointwise control does not have the controllability property, since there exists a function $\psi(x)$ (independent of the controls) such that

$$y(v; x, T) \leq \psi(x), \quad \forall\, x \in I.$$

This *obstruction* phenomenon was already pointed out in Ref. 7 and later proved in Ref. 8. This result is also true with other types of controls (such as Dirichlet control at one of the ends of the interval I or distributed control in a subinterval of I) and with other types of equations (some numerical experiments showing this phenomenon for a semilinear parabolic equation with a superlinear reaction term can be seen in Ref. 9). Therefore, we cannot expect to be able to drive our solution, as close as we want, to any target function. Nevertheless, if the target function is, at time T, sufficiently close to the uncontrolled solution, our approach seems operational, in general.

We use, for our algorithm, $\theta = 3/2$. Further, if v^k $(k = 1, 2, \cdots)$ is the sequence of controls we get from the BFGS algorithm, we use the following stopping criteria: We stop iterating after step k if either

$$\| \frac{\partial J_h^{\Delta t}}{\partial v}(u^k) \|_\infty \leq 10^{-5}$$

or

$$\frac{J_h^{\Delta t}(u^{k-1}) - J_h^{\Delta t}(u^k)}{\max\{|J_h^{\Delta t}(u^{k-1})|, |J_h^{\Delta t}(u^k)|, 1\}} \leq 2 \cdot 10^{-9}.$$

On Figure 1 (resp., 3 and 5) we have shown the uncontrolled state solution $y(T)$ (...), the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to a single control point at $a = 1/5$ (resp., $a = 2/3$ and $3/5$)([7]). The corresponding control functions have been represented on Figures 2, 4 and 6.

[7]To be precise: the control points were put on the grid points *nearest* to these values

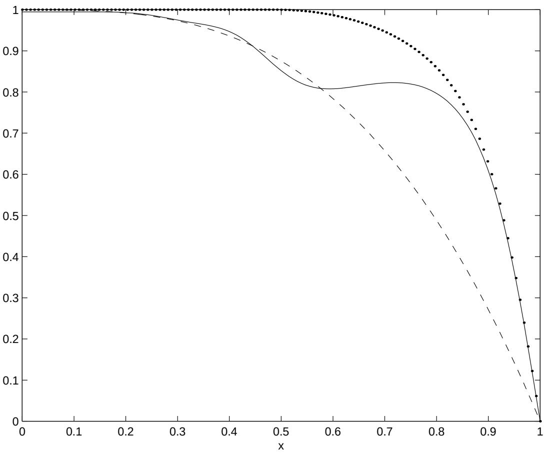

Fig. 1: The target function (- -), the uncontrolled (..) and controlled (–) states at time T, for $a = 1/5$.

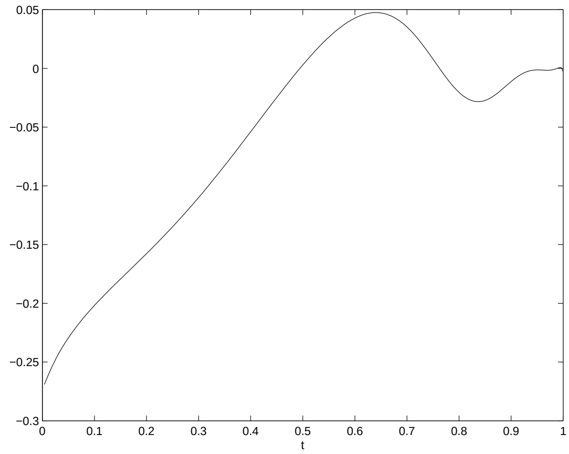

Fig. 2: The computed optimal control for $a = 1/5$.

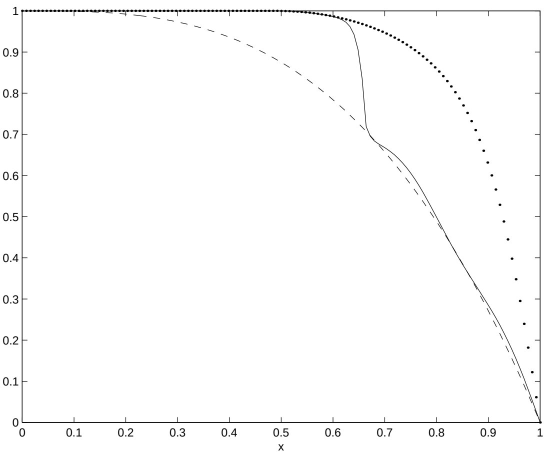

Fig. 3: The target function (- -), the uncontrolled (..) and controlled (–) states at time T, for $a = 2/3$.

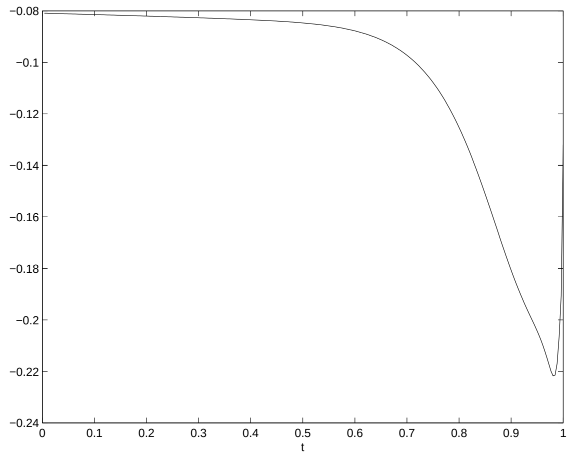

Fig. 4: The computed optimal control for $a = 2/3$.

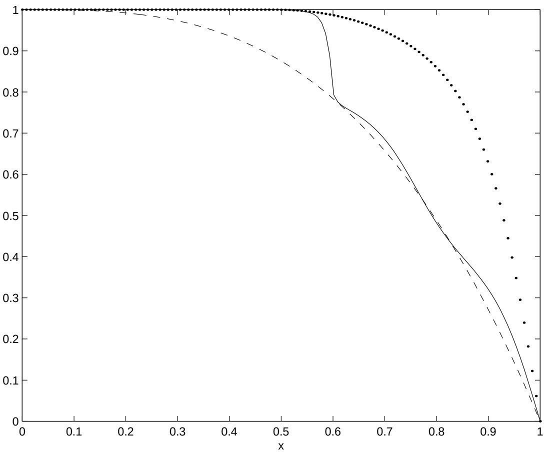

Fig. 5: The target function (- -), the uncontrolled (..) and controlled (–) states at time T, for $a = 3/5$.

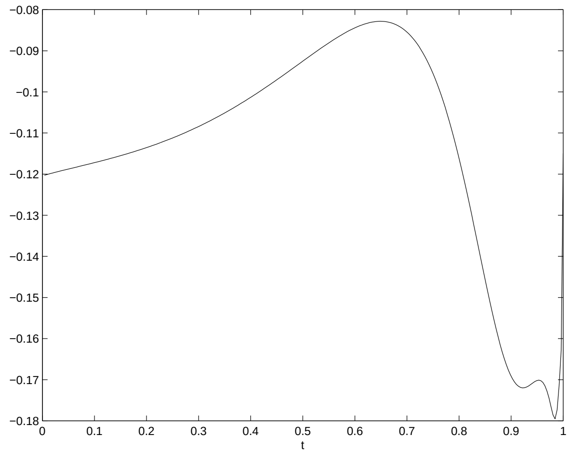

Fig. 6: The computed optimal control for $a = 3/5$.

Figure 7 shows the uncontrolled state solution $y(T)$ (...), the target function y_T (- - -), and the controlled state solution $y(T)$ (—), when controlling at $a_1 = 1/5$ and $a_2 = 3/5$, simultaneously, while Figures 8 shows the computed optimal controls.

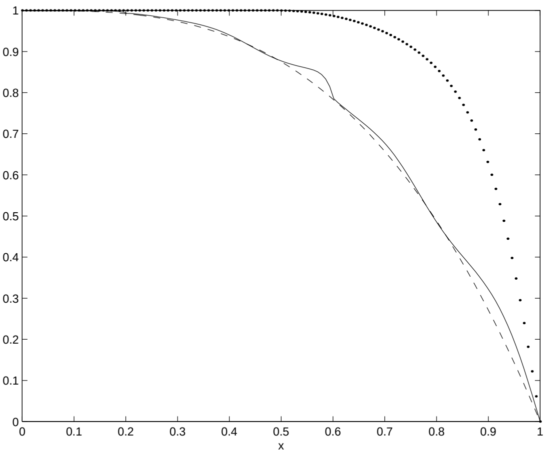

Fig. 7: The target function (- -), the uncontrolled (..) and controlled (–) states at time T, for $a_1 = 1/5$ and $a_2 = 3/5$.

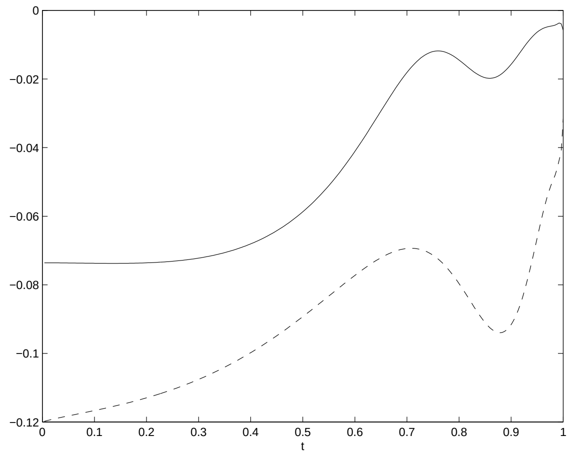

Fig. 8: The computed optimal controls u_1 (–) and u_2 (- -) for $a_1 = 1/5$ and $a_2 = 3/5$.

Our numerical results are consistent with those obtained in Refs. 3, 4 and 5. The methods employed by these authors are based on *first order accurate time discretization schemes* and the iterative solution of the discrete control problem is achieved by quasi-Newton algorithms in Ref. 3 and by conjugate gradient algorithms in Refs. 4 and 5.

In Table 1 we give some further results about our solutions and we do the comparison with the results in Ref. 4. The norms considered in all the tables of the present article refer to the L^2-norm of the discrete entries. Further, $y(v;T)$ represents the solution at time T, associated to the control v ($y(0,T)$ represents the solution without control, at time T).

Control Points	Number of discrete parabolic eqs. solved	$\frac{\|y(u;T)-y_T\|}{\|y_T\|}$	$\frac{\|y(0;T)-y_T\|}{\|y_T\|}$	$\parallel u \parallel$
a=1/5	22 (**178**)	0.1924 (**0.20**)		0.1073 (**0.11**)
a=2/3	20 (**94**)	0.0983 (**0.091**)		0.1122 (**0.11**)
a=3/5	20	0.0744	0.2522	0.1162
$a_1 = 1/5,\ a_2 = 3/5$	26 (**172**)	0.0241 (**0.0025**)		$\parallel u_1 \parallel = 0.0541$ $\parallel u_2 \parallel = 0.0944$

Table 1: Computational results. In bold the results obtained in Ref. 4

3 Nash Equilibria.

3.1 Problem Formulation.

Nash equilibria define a *non-cooperative multiple objective optimization approach* first proposed by J.F. Nash (see Ref. 10). Since it originated in *Games Theory* and *Economics*, the notion of *player* is often used. For an optimization problem with G objectives (or functionals J_i to minimize), a *Nash strategy* consists in having G *players* (or controls v_i), each optimizing its own criterion. However, each player has to optimize its own criterion given that all the other criteria are fixed by the rest of the *players*. When no *player* can further improve its criterion, it means that the system has reached a *Nash Equilibrium* state.

Remark 3.1 *There are different strategies for* Multi-objective Optimization*, as the* Pareto *(cooperative) strategy (see Ref. 11) and the* Stackelberg *(hierarchical non-cooperative) strategy (see Ref. 12).*

All the results to follow are also valid for more than two control points but for simplicity we shall consider the case of only two control points a_1 and a_2. Then, the state equation is

$$\begin{cases} y_t - \nu y_{xx} + y y_x = f + v_1 \delta(x-a_1) + v_2 \delta(x-a_2) & \text{in } Q, \\ \\ y_x(0,t) = 0,\ y(1,t) = 0 & \text{in } (0,T), \\ \\ y(0) = y_0 & \text{in } (0,1). \end{cases}$$

Let us consider $\omega_{di}, \omega_{Ti} \subset (0,1)$ $(i = 1,2)$ and the target functions $y_{di} \in L^2(\omega_d \times (0,T))$ and $y_{Ti} \in L^2(\omega_T)$ $(i = 1,2)$. We take as the control space $\mathcal{U}_1 = \mathcal{U}_2 = L^2(0,T)$.

The goal of each control v_i $(i = 1,2)$ is to drive the solution y close to y_{di} in $\omega_{di} \times (0,T)$ and $y(T)$ close to y_{Ti} in ω_{Ti} at a minimal cost for the control v_i. To do this, we define two *cost functions* by

$$J_i(v_1,v_2) = \frac{\alpha_i}{2} \parallel v_i \parallel_{\mathcal{U}}^2 + \frac{k_i}{2} \parallel y(v_1,v_2) - y_{di} \parallel_{L^2(\omega_{di}\times(0,T))}^2 + \frac{l_i}{2} \parallel y(v_1,v_2;T) - y_{Ti} \parallel_{L^2(\omega_{Ti})}^2,$$

$i = 1, 2$, where $\alpha_i > 0$, $k_i, l_i \geq 0$ and $k_i + l_i > 0$.

For every $w_2 \in \mathcal{U}_2$ we consider the optimal control problem

$$(\mathcal{CP}_1(w_2)) \quad \begin{cases} \text{Find } u_1(w_2) \in \mathcal{U}_1, \text{ such that} \\ \\ J_1(u_1(w_2), w_2) \leq J_1(v_1, w_2), \ \forall v_1 \in \mathcal{U}_1. \end{cases}$$

For every $w_1 \in \mathcal{U}_1$ we consider the optimal control problem

$$(\mathcal{CP}_2(w_1)) \quad \begin{cases} \text{Find } u_2(w_1) \in \mathcal{U}_2, \text{ such that} \\ \\ J_2(w_1, u_2(w_1)) \leq J_2(w_1, v_2), \ \forall v_2 \in \mathcal{U}_2. \end{cases}$$

A solution $u_1(w_2)$ of problem $(\mathcal{CP}_1(w_2))$ is characterized by $\dfrac{\partial J_1}{\partial v_1}(u_1(w_2), w_2) = 0$.

A solution $u_2(w_1)$ of problem $(\mathcal{CP}_2(w_1))$ is characterized by $\dfrac{\partial J_2}{\partial v_2}(w_1, u_2(w_1)) = 0$.

A Nash equilibrium is a pair $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that $u_1 = u_1(u_2)$ and $u_2 = u_2(u_1)$, i.e., (u_1, u_2) is a solution of the *coupled (optimality) system*:

$$\begin{cases} \dfrac{\partial J_1}{\partial v_1}(u_1, u_2) = 0, \\ \\ \dfrac{\partial J_2}{\partial v_2}(u_1, u_2) = 0. \end{cases} \tag{2}$$

Remark 3.2 *If $y_{d1} = y_{d2} = y_d$, $y_{T1} = y_{T2} = y_T$, $\alpha_1 = \alpha_2 = \alpha$, $k_1 = k_2 = k$ and $l_1 = l_2 = l$, then the Nash Equilibria problem (2) is equivalent to the classical control problem:*

$$(\mathcal{CP}) \quad \begin{cases} \textit{Find } (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2, \ \textit{such that} \\ \\ J(u_1, u_2) \leq J(v_1, v_2), \ \forall (v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2, \end{cases}$$

where

$$J(v_1, v_2) = \frac{\alpha}{2} \parallel v \parallel_{\mathcal{U}}^2 + \frac{k}{2} \parallel y(v_1, v_2) - y_d \parallel_{L^2(\omega_d \times (0,T))}^2 + \frac{l}{2} \parallel y(v_1, v_2; T) - y_T \parallel_{L^2(\omega_T)}^2 .$$

This is easy to prove, since the solution of $(\mathcal{CP})$ is equivalent to

$$\begin{cases} \dfrac{\partial J}{\partial v_1}(u_1, u_2) = 0, \\ \\ \dfrac{\partial J}{\partial v_2}(u_1, u_2) = 0, \end{cases}$$

and

$$\frac{\partial J}{\partial v_i}(v_1, v_2) = \frac{\partial J_i}{\partial v_i}(v_1, v_2), \quad \forall (v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2, \qquad \forall \ i = 1, 2.$$

Obviously, the computational cost in this case will be bigger if we follow the Nash strategy. This can be seen from the first numerical experiment of Section 3.2.

Once w_2 is fixed, we can follow the same approach as in Section 2 to solve problem $(\mathcal{CP}_1(w_2))$ and, similarly, once w_1 is fixed, we can follow the same approach to solve problem $(\mathcal{CP}_2(w_1))$. The difficulty is that we do not know a priori a suitable $w_2 (= u_2)$ and $w_1 (= u_1)$.

The algorithm we propose is the following:

Step 1. (u_1^0, u_2^0) is given in $\mathcal{U}_1 \times \mathcal{U}_2$.
Step 2. We get u_1^1 as the solution of $(\mathcal{CP}_1(u_2^0))$.
Step 3. We get u_2^1 as the solution of $(\mathcal{CP}_2(u_1^0))$.

Then, for $k \geq 1$, assuming that $(u_1^k, u_2^k) \in \mathcal{U}_1 \times \mathcal{U}_2$ is known, we compute (u_1^{k+1}, u_2^{k+1}) as follows:

Step 4. If $u_2^k = u_2^{k-1}$ then $u_1^{k+1} = u_1^k$;
else get u_1^{k+1} as the solution of $(\mathcal{CP}_1(u_2^k))$.
Step 5. If $u_1^k = u_1^{k-1}$ then $u_2^{k+1} = u_2^k$;
else get u_2^{k+1} as the solution of $(\mathcal{CP}_2(u_1^k))$.
Step 6. If $u_1^{k+1} = u_1^k$ and $u_2^{k+1} = u_2^k$ then take $(u_1, u_2) = (u_1^{k+1}, u_2^{k+1})$;
else do $k = k + 1$ and go to Step 4.

3.2 Numerical Experiments.

We consider the same data as those considered in Section 2.4. Namely, $T = 1$, $I = 128$, $N = 256$, $\nu = 10^{-2}$,

$$f(x,t) = \begin{cases} 1 & \text{if } (x,t) \in (0, 1/2) \times (0, T), \\ 2(1-x) & \text{if } (x,t) \in [1/2, 1) \times (0, T), \end{cases}$$

$y_0 \equiv 0$. We consider again $\theta = 3/2$ in the discretization of each control problem. We also consider $k_i = 0$ and $l_i = 8$ $(i = 1, 2)$. All the numerical experiments have been done with $a_1 = 1/5$ and $a_2 = 3/5$.

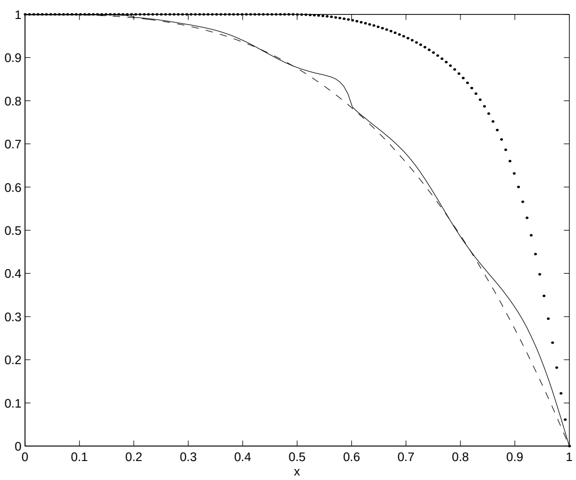

Fig. 9: The target function (- -), the uncontrolled (..) and controlled (–) states, for the Nash strategy, at time T.

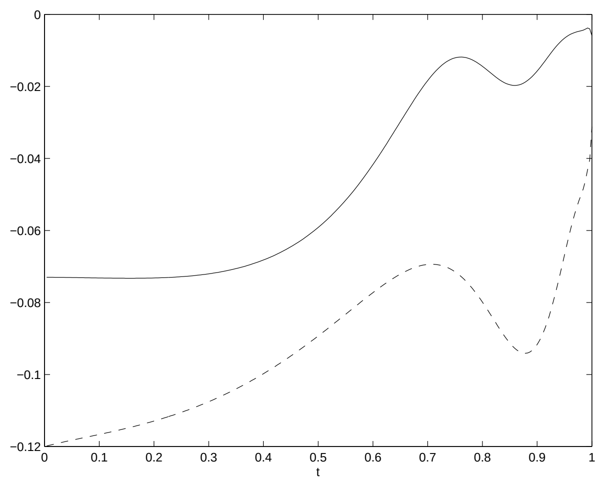

Fig. 10: The computed controls u_1 (–) and u_2 (- -) for the Nash strategy.

For the case $y_{T1}(x) = y_{T2}(x) = y_T(x) = 1 - x^3$, Figure 9 shows the uncontrolled state solution $y(T)$ (...), the target function y_T (- - -), and the controlled state solution $y(T)$ (—), when controlling with a Nash strategy. Figure 10 shows the computed controls. In Table 2 we give some further information about our solution. We point out that the results are consistent with Remark 3.2.

Quasi-Newton Meth. for J_1 / J_2	Parabolic eqs. solved for J_1 / J_2	$\dfrac{\frac{\|y(u;T)-y_T\|}{\|y_T\|}}{\frac{\|y(0;T)-y_T\|}{\|y_T\|}}$	$\| u_1 \|$	$\| u_2 \|$
19 / 18	286 / 232	$\dfrac{0.0241}{0.2522}$	0.0540	0.0944

Table 2: Computational results for the Nash strategy with $y_{T1}(x) = y_{T2}(x) = y_T(x) = 1 - x^3$.

For the case $y_{T1}(x) = \frac{1}{2}(1 - x^3)$, $y_{T2}(x) = 1 - x^3$, Figure 11 shows the uncontrolled state solution $y(T)$ (...), the target functions y_{T1} (- - -), y_{T2} (- . -), and the controlled state solution $y(T)$ (—), when controlling with a Nash strategy. Figure 12 shows the computed controls. In Table 3 we give some further information about our solution.

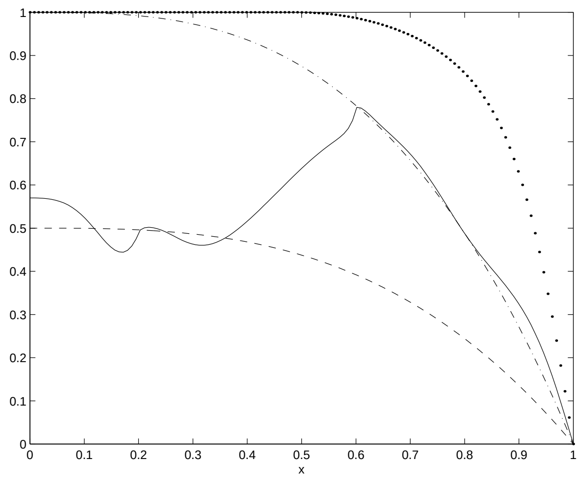

Fig. 11: The target functions y_{T1} (- -), y_{T2} (- . -), the uncontrolled (..) and controlled (–) states, for the Nash strategy, at time T.

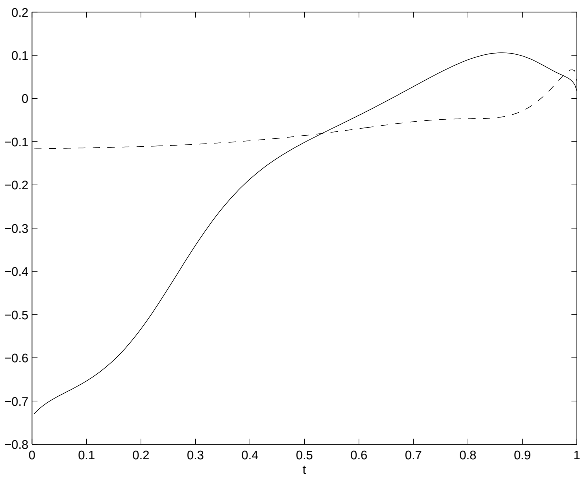

Fig. 12: The computed controls u_1 (–) and u_2 (- -) for the Nash strategy.

Quasi-Newton Meth. for J_1 / J_2	Parabolic eqs. solved for J_1 / J_2	$\dfrac{\frac{\|y(u;T)-y_{T1}\|}{\|y_{T1}\|}}{\frac{\|y(0;T)-y_{T1}\|}{\|y_{T1}\|}}$	$\dfrac{\frac{\|y(u;T)-y_{T2}\|}{\|y_{T2}\|}}{\frac{\|y(0;T)-y_{T2}\|}{\|y_{T2}\|}}$	$\| u_1 \|$	$\| u_2 \|$
5 / 4	188 / 54	$\dfrac{0.4921}{1.3308}$	$\dfrac{0.4110}{0.2522}$	0.3371	0.0850

Table 3: Computational results for the Nash strategy with $y_{T1}(x) = \frac{1}{2}(1 - x^3)$ and $y_{T2}(x) = 1 - x^3$.

For the case $y_{T1}(x) = 1 - x^3$, $y_{T2}(x) = \frac{9}{8}(1 - x^6)$, Figure 13 shows the uncontrolled state solution $y(T)$ (...), the target functions y_{T1} (- - -), y_{T2} (- . -), and the controlled state solution $y(T)$ (—), when controlling with a Nash strategy. Figure 14 shows the computed controls. In Table 4 we give some further information about our solution.

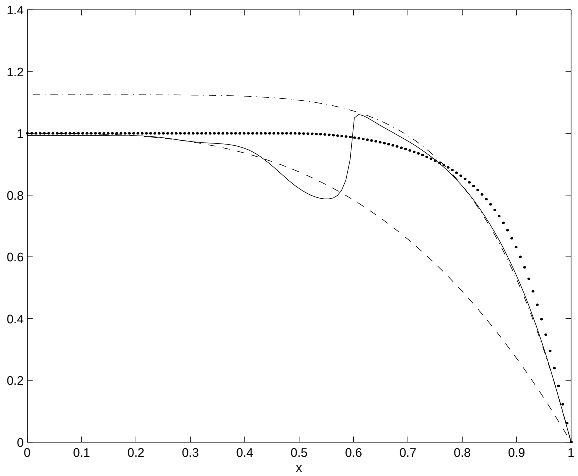

Fig. 13: The target functions y_{T1} (- -), y_{T2} (- . -), the uncontrolled (..) and controlled (–) states, for the Nash strategy, at time T.

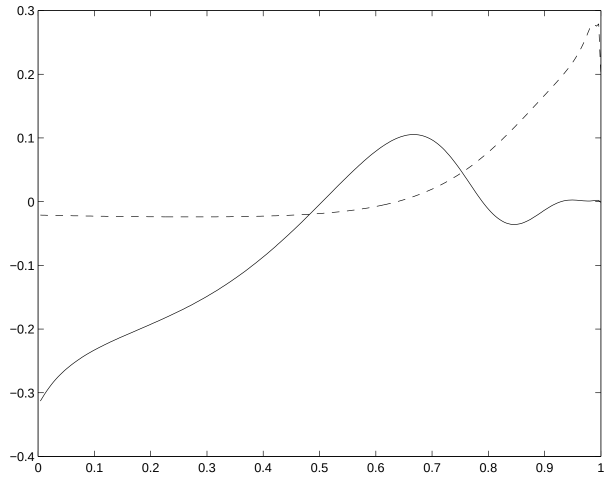

Fig. 14: The computed controls u_1 (–) and u_2 (- -) for the Nash strategy.

Quasi-Newton Meth. for J_1 / J_2	Parabolic eqs. solved for J_1 / J_2	$\frac{\|y(u;T)-y_{T1}\|}{\|y_{T1}\|}$ ——— $\frac{\|y(0;T)-y_{T1}\|}{\|y_{T1}\|}$	$\frac{\|y(u;T)-y_{T2}\|}{\|y_{T2}\|}$ ——— $\frac{\|y(0;T)-y_{T2}\|}{\|y_{T2}\|}$	$\| u_1 \|$	$\| u_2 \|$
6 / 6	78 / 76	0.2288 ——— 0.2522	0.1445 ——— 0.1001	0.1334	0.0849

Table 4: Computational results for the Nash strategy with $y_{T1}(x) = 1 - x^3$ and $y_{T2}(x) = \frac{9}{8}(1 - x^6)$.

For the case $y_{T1}(x) = \frac{9}{8}(1-x^6)$, $y_{T2}(x) = 1-x^3$, Figure 15 shows the uncontrolled state solution $y(T)$ (...), the target functions y_{T1} (- - -), y_{T2} (- . -), and the controlled state $y(T)$ (—), when controlling with a Nash strategy. Figure 16 shows the computed controls. In Table 5 we give some further information about our solution.

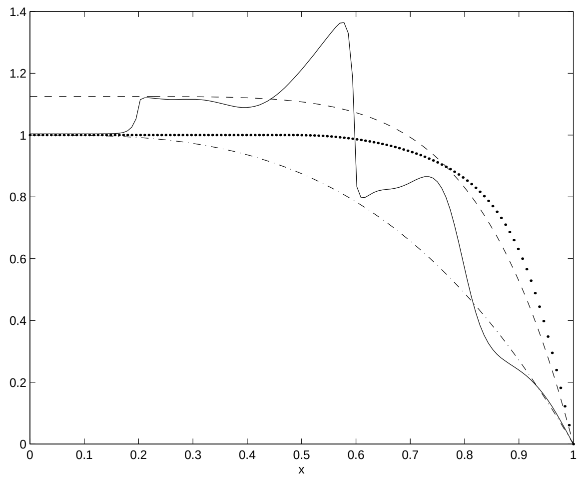

Fig. 15: The target functions y_{T1} (- -), y_{T2} (- . -), the uncontrolled (..) and the controlled (–) states, for the Nash strategy, at time T.

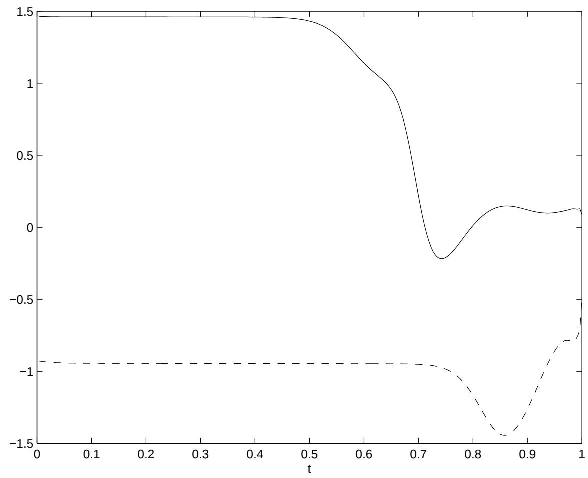

Fig. 16: The computed controls u_1 (–) and u_2 (- -) for the Nash strategy.

Quasi-Newton Meth. for J_1 / J_2	Parabolic eqs. solved for J_1 / J_2	$\frac{\|y(u;T)-y_{T1}\|}{\|y_{T1}\|}$ —— $\frac{\|y(0;T)-y_{T1}\|}{\|y_{T1}\|}$	$\frac{\|y(u;T)-y_{T2}\|}{\|y_{T2}\|}$ —— $\frac{\|y(0;T)-y_{T2}\|}{\|y_{T2}\|}$	$\| u_1 \|$	$\| u_2 \|$
55 / 55	1576 / 700	0.1702 —— 0.1001	0.2395 —— 0.2522	1.1486	0.9983

Table 5: Computational results for the Nash strategy with $y_{T1}(x) = \frac{9}{8}(1 - x^6)$ and $y_{T2}(x) = 1 - x^3$.

4 Conclusions.

For the single-objective control problems considered in this article, the computed solutions that have been obtained are practically the same that those obtained in Refs. 3, 4 and 5 but the computational cost of our algorithm is much smaller (see Figures 1–8 and Table 1).

For the Nash multi-objective control problems considered here, the numerical results obtained for the Burgers equation (a nonlinear model) are consistent with those obtained in Ref. 1 (for a linear problem) and with what we can expect from a non-cooperative strategy such as the Nash's one. Namely, we observe that:

1. In the neighborhood of each control point a_i, the solution, at time T, is close to the target function y_{Ti}.

2. There are cases, when the target functions are not compatible (i.e. when being close to one of the target functions implies to be far from the others), where the solution without control is closer to some of the target functions than the solution obtained with the Nash strategy. This can be seen in the numerical results comparing $\frac{\|y(u;T)-y_{Ti}\|}{\|y_{Ti}\|}$ with $\frac{\|y(0;T)-y_{Ti}\|}{\|y_{Ti}\|}$, for $i = 1, 2$, shown in Tables 3–5.

The results obtained in this article, for the Burgers equation, call for an investigation of Nash equilibria for more complicated models than Burgers, such as the Navier-Stokes equations, for example.

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