# Numerical Experiments Regarding the Distributed Control of Semilinear Parabolic Problems*

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**Abstract**

This work deals with some numerical experiments regarding the distributed control of semilinear parabolic equations of the type

$$y_t - y_{xx} + f(y) = u\chi_\omega \text{ in } (0,1)\times(0,T),$$

with Neumann and initial auxiliary conditions, where $\omega$ is an open subset of $(0,1)$, $f$ is a $\mathcal{C}^1$ nondecreasing real function, $u$ is the output control and $T>0$ is (arbitrarily) fixed. Given a target state $y_T$ we study the associated *approximate controllability problem* (given $\epsilon>0$, find $u\in L^2(0,T)$ such that $\|y(T;u)-y_T\|_{L^2(0,1)}\leq\epsilon$) by passing to the limit (when $k\to\infty$) in the *penalized optimal control problem* (find $u_k$ as the minimum of $J_k(u)=\frac{1}{2}\|u\|^2_{L^2(0,T)}+\frac{k}{2}\|y(T;u)-y_T\|^2_{L^2(0,1)}$). In the superlinear case (e.g. $f(y)=|y|^{n-1}y$, $n>1$) the existence of *two obstruction functions* $Y_{\pm\infty}$ shows that the approximate controllability is only possible if $Y_{-\infty}(x,T)\leq y_T(x)\leq Y_\infty(x,T)$ for a.e. $x\in(0,1)$. We carry out some numerical experiences showing that, for a fixed $k$, the "minimal cost" $J_k(u)$ (and the norm of the optimal control $u_k$) for a superlinear function $f$ becomes much larger when this condition is not satisfied. We also compare the values of $J_k(u)$ (and the norm of the optimal control $u_k$) for a fixed $y_T$ associated with two nonlinearities: one sublinear and the other one superlinear.

## 1 INTRODUCTION

This work deals with some numerical experiences regarding the control of semilinear equations of the type

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$$P(u) \begin{cases} y_t - y_{xx} + f(y) = u\chi_\omega & \text{in } (0,1)\times(0,T), \\ \frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(1,t) = 0 & \text{for } t\in(0,T), \\ y(x,0) = y_0(x) & \text{in } (0,1), \end{cases}$$

where $f$ is a $\mathcal{C}^1$ nondecreasing real function, $u$ is the output control, $\omega$ is an open subset of $(0,1)$, $T>0$ is (arbitrarily) fixed, and $y_0$ is a given function (for instance $y_0 \in H_0^1(0,1)$).

Given an arbitrary target state $y_T$ (we can assume, for simplicity, that $y_T \in C^0([0,1])$), the associated *approximate controllability problem* consists of, given an arbitrary $\varepsilon > 0$, find $u \in L^2(\omega \times (0,T))$ such that $\|y(T;u) - y_T\|_{L^2(0,1)} \leq \varepsilon$, where $y(T;u)$ denotes the solution of $P(u)$ at time $T$.

It is well known (see Fabre-Puel-Zuazua [9] and Díaz-Ramos [7]) that the answer is positive if "$f$ is sublinear at infinity" ($|f(s)| \leq M(|s|+1)$ for $|s|$ large and for some $M>0$). In the "superlinear at infinity" case

$$|f(s)| \geq M(|s|^n + 1) \text{ for } |s| \text{ large and for some } n>1 \text{ and } M>0, \tag{1}$$

the answer is negative. This type of negative results can be proved in different ways: via an energy argument (see, e.g. the case of control on the Neumann boundary condition, due to A. Bamberger, in Henry [14]) or via some pointwise obstruction phenomenon (see Díaz [4] for problem $P(u)$ and Díaz-Ramos [7] for other problems).

It is also well known that in the sublinear case, the solution to the controllability problem can be obtained by passing to the limit (as $k\to\infty$) in the *penalized optimal control problem* in which the control $u_k$ is found as the minimum of the functional

$$J_k(u) = \frac{1}{2}\|u\|^2_{L^2(\omega\times(0,T))} + \frac{k}{2}\|y(T;u) - y_T\|^2_{L^2(0,1)} \tag{2}$$

(see Lions [16] for the linear case and Fernández-Zuazua [11] for the semilinear case).

For the superlinear case the approximate controllability was obtained in Díaz [5] under the assumption

$$Y_{-\infty}(x,T) \leq y_T(x) \leq Y_\infty(x,T) \text{ for a.e. } x\in(0,1), \tag{3}$$

where $Y_{\pm\infty}$ are the "largest solutions". In our case, $Y_{\pm\infty}$ are the solutions of the problem

$$P(\pm\infty) \begin{cases} y_t - y_{xx} + f(y) = 0 & \text{in } ((0,1)\backslash\overline{\omega})\times(0,T), \\ \frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(1,t) = 0 & \text{for } t\in(0,T), \\ y(\cdot,t) = \pm\infty & \text{on } \partial\omega\times(0,T), \\ y(x,0) = y_0(x) & \text{in } (0,1) \end{cases}$$

(the existence of such large solutions requires $f$ to be superlinear, i.e, to satisfy (1)). Notice that the special case of $y_T \equiv 0$ is included (see, e.g., Fernández-Cara [10], for other results on null controllability).

More recently, some results on the approximate controllability of the projections on finite dimensional subspaces were obtained by Khapalov [15] (see also its references) for the superlinear case (1). Global exact steady-state controllability results have been obtained in Coron-Trélat [3].

A remarkable fact is that, for the sublinear case, the approximate controllability property holds for any open subset $\omega$ (as small as we want), but it may fail when the control domain $\omega$ is reduced to a single point (pointwise control). Furthermore, for linear cases it can be proved (see [12]) that the controllability property in the pointwise control is true for what is called *strategic* control points. When looking for these points one find that, for instance, for the linear heat equation, the non-rationals numbers are strategic and, if the problem is symmetric with respect to a rational number, this number is also an strategic point for that problem. Anyway, it is always possible to consider the associate optimal control problems similar to that with cost functional (2) and perform a similar analysis. This analysis has been carried out in [8] for several problems with the control point $x = 1/2$ and the suitable symmetries referred above holding (althought for nonlinear cases it is not guaranteed that this point is strategic).

The main goal of this work is to carry out some numerical experiments on the *penalized optimal control problem* for difference target states $y_T$ and different nonlinear terms $f(y)$. We illustrate the fact that, for a fixed $k$, the "minimal cost" $J_k(u)$ (and the norm of the optimal control $u_k$) for a superlinear function $f$ becomes much larger when (3) is not satisfied (see numerical test # 1, # 2 and # 3 below). We also compare the values of $J_k(u)$, the norm of the optimal control $u_k$, and $\|y(T;u) - y_T\|$, for a fixed $y_T$, associated to two different nonlinearities: one sublinear ($f(y) = \text{arctg}(y)$) and the other one superlinear ($f(y) = y^3$).

# 2 PROBLEM FORMULATION.

Let us consider a given target function $y_T \in L^2(0,1)$. We define the control space as $\mathcal{U} = L^2(\omega \times (0,T))$. The goal is to find a control $u \in \mathcal{U}$ so that $y(T)$ is *close* to $y_T$ at a minimal cost for the control, where $y(x,t)$ is the (unique) solution of $P(u)$). We recall that a *weak* formulation of $P(u)$ is provided by $y \in L^2(0,T;H^1(0,1)) \cap H^1(0,T;(H^1(0,1))') \subset \mathcal{C}([0,T] : L^2(\Omega))$ such that

$$\begin{cases} f(y) \in L^1(0,T;L^1(0,1)), \text{ and } \forall\ z \in L^2(0,T;H^1(0,1)) \cap L^\infty(Q) \\[2mm] \displaystyle\int_0^T < y_t, z >_{(H^1)' \times H^1} \mathrm{d}t + \int_0^T \int_0^1 y_x z_x \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^1 f(y) z \mathrm{d}x\mathrm{d}t \\ \displaystyle\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_0^T \int_\omega uz \mathrm{d}x\mathrm{d}t, \\[2mm] y(x,0) = y_0(x). \end{cases}$$

The existence and uniqueness of weak solution becomes standard after the work by Brezis-Browder [2]. Moreover, we can prove the boundedness of the solution even for unbounded controls.

**Proposition 1** *The weak solution $y$ of problem $\mathcal{P}(u)$ satisfies $y \in L^\infty((0,1) \times (0,T))$.*

*Proof.* Due to the monotonicity of function $f$, it is well-known that

$$|y(t,x)| \leq |h(t,x)| \quad \text{for any } t \in [0,T] \text{ and a.e. } x \in \Omega, \tag{4}$$

where $h$ is the (unique) solution of the linear equation

$$(\text{LHE}) \begin{cases} h_t - h_{xx} = u\chi_\omega & \text{in } (0,1) \times (0,T), \\ \frac{\partial h}{\partial x}(0,t) = \frac{\partial h}{\partial x}(1,t) = 0 & \text{for } t \in (0,T), \\ h(x,0) = y_0(x) & \text{in } (0,1), \end{cases}$$

But this equation has a unique solution

$$h \in L^2(0,T;H^2(0,1)) \cap H^1(0,T;L^2(0,1)) \subset \mathcal{C}([0,T]:H^1(0,1)) \subset L^\infty(Q).$$

The proof is based on Theorem 1.1 of Chapter 4 and Theorem 3.1 of Chapter 1 of Lions-Magenes [17]. This proves the result thanks to (4). □

For every $k \in I\!N$, we define the *cost function* $J_k$ by

$$J_k(v) = \frac{1}{2} \parallel v \parallel^2_{\mathcal{U}} + \frac{k}{2} \parallel y(T) - y_T \parallel^2_{L^2(0,1)}, \ \forall \ v \in \mathcal{U}.$$

The control problem is then

$$(\mathcal{CP}_k) \begin{cases} \text{find } u_k \in \mathcal{U}, \text{ such that} \\ \\ J_k(u_k) \le J_k(v), \quad \forall \ v \in \mathcal{U}. \end{cases}$$

A common way to solve this problem is to solve the problem

$$J'_k(u) = 0,$$

where $J'_k$ denotes the Gateaux differential of $J_k$.

Now, it is easy to prove (see, e.g., Glowinski-Lions [12] and Ramos-Glowinski-Periaux [19]) that

$$J'_k(v) = v + p|_\omega,$$

i.e.,

$$(J'_k(v), w) = \int_0^T \int_\omega (v+p) w \mathrm{d}x\mathrm{d}t, \ \forall \ w \in \mathcal{U},$$

where $p$ is the solution of the adjoint system

$$\begin{cases} -p_t - p_{xx} + f'(y)p = 0 \quad \text{in } Q, \\ \\ \dfrac{\partial p}{\partial x}(0) = \dfrac{\partial p}{\partial x}(1) = 0, \\ \\ p(T) = k(y(T;v) - y_T) \quad \text{in } (0,1) \end{cases}$$

and $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{U}$ defined by $(u,v) = \int_0^T \int_\omega uv \mathrm{d}x\mathrm{d}t$. Notice that Proposition 1 guarantees that $f'(y) \in L^\infty((0,1)\times(0,T))$.

# 3 TIME DISCRETIZATION.

We consider the time discretization step $\Delta t$, defined by $\Delta t = T/N$, where $N$ is a positive integer. Then, if $t^n = n\Delta t$, we have $0 < t^1 < t^2 < \cdots < t^N = T$. We approximate then problem $(\mathcal{CP})$ by the following finite-dimensional minimization problem:

$$(\mathcal{CP}_k)^{\Delta t} \begin{cases} \text{Find } u^{\Delta t} = \{u^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \text{ such that} \\ \\ J_k^{\Delta t}(u) \le J_k^{\Delta t}(v), \ \forall \ v = \{v^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \end{cases}$$

with the *time discrete control space* $\mathcal{U}^{\Delta t} = L^2(\omega) \times I\!R^N$ and

$$\begin{aligned} J_k^{\Delta t}(v) &= \frac{\Delta t}{2} \sum_{n=1}^{N} \| v^n \|^2_{L^2(\omega)} \\ &\quad + \frac{k}{2} \left( (1-\theta) \| y^{N-1} - y_T \|^2_{L^2(0,1)} + \theta \| y^N - y_T \|^2_{L^2(0,1)} \right), \end{aligned}$$

where $\theta \in (0,1]$ and $\{y^n\}_{n=1}^N$ is defined from the solution of the following *second order accurate time discretization* scheme of problem $(P(u))$:

$$y^0 = y_0,$$

$$\left\{ \begin{array}{l} \dfrac{y^1 - y^0}{\Delta t} - \dfrac{\partial^2}{\partial x^2}(\dfrac{2}{3}y^1 + \dfrac{1}{3}y^0) + f(y^1) = \dfrac{2}{3}v^1 \chi_\omega \quad \text{in } (0,1), \\ \\ \dfrac{\partial y^1}{\partial x}(0) = \dfrac{\partial y^1}{\partial x}(1) = 0, \end{array} \right.$$

and for $n \geq 2$,

$$\left\{ \begin{array}{l} \dfrac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} - \dfrac{\partial^2}{\partial x^2} y^n + f(y^n) = v^n \chi_\omega \quad \text{in } (0,1), \\ \\ \dfrac{\partial y^n}{\partial x}(0) = \dfrac{\partial y^n}{\partial x}(1) = 0. \end{array} \right.$$

*Remark.* We have used an *implicit* scheme. We could also have used a *semi-implicit* scheme, treating *implicitly* the diffusion term and *explicitly* the reaction term (as done in [1], [12] and [19] for the case of the diffusion and advection terms of the Burgers equation), but this choice may imply the necessity of choosing a very small time step $\Delta t$, in particular for reaction-dominated problem as the one we are treating.

# 4 FULL DISCRETIZATION.

We consider the space discretization step $h$, defined by $h = 1/I$, where $I$ is a positive integer. Then, if $x_i = (i-1)h$, we have $0 = x_1 < x_2 < \cdots < x_I < x_{I+1} = 1$. We approximate $H^1(0,1)$ by

$$V_h = \{z \in \mathcal{C}^0[0,1] : z|_{(x_i, x_{i+1})} \in P_1, i = 1, \cdots, I\},$$

where $P_1$ is the space of the polynomials of degree least or equal than one and $\mathcal{U}$ by $\mathcal{U}_h^{\Delta t} = (\mathcal{U}_h)^N$, where

$$\mathcal{U}_h = \{z : z \in \mathcal{C}^0(\overline{\omega}) : z|_{(x_i, x_{i+1})} \in P_1, \ \forall \ i = 1, \cdots, I \text{ such that } (x_i, x_{i+1}) \subset \omega\}.$$

We define $a_h$ by

$$a_h(y, z) = \int_0^1 y_x z_x \mathrm{d}x.$$

We approximate then problem $(\mathcal{CP}_k)$ by the following finite-dimensional minimization problem:

$$(\mathcal{CP}_k)_h^{\Delta t} \left\{ \begin{array}{l} \text{Find } u_h^{\Delta t} = \{u^n\}_{n=1}^N \in \mathcal{U}_h^{\Delta t}, \text{ such that} \\ \\ J_{k,h}^{\Delta t}(u_h^{\Delta t}) \leq J_{k,h}^{\Delta t}(v), \ \forall \ v = \{v^n\}_{n=1}^N \in \mathcal{U}_h^{\Delta t}; \end{array} \right.$$

with

$$J_{k,h}^{\Delta t}(v) = \frac{\Delta t}{2} \sum_{n=1}^{N} \| v^n \|_{L^2(\omega)}^2 + \frac{k}{2} \left( (1-\theta) \| y_h^{N-1} - y_T \|_{L^2(0,1)}^2 + \theta \| y_h^N - y_T \|_{L^2(0,1)}^2 \right),$$

where $\theta \in (0,1]$ and $\{y_h^n\}_{n=1}^N$ is defined from the solution of the following full discretization of problem $(P(u))$:

$$\begin{cases} y_h^0 \in V_h, \\ (y_h^0, z) = (y_0, z), \quad \forall z \in V_h; \end{cases}$$

$$\begin{cases} y_h^1 \in V_h, \\ \left( \dfrac{y_h^1 - y_h^0}{\Delta t}, z \right) + a_h(\frac{2}{3} y_h^1 + \frac{1}{3} y_h^0, z) + (f(y_h^1), z) = \dfrac{2}{3} \displaystyle\int_\omega v^1 z \mathrm{d}x, \qquad \forall z \in V_h; \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} y_h^n \in V_h, \\ \left( \dfrac{\frac{3}{2} y_h^n - 2 y_h^{n-1} + \frac{1}{2} y_h^{n-2}}{\Delta t}, z \right) + a_h(y_h^n, z) + (f(y_h^n), z) = \int_\omega v^n z \mathrm{d}x, \qquad \forall z \in V_h. \end{cases}$$

In the above algorithm $(\cdot, \cdot)$ denotes the scalar product in $L^2(0,1)$, that is,

$$(f, g) = \int_0^1 f(x) g(x) \mathrm{d}x \quad \forall\ f, g \in L^2(0,1).$$

As for the continuous case, to solve problem $(\mathcal{CP})_h^{\Delta t}$, we look for the solution $u_h^{\Delta t}$ of

$$\frac{\partial J_h^{\Delta t}}{\partial v}(u_h^{\Delta t}) = 0.$$

Computing $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$ is more complicated than in the continuous case but, following the same approach, we can show that

$$< \frac{\partial J_{k,h}^{\Delta t}}{\partial v}(v), w >= \Delta t \sum_{n=1}^{N} \int_\omega (v^n + p^n) w^n \mathrm{d}x,$$

where $\{p_h^n\}_{n=1}^{N+2}$ is the solution of

$$\begin{cases} p_h^{N+2} \in V_h, \\ (p_h^{N+2}, z) = -8l(1-\theta) \displaystyle\int_0^1 (y_h^{N-1} - y_T) z \mathrm{d}x - 2l\theta \int_0^1 (y_h^N - y_T) z \mathrm{d}x, \quad \forall z \in V_h; \end{cases}$$

$$\begin{cases} p_h^{N+1} \in V_h, \\ (p_h^{N+1}, z) = -2l(1-\theta) \displaystyle\int_0^1 (y_h^{N-1} - y_T) z \mathrm{d}x, \quad \forall z \in V_h; \end{cases}$$

and for $n = N, \cdots, 1$,

$$\begin{cases} p_h^n \in V_h, \\ \left( \dfrac{\frac{3}{2}p_h^n - 2p_h^{n+1} + \frac{1}{2}p_h^{n+2}}{\Delta t}, z \right) + a_h(p_h^n, z) + (f'(y_h^n)p_h^n, z) = 0, \quad \forall z \in V_h. \end{cases}$$

Now, once we know how to compute $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$, we use a *quasi-Newton method* à la BFGS (see, e.g., [18] for BFGS algorithms and their implementations) to compute the solution of the fully discrete control problem $(\mathcal{CP})_h^{\Delta t}$.

# 5 NUMERICAL EXPERIMENTS.

In all the tests considered we have taken $\omega = (0.4, 0.5)$, $T = 1$, $I = 100$, $N = 500$, $k = 10^{12}$ and $y_0 = 0$ (notice that this implies $y(x, t; 0) \equiv 0$). We use, for our algorithm, $\theta = 3/2$. Further, if $v_p$ $(p = 1, 2, \cdots)$ is the sequence of controls we get from the BFGS algorithm, we use the following stopping criteria: we stop iterating after step $p$ if either

$$\| \frac{\partial J_h^{\Delta t}}{\partial v}(u_p) \|_\infty \leq 10^{-5}$$

or

$$\frac{J_h^{\Delta t}(u_{p-1}) - J_h^{\Delta t}(u_p)}{\max\{|J_h^{\Delta t}(u_{p-1})|, |J_h^{\Delta t}(u_p)|, 1\}} \leq 2 \cdot 10^{-9}.$$

We have considered three different tests, depending on the target function.

## 5.1 Test 1: $y_T \equiv 5$.

On Figure 1 (resp., 2) we have shown the super-solution $Y_\infty(T)$ (...), the target function $y_T$ (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (resp. $f(y) = \mathrm{arctg}(y)$). The corresponding control functions have been represented on Figures 3 and 4.

On Figure 5 (resp., 6) we have shown the graphic of $\| y(t) - y_T \|_{L^2(0,1)}$, $t \in [0, 1]$, when $f(y) = y^3$ (resp. $f(y) = \mathrm{arctg}(y)$).

On Figure 7 (resp., 8) we have shown a 3D graphic of $y(x, t)$ when $t \in [0.98, 1]$ and $f(y) = y^3$ (resp. when $t \in [0.95, 1]$ and $f(y) = \mathrm{arctg}(y)$).

In Table 1 we give some further results about our solutions. The norms considered in all the tables of the present article refer to the $L^2-$norm of the discrete entries. One of the entries of the table shows the number of discrete parabolic equations the BFGS algorithm has needed to solve (a half of this number corresponds to the nonlinear state system and the other half corresponds to the linear adjoint system). Further, $y(v; T)$ represents the solution at time $T$, associated with the control $v$ ($y(0, T)$ represents the solution without control, at time $T$).

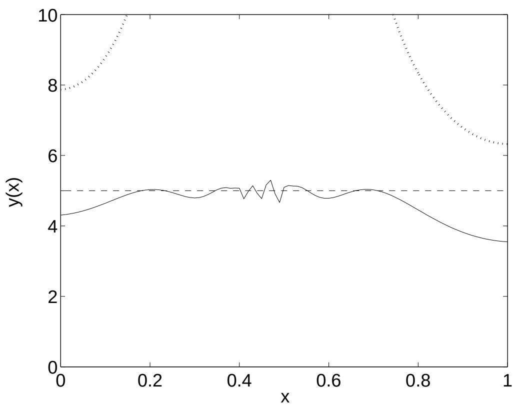


Figure 1: The target function (- -), the large solution (..) and controlled (–) states at time $T$, for $f(y) = y^3$.

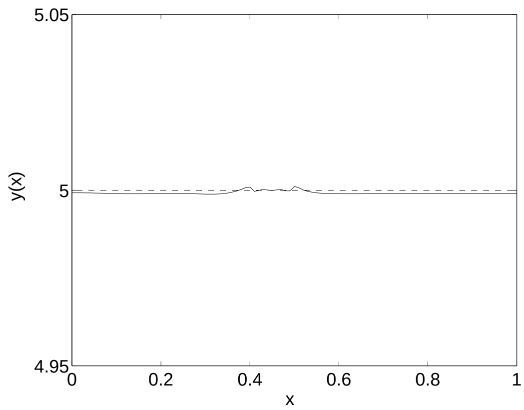


Figure 2: The target function (- -) and the controlled (–) state at time $T$, for $f(y) = \operatorname{arctg}(y)$.

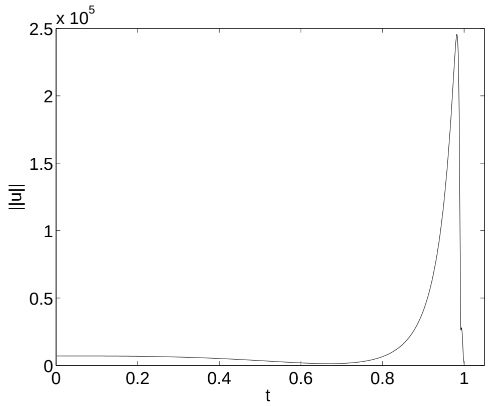


Figure 3: $\|u(t)\|$, for $f(y) = y^3$.

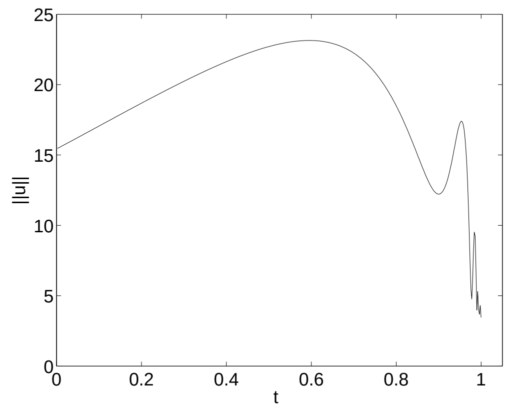


Figure 4: $\|u(t)\|$, for $f(y) = \operatorname{arctg}(y)$.

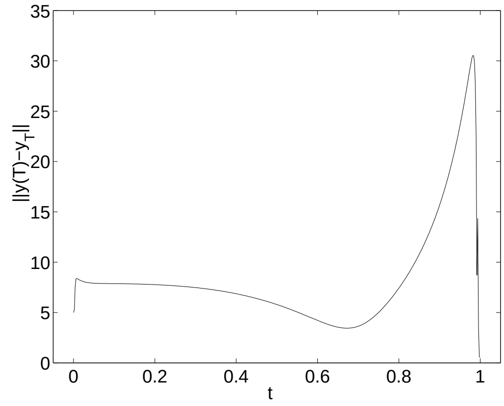


Figure 5: $\| y(t) - y_T \|$, for $f(y) = y^3$.

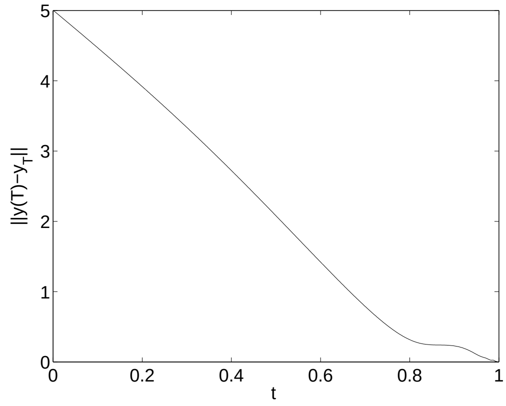


Figure 6: $\| y(t) - y_T \|$, for $f(y) = \text{arctg}(y)$.

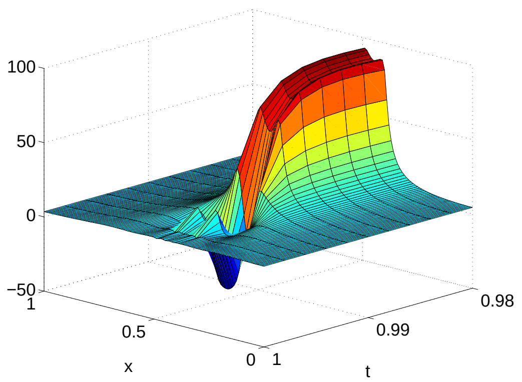


Figure 7: Graphic of $y(x, t)$ ($t \in [0.98, 1]$), for $f(y) = y^3$.

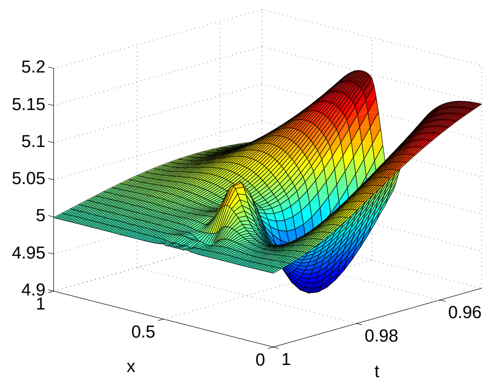


Figure 8: Graphic of $y(x, t)$ ($t \in [0.95, 1]$), for $f(y) = \text{arctg}(y)$.

| | $f(y) = y^3$ | $f(y) = \mathrm{arctg}(y)$ |
|---|---|---|
| $\parallel y(0;T) - y_T \parallel \ (=\parallel y_T \parallel)$ | 5 | 5 |
| $\parallel y(u;T) - y_T \parallel$ | 0.5613 | $8.69 \cdot 10^{-4}$ |
| $\parallel u \parallel$ | $4.2476 \cdot 10^4$ | 19.4185 |
| $J(0)$ | $1.25 \cdot 10^{13}$ | $1.25 \cdot 10^{13}$ |
| $J(u)$ | $1.5933 \cdot 10^{11}$ | $3.7797 \cdot 10^5$ |

Table 1: Computational results.

## 5.2 Test 2: $y_T \equiv 50$.

On Figure 9 (resp., 10) we illustrate the super-solution $Y_\infty(T)$ (...), the target function $y_T$ (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (resp. $f(y) = \mathrm{arctg}(y)$). The corresponding control functions have been represented on Figures 11 and 12.

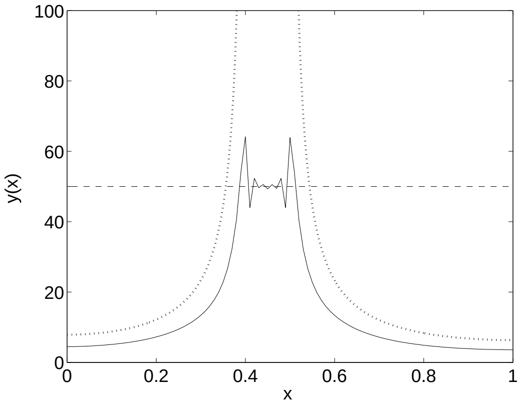


Figure 9: The target function (- -), the large solution (..) and controlled (–) states at time $T$, for $f(y) = y^3$.

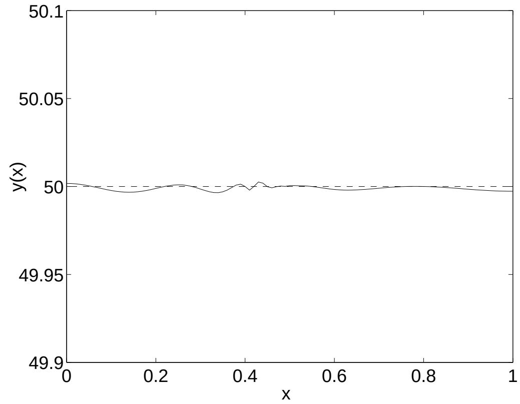


Figure 10: The target function (- -) and the controlled (–) state at time $T$, for $f(y) = \mathrm{arctg}(y)$.

On Figure 13 (resp., 14) we illustrate the graphic of $\parallel y(t) - y_T \parallel_{L^2(0,1)}$, $t \in [0,1]$, when $f(y) = y^3$ (resp. $f(y) = \mathrm{arctg}(y)$).

On Figure 15 (resp., 16) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.98, 1]$ and $f(y) = y^3$ (resp. when $t \in [0.95, 1]$ and $f(y) = \mathrm{arctg}(y)$).

In Table 2 we give some further results about our solutions.

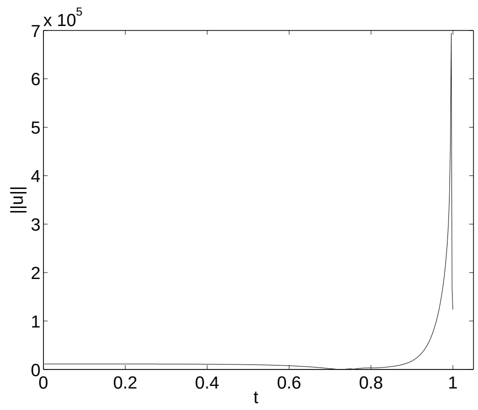


Figure 11: The computed optimal control for $f(y) = y^3$.

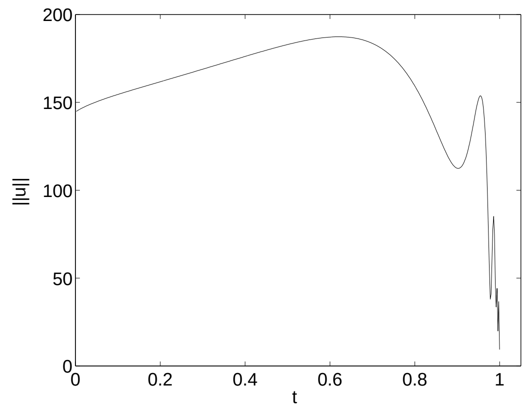


Figure 12: The computed optimal control for $f(y) = \mathrm{arctg}(y)$.

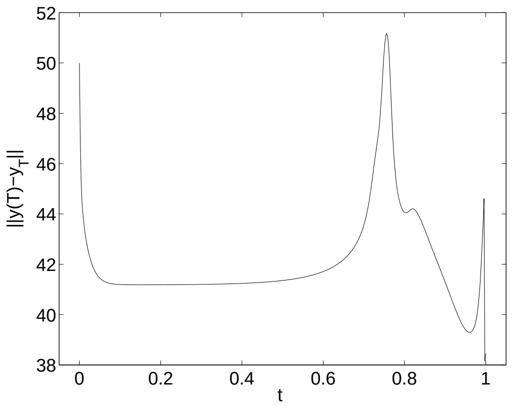


Figure 13: $\| y(t) - y_T \|$, for $f(y) = y^3$.

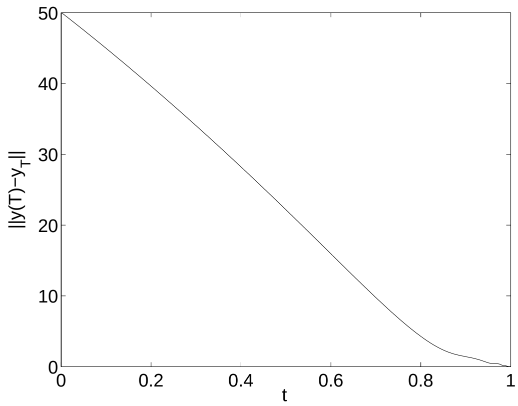


Figure 14: $\| y(t) - y_T \|$, for $f(y) = \mathrm{arctg}(y)$.

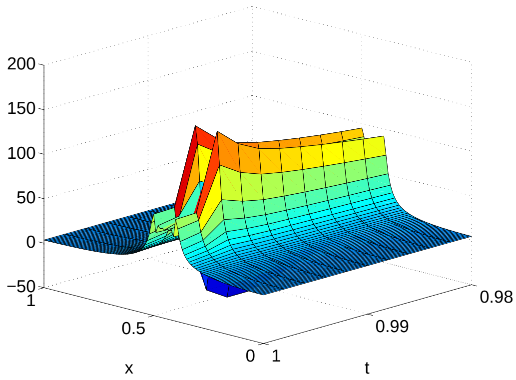


Figure 15: Graphic of $y(x,t)$ ($t \in [0.98, 1]$), for $f(y) = y^3$.

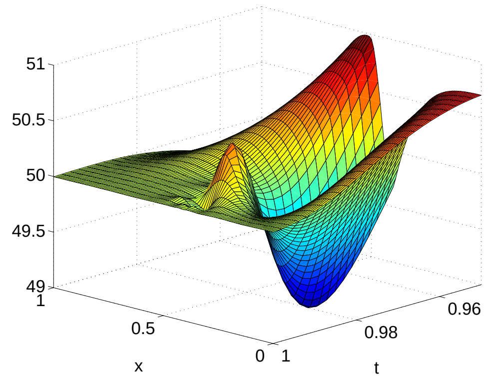


Figure 16: Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = \text{arctg}(y)$.

| | $f(y) = y^3$ | $f(y) = \text{arctg}(y)$ |
|---|---|---|
| $\| y(0;T) - y_T \|$ $(= \| y_T \|)$ | 50 | 50 |
| $\| y(u;T) - y_T \|$ | 38.4450 | $1.615 \cdot 10^{-3}$ |
| $\| u \|$ | $5.6426 \cdot 10^4$ | $1.6364 \cdot 10^2$ |
| $J(0)$ | $1.25 \cdot 10^{15}$ | $1.25 \cdot 10^{15}$ |
| $J(u)$ | $7.3901 \cdot 10^{14}$ | $1.3312 \cdot 10^6$ |

Table 2: Computational results.

### 5.3 Test 3:

$$y_T(x) = \begin{cases} 0 & \text{if } x \in (0, 0.5), \\ 8x - 4 & \text{if } x \in (0.5, 0.75), \\ -8x + 8 & \text{if } x \in (0.75, 1). \end{cases}$$

On Figure 17 (resp., 18) we have shown the super-solutions $Y_\infty(T)$ (...), the target function $y_T$ (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (resp. $f(y) = \text{arctg}(y)$). The corresponding control functions have been represented on Figures 19 and 20.

On Figure 21 (resp., 22) we have shown the graphic of $\| y(t) - y_T \|_{L^2(0,1)}$, $t \in [0, 1]$, when $f(y) = y^3$ (resp. $f(y) = \text{arctg}(y)$).

On Figure 23 (resp., 24) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.98, 1]$ and $f(y) = y^3$ (resp. when $t \in [0.99, 1]$ and $f(y) = \text{arctg}(y)$).

In Table 3 we give some further results about our solutions.

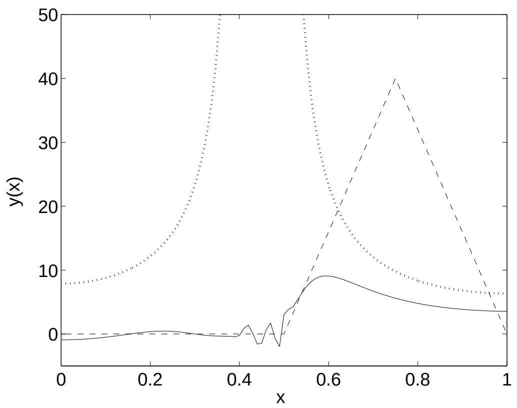


Figure 17: The target function (- -), the large solutions (..) and controlled (–) states at time $T$, for $f(y) = y^3$.

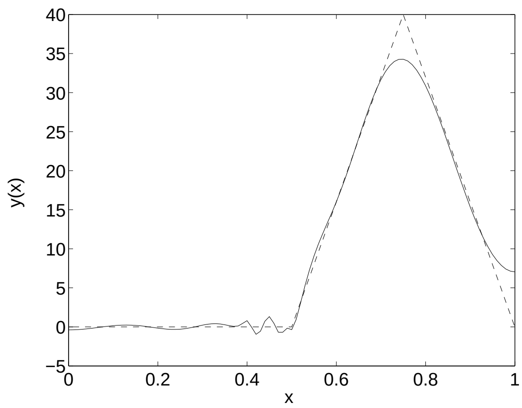


Figure 18: The target function (- -) and the controlled (–) state at time $T$, for $f(y) = \mathrm{arctg}(y)$.

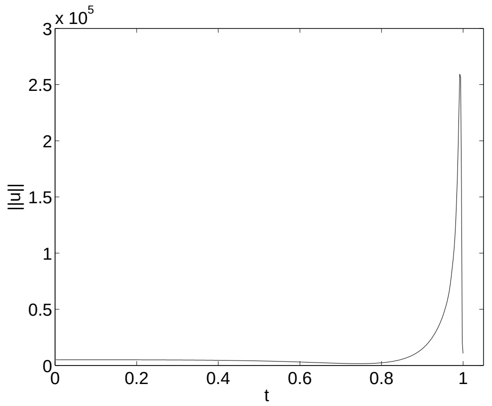


Figure 19: The computed optimal control for $f(y) = y^3$.

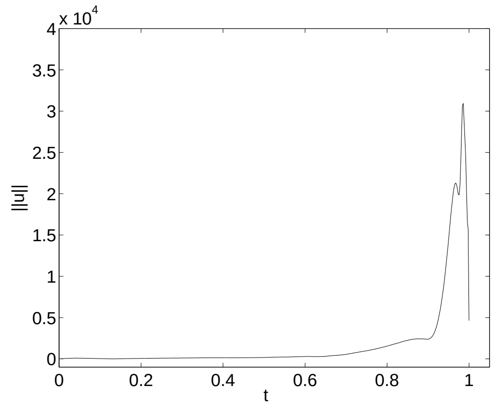


Figure 20: The computed optimal control for $f(y) = \mathrm{arctg}(y)$.

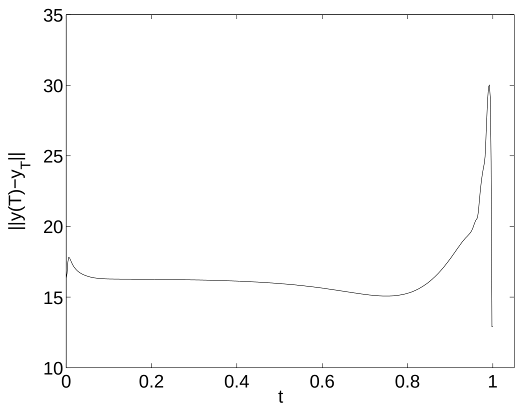


Figure 21: $\| y(t) - y_T \|$, for $f(y) = y^3$.

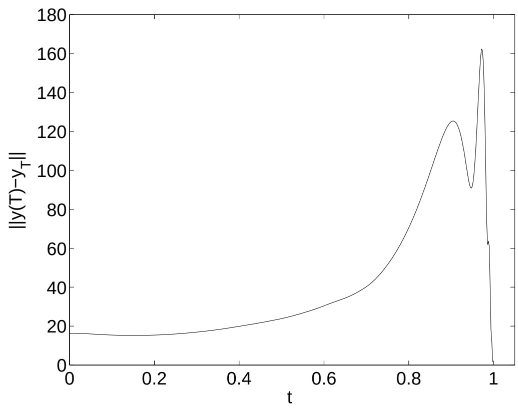


Figure 22: $\| y(t) - y_T \|$, for $f(y) = \text{arctg}(y)$.

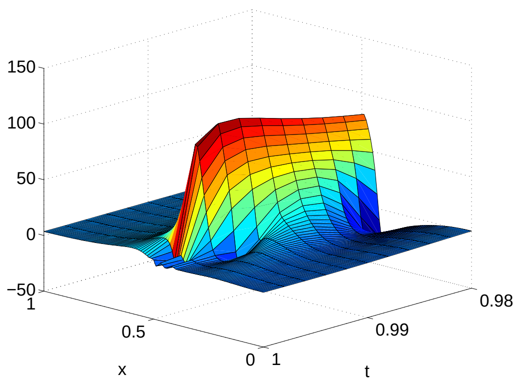


Figure 23: Graphic of $y(x,t)$ $(t \in [0.95, 1])$, for $f(y) = y^3$.

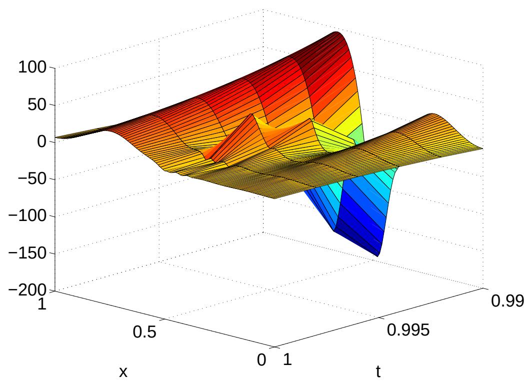


Figure 24: Graphic of $y(x,t)$ $(t \in [0.99, 1])$, for $f(y) = \text{arctg}(y)$.

| | $f(y) = y^3$ | $f(y) = \mathrm{arctg}(y)$ |
|---|---|---|
| $\Vert y(0;T) - y_T \Vert \ \ (=\Vert y_T \Vert)$ | 16.3299 | 16.3299 |
| $\Vert y(u;T) - y_T \Vert$ | 12.920120 | 1.418373 |
| $\Vert u \Vert$ | $2.9238 \cdot 10^4$ | $5.1621 \cdot 10^3$ |
| $J(0)$ | $1.3333 \cdot 10^{14}$ | $1.3333 \cdot 10^{14}$ |
| $J(u)$ | $8.3466 \cdot 10^{13}$ | $1.0059 \cdot 10^{12}$ |

Table 3: Computational results.

# 6 CONCLUSIONS AND CONJECTURES.

Our numerical results give some quantitaive information on a result theoretically showed in [7]: when we consider a superlinear at infinity nonlinearity (e.g. $f(y) = y^3$) and the target function $y_T$ does not satisfy (3), then the approximate controllability property fails.

We also (numerically) show the obstruction phenomenon does not appear when $f$ is sublinear at infinity (e.g. $f(y) = \mathrm{arctg}(y)$) and get suitable controls. This is consistent with the theoretical approximate controllability results obtained in [9] (see also [7]).

For the superlinear case, our experiments confirm that, as theoretically proved in [5], when the target function satisfies (3), the controllability property holds. The above mentioned proof in [5] is not constructive and follows a different scheme to the successive *penalized optimal control problems* used in this paper.

A remarkable fact is that, in superlinear cases (and occasionally also in sublinear cases), the solution $y$ oscillates very fast for times $t \in (T - \delta, T)$, getting away from the target state $y_T$ and finally approaching $y_T$ at time $T$. This is an unstable phenomenon typical of optimal control problems of controllability type, in contrast with the non-oscillating behavior of the solution of stabilization type problems (see, e.g. Glowinski-Ramos [13]).

Finally, we point out that the optimal controls obtained in our experiments follow the typical pattern of remaining *close* to zero until the last part of the time interval.

The above numerical experiences lead us to formulate the following conjectures:

A. A theoretical proof of the approximate controllability property for problems with superlinear at infinity nonlinearities and target states satisfying (3) can be also obtained in a constructive way, by means of the *penalized optimal control problems* $(\mathcal{CP}_k)$ used in this paper.

B. Fixed a target function $y_T$ satisfying (3), the *cost* (in terms of the norm of the controls) to approximate this function is, in general, much bigger for superlinear cases than for sublinear cases. However, this result can be false if $y_T$ is *small* enough. For instance, when $f(y) = |y|^{p-1}y$, the cost to approximate $y_T$ is much bigger when $p > 1$, except for target functions satisfying $|y_T(x)| \leq 1$. This conjecture is exactly the opposite of the results obtained in Díaz-Lions [6] for the case of initial value control problems with nonlinearities of the type $f(y) = -y^3$.

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## References


[1] M. Berggren, R. Glowinski and J.L. Lions, "A Computational Approach to Controllability Issues for Flow-Related Models. (I): Pointwise Control of the Viscous Burgers Equation", *International Journal of Computational Fluid Dynamics*, **7**, 237–252 (1996).

[2] H. Brezis and F. Browder, "Strongly nonlinear parabolic initial-boundary value problems", *Proc. Nat. Acad. Sci.*, 76, 38-40 (1979).

[3] J.M. Coron and E. Trélat, "Global steady-state controllability of 1-D semilinear heat equations". Preprint, 2003.

[4] J.I. Díaz, "Mathematical Analysis of some diffusive energy balance models in Climatology". In *Mathematics, Climate and Environment*, J.I. Díaz and J.L. Lions Eds., Masson, 28–56 (1993).

[5] J.I. Díaz, "Controllability and Obstruction for Some Nonlinear Parabolic Problems in Climatology". In *Modelado de sistemas en Oceanografía, Climatología y Ciencias Medio-Ambientales*, Univ. de Málaga Eds., 43–58 (1994).

[6] J.I. Díaz and J.L. Lions, "On the Approximate Controllability for some Explosive Parabolic Problems". In *International Series of Numerical Mathematics*, Vol. 133, 115–132 (1999).

[7] J.I. Díaz and A.M. Ramos, "Positive and negative approximate controllability results for semilinear parabolic equations", *Rev. Real Academia de Ciencias Exactas, Físicas y Naturales, de Madrid*, Tomo LXXXIX, 11–30 (1995).

[8] J.I. Díaz and A.M. Ramos, "Numerical Experiments Regarding the Localized Control of Semilinear Parabolic Problems". In the Proceedings of the *European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2000)*, Barcelona, September 11–14, 2000. CD-ROM format, ISBN: 84-89925-70-4.

[9] C. Fabre, J.P. Puel and E. Zuazua, "Contrôlabilité approchée de l'équation de la chaleur semi-linéaire", *C.R. Acad. Sci. Paris*, t. 315, Serie I, 807–812 (1992).

[10] E. Fernández-Cara, "Null Controllability for Semilinear Heat Equation", *ESSAIM: Control, Optimization and Calculus of Variations*, 87–103 (1997).

[11] L.A. Fernández and E. Zuazua, "Approximate Controllability of the Semilinear Heat Equation via Optimal Control", Preprint de la Univ. de Cantabria (1996).

[12] R. Glowinski and J.L. Lions, "Exact and approximate controllability for distributed parameter systems, Part II", *Acta Numerica*, 159–333 (1995).

[13] R. Glowinski and A.M. Ramos, "Control and Stabilization of the Cahn-Hilliard Equation. A Numerical Approach". In *Actas del XVI C.E.D.Y.A./ VI Congreso de Matemática Aplicada*, Vol. III (Anexo), Servicio de Publicaciones de la Universidad de Las Palmas de Gran Canaria, R. Montenegro, G. Montero y G. Winter Eds. (1999).

[14] J. Henry, *Etude de la contrôlabilité de certaines équations paraboliques*, Tèse d'Etat, Université Paris VI, (1978).

[15] A.Y. Khapalov, "Global controllability properties for the semilinear heat equation with superlinear terms", *Revista Matemática Complutense*, **12**, 511–535 (1999).

[16] J.L. Lions, "Remarques sur la contrôlabilité approchée". In *Proceedings of Jornadas Hispano-Francesas sobre control de sistemas distribuidos*, Univ. de Málaga Eds., 77–87 (1991).

[17] J.L Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications*, Vol. 1 and 2, Dunod, Paris, 1968.

[18] D.C. Liu and J. Nocedal, "On the limited memory BFGS method for large scale optimization", *Mathematical Programming*, **45**, 503–528 (1989).

[19] A.M. Ramos, R. Glowinski and J. Periaux, "Nash Equilibria for the Multi-Objective Control of Linear Partial Differential Equations". *Journal of Optimization, Theory and Applications*, Vol. 112, N. 3, 2002, pp. 457–498.