Factorization of Second Order Elliptic Boundary Value Problems by Dynamic Programming

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Abstract: We present a method to factorize a second order boundary value problem in a cylindrical domain in a system of uncoupled first order initial value problems, together with a nonlinear Riccati type equation for functional operators. This uncoupling is obtained by a space invariant embedding technique along the axis of the cylinder. This method can be viewed as an infinite dimensional generalization of the block Gauss LU factorization.

Keywords: Factorization; boundary value problem; Riccati equation; invariant embedding; Neumann-to-Dirichlet (NtD) operator; Dirichlet-to-Neumann (DtN) operator

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1 Introduction

In [1] Angel and Bellman proposed a method based on invariant embedding to transform a second order elliptic boundary value problem in a rectangle in a system of first order decoupled initial value problems which can be solved by a two sweep process (see also [2]). This formulation was derived only formally with the use of the Neumann-to-Dirichlet (NtD) map. Here we study this method for a model problem : the Poisson equation. In Section 2 we present the formal derivation of the factorization, extending previous results to an n-dimensional cylindrical domain with axis parallel to the x_1 coordinate, for various boundary conditions, using also the Dirichlet-to-Neumann (DtN) map. The first objective

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of the paper, carried out in Section 3, is to give a functional space framework and a mathematical justification of the derivation of the factorization. Secondly, in Section 4 we show the relation of the use of invariant embedding for this problem and for optimal control problems associated to evolution equations. In particular we show the relation between the time dependent Riccati equation providing the feedback law of such optimal control problems (see e.g. [9, 5]) and the x_1 Riccati equation satisfied by the NtD or DtN maps in our case. We also show, in Section 5, that this factorization can be viewed as the extension to the infinite dimensional problem of the well known block Gauss LU factorization of the matrix of the discretized problem. Section 6 gives some clues about the interest of such a factorization for the study of elliptic boundary value problems, presenting some situations where one can take advantage from the factorized form of the problem. It is believed that the method of factorization of boundary value problems is more general and can be applied to more complex situation than the Poisson equation in a cylindrical domain. We found this case convenient to present the method and give full mathematical justifications. Other results can be found in [15]. In [7], the authors use these techniques (in a formal way) to solve an optimal control problem associated to an elliptic equation and get the optimal control in an explicit way. In [6] the method is applied to the factorization of the linear elasticity system. Furthermore, similar techniques have been used recently in acoustics in order to compute generalized impedance in waveguides (see [13], [12]). Specific numerical schemes are developped from this approach ([11]).

2 Elliptic problem in a cylindrical domain.

We consider the Poisson equation in a cylindrical domain along the x_1-coordinate. This coordinate plays the role of time for a parabolic equation. We shall make a strong analogy with the uncoupling of the optimality conditions associated to an optimal control problem of such systems.

2.1 Statement of the problem and formal resolution

Let $\mathcal{O}$ be a smooth bounded open set in $I\!R^{n-1}$, Ω be the cylinder $\Omega =]0, a[\times\mathcal{O}$ in $I\!R^n$, $\Gamma_0 = \{0\} \times \mathcal{O}$, $\Gamma_a = \{a\} \times \mathcal{O}$ and $f \in L^2(\Omega)$. The lateral boundary of the cylinder is denoted by $\Sigma = \partial\mathcal{O}\times]0, a[$. The regularity of the data y_0 and y_1 is defined below. Let us denote $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x_1^2} + \Delta_z$, where z denotes the independent variables $x_2, ..., x_n$. We consider the problem

$$(\mathcal{P}_0) \begin{cases} -\Delta y = f \quad \text{in } \Omega, \\ y|_\Sigma = 0, \quad -\dfrac{\partial y}{\partial x_1}|_{\Gamma_0} = y_0, \quad y|_{\Gamma_a} = y_1. \end{cases}$$

We recall the Sobolev space $H^{1/2}_{00}(\mathcal{O})$ defined in Theorem 11.7, p. 72 of [10]

as the 1/2 interpolate between $H_0^1(\mathcal{O})$ and $L^2(\mathcal{O})$. In problem $(\mathcal{P}_0)$ we take $y_0 \in H_{00}^{1/2}(\mathcal{O})'$ (dual space of $H_{00}^{1/2}(\mathcal{O})$) and $y_1 \in H_{00}^{1/2}(\mathcal{O})$.
Using the technique of invariant embedding introduced by R. Bellman (see [4]), we embed problem $(\mathcal{P}_0)$ in a family of similar problems $(\mathcal{P}_{s,h})$ defined on $\Omega_s =]0, s[\times \mathcal{O}$ for $s \in]0, a]$ (see Fig. 1).

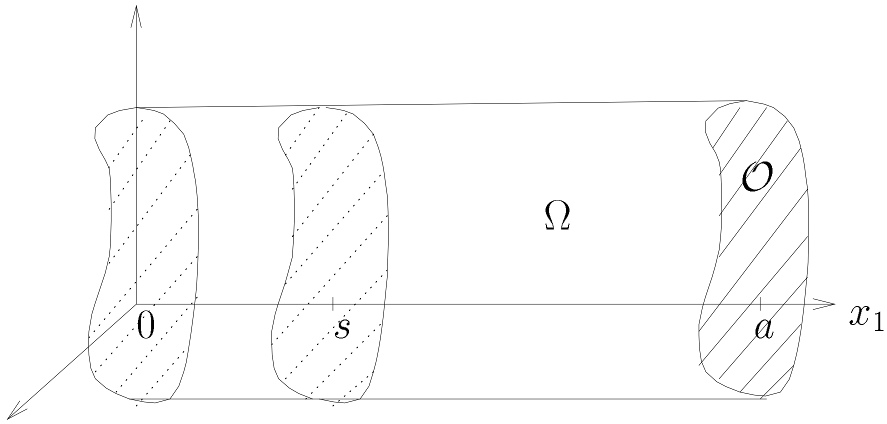

Figure 1: Domain of the problem.

For each problem we impose the Dirichlet boundary condition $y|_{\Gamma_s} = h$, where $\Gamma_s = \{s\} \times \mathcal{O}$.

$$(\mathcal{P}_{s,h}) \begin{cases} -\Delta y = f \quad \text{in } \Omega_s, \\ y|_\Sigma = 0, \quad -\dfrac{\partial y}{\partial x_1}|_{\Gamma_0} = y_0, \quad y|_{\Gamma_s} = h. \end{cases}$$

Clearly $(\mathcal{P}_0)$ is exactly $(\mathcal{P}_{s,h})$ for $s = a, h = y_1$.
For $s \in]0, a]$ we define $Y_s = \{v \in H^1(\Omega_s) : \frac{\partial^2 v}{\partial x_1^2} \in L^2(0, s; H^{-1}(\mathcal{O}))$ and $v|_\Sigma = 0\}$.
We then apply a method quite similar to the one used by Lions ([9]) for deriving the optimal feedback for an optimal control problem of a parabolic equation.

Definition 2.1 *For every $s \in]0, a]$ we define the DtN map $Q(s)$ by $Q(s)h = \frac{\partial \gamma}{\partial x_1}|_{\Gamma_s}$, where $h \in H_{00}^{1/2}(\mathcal{O})$ and $\gamma \in Y_s$ is the solution of*

$$\begin{cases} -\Delta \gamma = 0 \quad \textit{in } \Omega_s, \\ \gamma|_\Sigma = 0, \quad \dfrac{\partial \gamma}{\partial x_1}|_{\Gamma_0} = 0, \quad \gamma|_{\Gamma_s} = h. \end{cases}$$

We set $Q(0)h = 0$ and we also define $w(s) = \frac{\partial \beta}{\partial x_1}|_{\Gamma_s}$, where $\beta \in Y_s$ is the solution of

$$\begin{cases} -\Delta \beta = f \quad \textit{in } \Omega_s, \\ \beta|_\Sigma = 0, \quad -\dfrac{\partial \beta}{\partial x_1}|_{\Gamma_0} = y_0, \quad \beta|_{\Gamma_s} = 0. \end{cases}$$

and we set $w(0) = -y_0$.

For every $s \in [0,a]$, $Q(s) : H^{1/2}_{00}(\mathcal{O}) \to H^{1/2}_{00}(\mathcal{O})'$ is a linear operator and $w(s) \in H^{1/2}_{00}(\mathcal{O})'$ because of the well-posedness of the problem in γ and of properties of trace application which will be proved in section 3. By linearity of $(\mathcal{P}_{s,h})$ we have $\frac{\partial y}{\partial x_1}|_{\Gamma_s} = Q(s)h + w(s)$.

Furthermore, the solution y of $(\mathcal{P}_0)$ restricted to $]0,s[$ satisfies $(\mathcal{P}_{s,y|_{\Gamma_s}})$ for $s \in]0,a[$ so that

$$\frac{\partial y}{\partial x_1}(x_1,z) = (Q(x_1)y|_{\Gamma_{x_1}})(z) + (w(x_1))(z). \tag{1}$$

Then, by formally taking the derivative with respect to x_1 of this formula, we obtain

$$\frac{\partial^2 y}{\partial x_1^2} = -\Delta_z y - f = \frac{dQ}{dx_1}y + Q\frac{\partial y}{\partial x_1} + \frac{\partial w}{\partial x_1}.$$

Therefore substituting $\frac{\partial y}{\partial x_1}$ from equation (1)

$$0 = \left(\frac{dQ}{dx_1} + Q^2 + \Delta_z\right)y + \frac{\partial w}{\partial x_1} + Qw + f,$$

and then, since y is arbitrary, we obtain the decoupled system

$$\begin{cases} \dfrac{dQ}{dx_1} + Q^2 + \Delta_z = 0, & Q(0) = 0, \\ \dfrac{dw}{dx_1} + Qw = -f, & w(0) = -y_0, \\ -\dfrac{dy}{dx_1} + Qy = -w, & y(a) = y_1. \end{cases} \tag{2}$$

Let us stress that Q is an operator on functions in z depending on x_1 which satisfies a Riccati equation. The system (2) is decoupled because one can integrate the first two equations in x_1 from 0 to a giving Q and w, then y is obtained by the integration backwards of the third equation. Formally, we have factorized $-\Delta y = f$ as $-\left(\frac{d}{dx_1} + Q\right)\left(\frac{d}{dx_1} - Q\right)y = f$. Since Q is self adjoint (as it will be shown further), it is clear that the two factors are adjoint of each other. Also, as $-Q$ is coercive, the equations for w and y are of parabolic type.

2.2 An equivalent formulation.

The solution y can also be obtained by invariant embedding in the complementary domain $\widetilde{\Omega}_s =]s,a[\times \mathcal{O}$.

For $s \in]0,a[$ we define $\widetilde{Y}_s = \{v \in H^1(\widetilde{\Omega}_s) : \frac{\partial^2 v}{\partial x_1^2} \in L^2(s,a;H^{-1}(\mathcal{O}))$ and $v_{|\Gamma_a} = v_{|\Sigma} = 0\}$.

$$(\widetilde{\mathcal{P}}_{s,h}) \begin{cases} -\Delta y = f & \text{in } \widetilde{\Omega}_s, \\ y|_\Sigma = 0, \quad \dfrac{\partial y}{\partial x_1}|_{\Gamma_s} = h, & y|_{\Gamma_a} = y_1. \end{cases}$$

In a similar way we define the NtD map $P(s)$ with respect to $\widetilde{\Omega}_s$ for every $s \in [0, a[$ and $h \in H^{1/2}_{00}(\mathcal{O})'$: we decompose y into its linear part in h, γ, and its part independent of h, β We have $P(s)h = \gamma|_{\Gamma_s}$, $r(s) = \beta|_{\Gamma_s}$ and $y(s) = P(s)h + r(s)$. We set $P(a) = 0$ and $r(a) = y_1$. For every $s \in [0, a]$, $P(s) : H^{1/2}_{00}(\mathcal{O})' \to H^{1/2}_{00}(\mathcal{O})$ is a continuous linear operator and $r(s) \in H^{1/2}_{00}(\mathcal{O})$. Further, the solution y of $(\mathcal{P}_0)$ is given by

$$y(x_1, z) = (P(x_1)\frac{\partial y}{\partial x_1}|_{\Gamma_{x_1}})(z) + (r(x_1))(z). \tag{3}$$

Formal calculation. By using $(\mathcal{P}_0)$, (3) and formal derivation we obtain

$$\frac{\partial y}{\partial x_1} = \left(\frac{dP}{dx_1} - P\Delta_z P\right)\frac{\partial y}{\partial x_1} + \frac{\partial r}{\partial x_1} - P\Delta_z r - Pf. \tag{4}$$

Therefore, since y is arbitrary, we “deduce” the following uncoupled system

$$\begin{cases} \dfrac{dP}{dx_1} - P\Delta_z P - I = 0, & P(a) = 0, \\ \dfrac{dr}{dx_1} - P\Delta_z r = Pf, & r(a) = y_1, \\ -P\dfrac{dy}{dx_1} + y = r, & y(0) = -P(0)y_0 + r(0). \end{cases} \tag{5}$$

Formally, we have factorized $-\Delta y = f$ as $\left(\frac{d}{dx_1} - P\Delta_z\right)\left(I - P\frac{d}{dx_1}\right)y = Pf$. As in the previous section, one can check that the factors are adjoint of each other using the Riccati equation.

Remark 2.2 *The link between these factorizations and Control Theory will be shown further. Then the relation between systems (2) and (5) is exactly the same as between the control and filtering problems respectively.*

3 A Justification for the factorization

In this section we give a precise derivation of the decoupled system (5) of the problem $(\mathcal{P}_0)$.

Definition 3.1 *We define X, X_0 and Y by*

$$\begin{gathered} X = L^2(0, a; H^1_0(\mathcal{O})) \cap H^1(0, a; L^2(\mathcal{O})) \\ X_0 = X \cap \{\varphi \in X : \ \varphi|_{\Gamma_a} = 0\} \\ Y = \{y \in X : \tfrac{\partial^2 y}{\partial x_1^2} \in L^2(0, a; H^{-1}(\mathcal{O}))\} \end{gathered}$$

We need the following trace theorem.

Proposition 3.2 *If $y \in Y$ it holds that*

$$(y|_{\Gamma_s}, \frac{\partial y}{\partial x_1}|_{\Gamma_s}) \in \mathcal{C}([0,a]; H_{00}^{1/2}(\mathcal{O}) \times H_{00}^{1/2}(\mathcal{O})'),$$

and the trace mapping $y \to (y|_{\Gamma_s}, \frac{\partial y}{\partial x_1}|_{\Gamma_s})$ is continuous from Y onto $H_{00}^{1/2}(\mathcal{O}) \times H_{00}^{1/2}(\mathcal{O})'$.

Proof. It is a direct application of theorem 3.1, p. 23 of [10], having into account that $H_{00}^{1/2}(\mathcal{O})'$ is the 1/2 interpolate between $L^2(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ (see theorem 12.4, p. 81 of [10]). □

By making a translation on y and the corresponding one on f we can discard the inhomogeneous Dirichlet boundary condition. From now on we assume $y_1 = 0$. The variational formulation for y is

$$\int_\Omega \nabla y \nabla \varphi \, dx_1 dz = \int_\Omega f \varphi \, dx_1 dz + < y_0, \varphi|_{\Gamma_0} >_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}, \quad \forall \varphi \in X_0. \tag{6}$$

3.1 Properties of P.

The following proposition collects some basic properties of operator P.

Proposition 3.3 *The linear operator $-P(s) : H_{00}^{1/2}(\mathcal{O})' \to H_{00}^{1/2}(\mathcal{O})$ is continuous, self-adjoint, positive for every $s \in [0,a]$ and coercive for every $s \in [0,a[$.*

Proof. We already noticed that $P(s)$ is continuous as the composition of continuous operators: $h \to \gamma \to \gamma|_{\Gamma_s}$. Let $h, \overline{h} \in H_{00}^{1/2}(\mathcal{O})'$ and $\gamma, \overline{\gamma}$ the corresponding solutions in $\widetilde{\Omega}_s$, then we get

$$\int_{\widetilde{\Omega}_s} \nabla \gamma \nabla \overline{\gamma} \, dx_1 dz = - < h, P(s)\overline{h} >_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})},$$

which gives the self-adjointness and positivity properties for $h = \overline{h}$. Then, by Poincaré inequality and Proposition 3.2,

$$- < h, P(s)h >_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} \geq c \parallel h \parallel^2_{H_{00}^{1/2}(\mathcal{O})'},$$

which proves the coercivity. □

3.2 Semi discretization.

At this stage, operator P and function r are clearly defined but the equations they satisfy have been derived only formally. We will justify these equations using the Galerkin method. Let $\{w_1, ..., w_n, ..\}$ be a Hilbert basis of $H_0^1(\mathcal{O})$ of eigenfunctions of the Dirichlet problem $-\Delta_z w_n = \lambda_n w_n$ for $z \in \mathcal{O}$ with the boundary conditions $w_n|_{\partial\mathcal{O}} = 0$. It has the following properties:

$$\left\{\begin{array}{l} (a)\ (w_n, w_m)_{L^2(\mathcal{O})} = \delta_{n,m} \quad \forall m, n. \\ (b)\ (w_n, w_m)_{H_0^1(\mathcal{O})} = \int_{\mathcal{O}} \nabla_z w_n(z) \nabla_z w_m(z) dz = \lambda_n \delta_{n,m}, \\ (c)\{\sum_{\text{finite}} \mu_j w_j,\ \mu_j \in I\!R\} \text{ is a dense subset of } H_0^1(\mathcal{O}). \end{array}\right. \tag{7}$$

We define $V^m = span\{w_1, ..., w_m\}$, and $X_0^m = \{\varphi \in H^1(0, a; V^m) :\ \varphi|_{\Gamma_a} = 0\}$. We equip X_0^m with the norm

$$\| \varphi \|_m^2 = \int_0^a \sum_{i=1}^m \left(\lambda_i \varphi_i^2(x_1) + \left| \frac{d\varphi(x_1)}{dx_1} \right|^2 \right) dx_1, \quad (\varphi(x_1, z) = \sum_{i=1}^m \varphi_i(x_1) w_i(z)).$$

Proposition 3.4 *$X_0^m \subset \mathcal{C}([0, a]; V^m)$ is a continuous injection and*

$$\bigcup_{m \in I\!N} \{\varphi \in H^1(0, a; V^m) :\ \varphi_{|\Gamma_a} = 0\}$$

is a dense set of X_0 (and so of $L^2(\Omega)$).

We define the approximation y^m of y by the solution of

$$\int_\Omega (\nabla y^m \nabla \varphi - f\varphi)\, dx_1 dz = < y_0, \varphi|_{\Gamma_0} >_{H_{00}^{1/2}{}' \times H_{00}^{1/2}} \quad \forall\, \varphi \in X_0^m. \tag{8}$$

Now it can be shown that the coordinates $c(x_1) = \{c_i(x_1)\}_{i=1}^m$ of $y^m(x_1)$ satisfy the uncoupled system of two-point boundary value problems

$$\left\{\begin{array}{l} -\dfrac{d^2 c_i}{dx_1^2}(x_1) + \lambda_i c_i(x_1) = \displaystyle\int_{\mathcal{O}} f(x_1) w_i dz \quad x_1 \in]0, a[, \quad i = 1, .., m \\ -\dfrac{dc_i}{dx_1}(0) = < y_0, w_i >_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}, \quad c_i(a) = 0. \end{array}\right. \tag{9}$$

Then, we have that $c \in (H^2(0, a))^m \subset (\mathcal{C}^1(0, a))^m$ and

$$\frac{\partial y^m}{\partial x_1}(x_1, z) = \sum_{i=1}^m \frac{dc_i}{dx_1}(x_1) w_i(z) \in X_0^m. \tag{10}$$

As in Section 2.2 we embed the problem in a family depending on s and h. For every $s \in [0, a[$ and for every $h \in V^m$ we consider the semi discrete approximation of $(\widetilde{\mathcal{P}}_{s,h})$ defined on $\widetilde{\Omega}_s$ with boundary data $\frac{dy^m}{dx_1}(s) = h \in V^m$. Let us define $X_{s,0}^m = \{\varphi \in H^1(s, a; V^m) : \varphi|_{\Gamma_a} = 0\}$ and we denote $\beta^m, \gamma^m \in X_{s,0}^m$ the part of y^m independent of h and depending linearly on h respectively, and $f^m = \sum_{i=1}^m (f, w_i) w_i$. They satisfy

$$\int_s^a\!\!\int_{\mathcal{O}} \nabla \beta^m \nabla \varphi\, dx_1 dz = \int_s^a\!\!\int_{\mathcal{O}} f^m \varphi\, dx_1 dz \quad \forall\, \varphi \in X_{s,0}^m, \tag{11}$$

$$\int_s^a\!\!\int_{\mathcal{O}} \nabla \gamma^m \nabla \varphi\, dx_1 dz = -< h, \varphi(s) >_{H_{00}^{1/2}{}' \times H_{00}^{1/2}} \quad \forall\, \varphi \in X_{s,0}^m. \tag{12}$$

Then we define the finite dimensional operator $P^m(s)$ by: $\gamma^m(s) = P^m(s)h$ and we fix $P^m(a) = 0$. Also we define $r^m(s) = \beta^m(s)$ and $r^m(a) = 0$. Then

$$P^m(s) \in \mathcal{L}(V^m, V^m) \text{ and } r^m(s) \in V^m.$$

Operator $P_m(s)$ has the same properties as $P(s)$ given by Proposition 3.3. Now if we take $h = \frac{dy^m}{dx_1}(s)$, the solution of (8), being unique, satisfies for any $s = x_1 \in [0, a]$

$$y^m(x_1) = P^m(x_1)\frac{dy^m}{dx_1}(x_1) + r^m(x_1). \tag{13}$$

Let us assume that $P^m(x_1)$ and $r^m(x_1)$ are derivable. Then, taking the derivative of (13) and the inner product with w_j in $L^2(\mathcal{O})$, which we denote $(.,.)$, gives

$$\left(\frac{dy^m}{dx_1}, w_j\right) = \left(\frac{dP^m}{dx_1}\frac{dy^m}{dx_1}, w_j\right) + \left(P^m\frac{d^2y^m}{dx_1^2}, w_j\right) + \left(\frac{dr^m}{dx_1}, w_j\right). \tag{14}$$

The second term of the right-hand side can be evaluated using (7) and (9)

$$\begin{aligned} \left(\frac{d^2y^m}{dx_1^2}, P^m w_j\right) &= \left(\sum_{i=1}^m([\lambda_i c_i - (F, w_i)]w_i), P^m w_j\right) \\ &= (y^{m.*}, P^m w_j)_{H_0^1(\mathcal{O})} - (f^m, P^m w_j), \end{aligned}$$

and using (13)

$$\left(P^m\frac{d^2y^m}{dx_1^2}, w_j\right) = -\left(P^m\Delta_z P^m\frac{dy^m}{dx_1} + P^m\Delta_z r^m + P^m f^m, w_j\right),$$

which we substitute in (14). By a controllability argument it can easily be shown that when g spans V^m, $\frac{dy^m}{dx_1}(x_1)$ also spans V^m. Now we have derived the Riccati equation for P^m and the equation for r^m

$$\begin{cases} \dfrac{dP^m}{dx_1}(x_1) - P^m(x_1)\Delta_z P^m(x_1) - I = 0, \\ P^m(a) = 0, \end{cases} \tag{15}$$

$$\begin{cases} \dfrac{dr^m}{dx_1}(x_1) - P^m(x_1)\Delta_z r^m(x_1) - P^m(x_1)f^m = 0, \\ r^m(a) = 0. \end{cases} \tag{16}$$

But this derivation is still formal since we have assumed the derivability of P^m and r^m. Then, by the theory of ordinary differential equations, we know that there exists a local solution P^m to (15) in $[a-\delta, a]$, with δ small enough. Further, P^m is $\mathcal{C}^1$ from $[a, a-\delta]$ with values in $\mathcal{L}(V^m)$. From here we deduce (see, for instance, Theorem 2 of [8]), for the case of general functions f, that r^m is solution of (16) in $[a-\delta, a]$ and that $r^m \in H^1(a-\delta, a; V^m)$. Furthermore, for these solutions and $s \in [a-\delta, a]$ (13) is satisfied. To go further, we need estimates on $P^m(s)$ independent of s.

Proposition 3.5 *The operator P^m is bounded uniformly in m, in $L^\infty(0,a;\mathcal{L}(H_{00}^{1/2}(\mathcal{O})', H_{00}^{1/2}(\mathcal{O})))$, in $L^\infty(0,a;\mathcal{L}(H_{00}^{1/2}(\mathcal{O}), H_0^1(\mathcal{O})\cap H^{3/2}(\mathcal{O})))$ and in $L^\infty(0,a;\mathcal{L}(L^2(\mathcal{O}), H_0^1(\mathcal{O}))$. Furthermore r^m is uniformly bounded in $L^\infty(0,a;H_{00}^{1/2}(\mathcal{O}))$.*

Proof. Let $\varphi(z) = \sum_{i=1}^m \varphi_i w_i(z) \in V^m$. We define on V^m the following norms

$$\begin{array}{ll} \| \varphi \|_{m,-1/2}^2 = \sum_{i=1}^m \frac{1}{\sqrt{\lambda_i}}\varphi_i^2, & \| \varphi \|_{m,1/2}^2 = \sum_{i=1}^m \sqrt{\lambda_i}\varphi_i^2, \\ \| \varphi \|_{m,1}^2 = \sum_{i=1}^m \lambda_i\varphi_i^2, & \| \varphi \|_{m,3/2}^2 = \sum_{i=1}^m \lambda_i^{3/2}\varphi_i^2, \end{array}$$

which are equivalent to the norms of $H_{00}^{1/2}(\mathcal{O})'$, $H_{00}^{1/2}(\mathcal{O})$, $H_0^1(\mathcal{O})$ and $H_0^1(\mathcal{O})\cap H^{3/2}(\mathcal{O})$ respectively (see [10]), uniformly in m. Using at the same time the classical proof of trace theorems and of Poincaré inequality we first show that

$$\| \gamma^m|_{\Gamma_s} \|_{m,1/2}^2 \leq 2 \left\| \frac{\partial \gamma^m}{\partial x_1} \right\|_{L^2(\tilde{\Omega}_s)} \|\nabla_z \gamma^m\|_{L^2(\tilde{\Omega}_s)} . \tag{17}$$

This is done by considering system (9) on $x_1 \in [s,a]$ with $f = 0$ and $y_0 = h$. We have

$$\begin{aligned} |c_i(s)|^2 &= -\int_s^a \frac{dc_i^2(x_1)}{dx_1} dx_1 \\ &\leq 2\left(\int_s^a |c_i(x_1)|^2 dx_1\right)^{1/2} \left(\int_s^a \left|\frac{dc_i(x_1)}{dx_1}\right|^2 dx_1\right)^{1/2} . \end{aligned}$$

Multiplying by $\sqrt{\lambda_i}$

$$\sqrt{\lambda_i}|c_i(s)|^2 \leq 2\left(\int_s^a \lambda_i |c_i(x_1)|^2 dx_1\right)^{1/2} \left(\int_s^a \left|\frac{dc_i(x_1)}{dx_1}\right|^2 dx_1\right)^{1/2} ,$$

then summing in i and applying Cauchy-Schwarz inequality gives (17). The usual estimate yields from (12)

$$\int_{\tilde{\Omega}_s} \left(\left|\frac{\partial \gamma^m}{\partial x_1}\right|^2 + |\nabla_z \gamma^m|^2 \right) dx_1 dz = - < h, \gamma^m|_{\Gamma_s} >_{H_{00}^{1/2}{}' \times H_{00}^{1/2}} . \tag{18}$$

Hence with (17), $\| \gamma^m|_{\Gamma_s} \|_{m,1/2} \leq C \| h \|_{m,-1/2}$, which proves the first inequality. Multiplying by $\lambda_i^{3/2}$ instead of $\sqrt{\lambda_i}$ yields

$$\| \gamma^m|_{\Gamma_s} \|_{m,3/2}^2 \leq 2 \left(\int_s^a \left\| \frac{\partial \gamma^m}{\partial x_1} \right\|_{m,1}^2 dx_1\right)^{1/2} \left(\int_s^a \| \nabla_z \gamma^m \|_{m,1}^2 dx_1\right)^{1/2} ,$$

instead of (17). Now considering system (9), multiplying each equation by $\lambda_i c_i$, summing in i and integrating in x_1, then integrating by parts yields

$$\begin{aligned} \int_s^a \left\| \frac{\partial \gamma^m}{\partial x_1} \right\|_{m,1}^2 dx_1 + \int_s^a \| \nabla_z \gamma^m \|_{m,1}^2 dx_1 &= -\sum_{i=1}^m \lambda_i h_i c_i(s) \\ &\leq \| h \|_{m,1/2} \| \gamma^m|_{\Gamma_s} \|_{m,3/2}, \end{aligned}$$

hence $\| \gamma^m|_{\Gamma_s} \|_{m,3/2} \leq C \| h \|_{m,1/2}$. The result in $H_0^1(\mathcal{O})$ is obtained by interpolation. The proof for r^m is similar using estimates on β^m. □

Then, by Proposition 3.5, we deduce that P^m is a global solution to (15) and $\mathcal{C}^1$ from $[0,a] \to \mathcal{L}(V^m, V^m)$. Therefore, applying again Theorem 2 of [8], we deduce that r^m is a global solution to (16) and $r^m \in H^1(0,a;V^m)$.

3.3 Passing to the limit

We now study the convergence of y^m as m goes to infinity.

Theorem 3.6 *Let $h \in V^{m_0}$ for a fixed m_0. As $m \to \infty$ we have $y^m \to y$ in X_0, and $y^m(x_1) \to y(x_1)$ in $H_{00}^{1/2}(\mathcal{O})$ $\forall x_1 \in [0,a]$, where y^m, y are respectively the solutions of (8),(6) for $g = h$.*

Proof. From (8) we get

$$\int_0^a \int_{\mathcal{O}} |\nabla y^m|^2 dx_1 dz \leq \| F \|_{L^2(\Omega)} \| y^m \|_{L^2(\Omega)} + \| h \|_{H_{00}^{1/2}(\mathcal{O})} \| y^m \|_{X_0} .$$

Therefore y^m is bounded in X_0, and by compactness one can extract a subsequence such that $y^m \to z$ in X_0 weak, and $y^m(x_1) \to z(x_1)$ in $H_{00}^{1/2}(\mathcal{O})$ weak. Using (6) it is easy to show that $z = y$. To prove the strong convergence we compute

$$\begin{aligned}
& \int_0^a \int_{\mathcal{O}} |\nabla (y^m - y)|^2 dx_1 dz = \\
= \quad & - \int_0^a \int_{\mathcal{O}} \nabla y \nabla (y^m - y) dx_1 dz - \int_0^a \int_{\mathcal{O}} \nabla y \nabla y^m dx_1 dz + \\
& + \int_0^a \int_{\mathcal{O}} f y^m dx_1 dz + < h, y^m|_{\Gamma_0} >_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} \\
\to \quad & - \int_0^a \int_{\mathcal{O}} (|\nabla y|^2 + f y) dx_1 dz + < h, y|_{\Gamma_0} >_{H_{00}^{1/2}{}' \times H_{00}^{1/2}} = 0.
\end{aligned}$$

□

Corollary 3.7 *As $m \to \infty$ we have $r^m(s) \to r(s)$ in $H_{00}^{1/2}(\mathcal{O})$ $\forall s \in]0,a[$, and for all $h \in V^{m_0}$ for a fixed m_0, $P^m(s)h \to P(s)h$ strongly in $H_{00}^{1/2}(\mathcal{O})$ and weakly in $H_0^1(\mathcal{O}) \cap H^{3/2}(\mathcal{O})$ for all $s \in]0,a[$.*

Proof. Applying Theorem 3.6 for all $s \in]0,a[$ gives $P^m(s)h + r^m(s) \to P(s)h + r(s)$ in $H_{00}^{1/2}(\mathcal{O})$ strong. Taking $h = 0$ gives the result for r, and then for $P(s)h$ in $H_{00}^{1/2}(\mathcal{O})$. Now by Proposition 3.5 $P^m(s)h$ is bounded in $H_0^1(\mathcal{O}) \cap H^{3/2}(\mathcal{O})$ and by compactness we can extract a subsequence converging weakly. By density it is easy to prove that the limit is $P(s)h$. □

We can now pass to the limit in (15).

Theorem 3.8 *Operator P satisfies the Riccati equation in the following sense: for every $h, \overline{h} \in L^2(\mathcal{O})$,*

$$\frac{d}{dx_1}(P(x_1)h, \overline{h}) + (\nabla_z P(x_1)h, \nabla_z P(x_1)\overline{h}) = (h, \overline{h}) \text{ in } \mathcal{D}'(]0, a[) \tag{19}$$

with the initial condition $P(a) = 0$.
Furthermore $P \in L^\infty(0, a; \mathcal{L}(L^2(\mathcal{O}), H_0^1(\mathcal{O})))$ and is weakly continuous in x_1, that is $(P(x_1)h, \overline{h})$ is continuous on $[0, a]$ $\forall$ h, $\overline{h} \in L^2(\mathcal{O})$.

Proof. Let $h, \overline{h} \in V^{m_0}$. From (15) we have

$$\left(\frac{dP^m}{dx_1}(x_1)h, \overline{h}\right) + (\nabla_z P^m(x_1)h, \nabla_z P^m(x_1)\overline{h}) = (h, \overline{h}),$$

and $P^m(0) = 0$. Let $\varphi \in \mathcal{C}^1([0, a])$ such that $\varphi(a) = 0$. We get

$$\begin{aligned} &-\int_0^a (P^m(x_1)h, \overline{h})\varphi(x_1)' dx_1 \\ &+ \int_0^a (\nabla_z P^m(x_1)h, \nabla_z P^m(x_1)\overline{h})\varphi(x_1) dx_1 = \int_0^a (h, \overline{h})\varphi(x_1) dx_1. \end{aligned} \tag{20}$$

The integrand of the first term converges to $(P(x_1)h, \overline{h})\varphi(x_1)'$ by Corollary 3.7, and is bounded by Proposition 3.5. Similarly we have the convergence of the integrand of the second term because, for example, $\nabla_z P^m(x_1)h$ converges in $H_{00}^{1/2}(\mathcal{O})'$ strong and $\nabla_z P^m(x_1)\overline{h}$ in $H_{00}^{1/2}(\mathcal{O})$ weak. It is also bounded by Proposition 3.5. Then by Lebesgue's theorem, we can pass to the limit in (20). For $\varphi \in \mathcal{D}(]0, a[)$ it yields (19) for $h, \overline{h}$ in V^{m_0}. The result for $h, \overline{h}$ in $L^2(\mathcal{O})$ is obtained by density as $m_0 \to \infty$. Then from (19), $\frac{d}{dx_1}(P(x_1)h, \overline{h})$ belongs to $L^\infty(0, a)$ and so $(P(x_1)h, \overline{h})$ is continuous in x_1, and for $\varphi \in \mathcal{C}^1([0, a])$ we can integrate (20) by part to recover the initial condition $P(0) = 0$. $\square$

We now turn to the convergence of r^m.

Theorem 3.9 *Function r belongs to $L^\infty(0, a; H_{00}^{1/2}(\mathcal{O}))$ and satisfies the following equation*

$$\begin{aligned} &< \frac{dr}{dx_1}(x_1), h >_{H_{00}^{1/2}\,' \times H_{00}^{1/2}} + < \nabla_z r(x_1), \nabla_z P(x_1)h >_{H_{00}^{1/2}\,' \times H_{00}^{1/2}} \\ &\qquad = (F, P(x_1)h) \quad \forall h \in H_{00}^{1/2}(\mathcal{O}), \end{aligned} \tag{21}$$

in $\mathcal{D}'(]0, a[)$, with initial condition $r(a) = 0$.

Proof. For $h \in V^{m_0}$, we have from (16)

$$(\frac{dr^m}{dx_1}(x_1), h) + (\nabla_z r^m(x_1), \nabla_z P^m(x_1)h) = (F, P^m(x_1)h),$$

with $r^m(a) = 0$. Let $\varphi \in \mathcal{C}^1([0,a])$ such that $\varphi(a) = 0$. We get

$$-\int_0^a (r^m(x_1), h)\varphi(x_1)' dx_1 + \int_0^a (\nabla_z r^m(x_1), \nabla_z P^m(x_1)h)\varphi(x_1)\, dx_1$$
$$= \int_0^a (F, P^m(x_1)h)\varphi(x_1)\, dx_1.$$

¿From Corollary 3.7, we have the convergence as in the proof of Theorem 3.8. Let us check the second term: $\nabla_z P^m(x_1)h$ converges in $H_{00}^{1/2}(\mathcal{O})$ weak, and $\nabla_z r^m(x_1)$ in $H_{00}^{1/2}(\mathcal{O})'$ strong. It yields (21) for $h \in V^{m_0}$ and $\varphi \in \mathcal{D}(]0,a[)$. By density, it is true for $h \in H_{00}^{1/2}(\mathcal{O})$, as for the second term r belonging to $H_{00}^{1/2}(\mathcal{O})$, h must also be in that space to guarantee the continuity of the bilinear form. Now from (21) $\frac{dr}{dx_1}(x_1)$ belongs to $L^2(0,a;H_{00}^{1/2}(\mathcal{O})')$, and so we can recover the initial condition $r(a) = 0$. □

Then y is solution of equation (3), with $\frac{dy}{dx_1}$ belonging to $\mathcal{C}(0,a;H_{00}^{1/2}(\mathcal{O})')$ and y to $\mathcal{C}([0,a];H_{00}^{1/2}(\mathcal{O}))$.

Remark 3.10 *It is possible to prove that the solution P of (19) is unique among the class satisfying Proposition 3.3. For that purpose one can show that from any solution of (19) one can construct r and y solution of (21) and (3) respectively, and y satisfies also (6) and is unique. The proof needs results on well posedness of (21) and (3) and will be presented elsewhere.*

4 Optimal control problem associated to the boundary value problem.

In this section we show the relation with Riccati equations appearing in optimal control theory (see for instance [9]). In fact we show that problem $(\mathcal{P}_0)$ can be formulated as an optimal control problem. We use the operator Q and the function w defined in Section 2.1, with $y_0 = 0$ (for the sake of simplicity). Let us consider the control space $\mathcal{U} = L^2(\Omega)$. For every $v \in \mathcal{U}$ the state $y(v) \in H^1(0,a;L^2(\mathcal{O}))$ is solution of

$$\begin{cases} \dfrac{\partial y}{\partial x_1} = v & \text{in } \Omega, \\ y(a) = y_1. \end{cases} \tag{22}$$

We also denote $\mathcal{U}_{ad} = \{v \in \mathcal{U} : \ y(v) \in X_{y_1}\}$ the space of admissible controls, where

$$X_{y_1} = \{h \in L^2(0,a;H_0^1(\mathcal{O})) \cap H^1(0,a;L^2(\mathcal{O})) : \ h(a) = y_1\}.$$

The desired state y_d is given almost everywhere in x_1 by the solution of the family of (n-1) dimensional problems

$$\begin{cases} -\Delta_z y_d(x_1) = f(x_1) & \text{in } \mathcal{O} \\ y_d|_{\partial\mathcal{O}} = 0. \end{cases} \tag{23}$$

Then y_d belongs to $L^2(0,a;H_0^1(\mathcal{O}))$. Now we look for $u \in \mathcal{U}_{ad}$ such that $J(u) = \inf_{v\in\mathcal{U}_{ad}} J(v)$, where, for every $v \in \mathcal{U}_{ad}$,

$$J(v) = \int_0^a \|\nabla_z y(v) - \nabla_z y_d\|^2_{L^2(\mathcal{O})}\, dx_1 + \int_0^a\!\!\int_{\mathcal{O}} v^2 dx_1 dz. \tag{24}$$

At this point we have the problem that $\mathcal{U}_{ad}$ is not a closed subset of $L^2(\Omega)$ and therefore we cannot use directly the classic techniques (see, for instance, [9]) in order to solve this problem. Nevertheless, since $\mathcal{U}_{ad} = \{\frac{\partial h}{\partial x_1} \;:\; h \in X_{y_1}\}$, $J(u) = \inf_{v\in\mathcal{U}_{ad}} J(v) = \inf_{h\in X_{y_1}} \overline{J}(h) = \overline{J}(y)$, where $\frac{\partial y}{\partial x_1} = u$ and

$$\overline{J}(h) = \int_0^a \|\nabla_z h - \nabla_z y_d\|^2_{L^2(\mathcal{O})}\, dx_1 + \int_0^a\!\!\int_{\mathcal{O}} \left|\frac{\partial h}{\partial x_1}\right|^2 dx_1 dz. \tag{25}$$

Now, X_{y_1} is a closed convex set in the Hilbert space $L^2(0,a;H_0^1(\mathcal{O}))\cap H^1(0,a;L^2(\mathcal{O}))$ and $\overline{J}(h)^{1/2}$ is a norm of that space. Then (see Theorem 1.3 of chapter I of [9]) there exists a unique $y \in X_{y_1}$ satisfying $\overline{J}(y) = \inf_{h\in X_{y_1}} \overline{J}(h)$, which is uniquely determined by

$$\overline{J}'(y)(h) = 0 \quad \forall\, h \in X_0. \tag{26}$$

Let us show that y is solution of $(\mathcal{P}_0)$. Developping (25), one gets

$$\overline{J}(y) = \int_\Omega |\nabla y|^2\, dx - 2\int_\Omega \nabla_z y\, \nabla_z y_d\, dx + \int_\Omega |\nabla_z y_d|^2\, dx.$$

But from (23), y_d satisfies almost everywhere in x_1

$$\int_{\mathcal{O}} \nabla_z y_d(x_1)\nabla_z y(x_1)\, dz = \int_{\mathcal{O}} f(x_1)y(x_1)\, dz,$$

Then

$$\overline{J}(y) = \int_\Omega |\nabla y|^2\, dx - 2\int_\Omega f y dx + \int_\Omega |\nabla_z y_d|^2\, dx.$$

Now it is clear that $\overline{J}(y)$ is the energy functional associated to $(\mathcal{P}_0)$ up to a constant term. We introduce the adjoint state p by

$$\begin{cases} \dfrac{\partial p}{\partial x_1} = -\Delta_z y - f \quad \text{in } \Omega, \\ p(0) = 0. \end{cases}$$

Then, since $-\Delta_z y - f \in L^2(0,a;H^{-1}(\mathcal{O}))$, we know (see Theorem 1.2 of chapter III of [9]) that $p \in H^1(0,a;H^{-1}(\mathcal{O}))$. Furthermore, since $y \in Y$, we also deduce that $\frac{\partial p}{\partial x_1} \in H^{-1}(0,a;L^2(\mathcal{O}))$ and therefore, $p \in L^2(\Omega)$. Now for every $h \in X_0$, we know that

$$\begin{aligned}\int_0^a < -\Delta_z y - f, h >_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})}\, dx_1 &= \int_0^a < \frac{\partial p}{\partial x_1}, h >_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})}\, dx_1 \\ &= -\int_0^a\!\!\int_{\mathcal{O}} p\frac{\partial h}{\partial x_1} dx_1 dz.\end{aligned}$$

Therefore, from optimality condition (26) we deduce that

$$\int_0^a < -\Delta_z y - f, h >_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})} dx_1 \quad + \quad \int_0^a\int_{\mathcal{O}} \frac{\partial y}{\partial x_1}\frac{\partial h}{\partial x_1} dx_1 dz =$$
$$\int_0^a\int_{\mathcal{O}} (-p + \frac{\partial y}{\partial x_1})\, u\, dx_1 dz \quad = \quad 0, \qquad \forall\, u \in \mathcal{U}_{ad}. \tag{27}$$

Then we have obtained the optimality system

$$\begin{cases} -\dfrac{\partial y}{\partial x_1} = -p, & y(a) = y_1, \\ \dfrac{\partial p}{\partial x_1} = -\Delta_z y - f, & p(0) = 0, \end{cases}$$

which has the same associated Riccati equation (see Section 4 of chapter III of [9]) that the system of equations for Q and w of Section 2.1.

5 Factorization of the discretized problem.

In this section we consider a finite difference discretization of problem $(\mathcal{P}_0)$. For the sake of simplicity, we present only the 2D case, i.e. Ω is the rectangle $]0, a[\times]0, b[$, but the same method could be applied in higher dimension. We show that the factorization method, suitably applied to the resulting linear system, leads to a discretized version of system (2). Furthermore this factorization turns out to be the Gauss LU block factorization of the block tridiagonal matrix representing the discretized Laplace operator on the rectangle. This allows to interpret the factorizations (2) or (5) as infinite dimensional versions of the block Gauss factorization.

5.1 Finite difference discretization of problem $(\mathcal{P}_0)$.

We consider f, y_0 and y_1 regular enough in order to obtain a solution $y \in \mathcal{C}^3(\Omega)$ and convergence of the discretization towards the result. Given an integer $N > 0$ we consider the following lattice, taking the same step $h = a/(N - \frac{1}{2})$ for both coordinates and assuming that $ph = b$

$$\begin{cases} a_{i,j} = ((-1/2 + i)h, jh) \text{ for } i \in \{0, ..., N\} \text{ and } j \in \{0, ..., p\}, \\ a_{1/2,j} = (0, jh) \text{ for } j \in \{0, ..., p\}. \end{cases}$$

We compute an approximation $y_{i,j}$ of $y(a_{i,j})$ by the usual five points scheme

$$\frac{1}{h^2}\left(4y_{i,j} - y_{i-1,j} - y_{i+1,j} - y_{i,j-1} - y_{i,j+1}\right) = f(a_{i,j}) = f_{i,j}.$$

Let $y_h, f_h \in I\!R^{(N-1)(p-1)}$ be the vector of the unknowns $y_{i,j}$, and of the right hand side $f_{i,j}$ respectively, when numbering for increasing j first. Let $y^i, f^i \in I\!R^{(p-1)}$

be the components of y_h and f_h respectively, corresponding to the nodes $a_{i,j}$ for $j \in \{1, \ldots, p-1\}$. Let $\nabla^2_{h,2}$ be the finite difference approximation to the second order derivative operator with respect to x_2 with Dirichlet boundary conditions. Then, taking into account the boundary conditions at $i = 1/2$ and $i = N$, if we put $B_1 = I - h^2\nabla^2_{h,2}$ and $B_i = 2I - h^2\nabla^2_{h,2}$ for $i = 2, ..., N-1$, we can define A_h and F_h by

$$A_h = \frac{1}{h^2}\begin{pmatrix} B_1 & -I & & & \\ -I & B_2 & -I & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -I & B_{N-2} & -I \\ & & & -I & B_{N-1} \end{pmatrix} \in \mathcal{M}_{(N-1)(p-1)\times(N-1)(p-1)},$$

$$F_h = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} f^1 + \frac{y_0(a_{1/2})}{h} \\ f^2 \\ \vdots \\ f^{N-1} + \frac{y_1(a_N)}{h^2} \end{pmatrix} \in I\!R^{(N-1)(p-1)},$$

then y_h satisfies the finite difference discretization $(\mathcal{P}_h)$ of problem $(\mathcal{P}_0)$

$$(\mathcal{P}_h) \qquad A_h y_h = F_h.$$

Here we have stressed the block tridiagonal structure of A_h.

5.2 The factorization method applied to $(\mathcal{P}_h)$.

We define

$$\xi^{i-1/2} = \frac{y^i - y^{i-1}}{h} \in I\!R^{p-1} \quad \forall\ i = 2, ..., N,$$

and $\xi^{1/2} = -y_0(a_{1/2})$. In a way similar to that followed in Section 2.1, for every $i_0 \in \{2, ..., N\}$ we define the problem $(\mathcal{P}^{i_0}_{h,g_h})$ on the domain $1 \le i \le i_0$ with the Dirichlet data $g_h \in I\!R^{p-1}$ on $i = i_0$ by

$$(\mathcal{P}^{i_0}_{h,g_h}) \qquad A_h^{(i_0)} y_h^{(i_0)}(g_h) = F_h^{(i_0)}(g_h),$$

where $A_h^{(i_0)} \in \mathcal{M}_{(i_0-1)(p-1)\times(i_0-1)(p-1)}$ is the principal submatrix extracted from A_h and $F_h^{(i_0)}(g_h)$ is defined by

$$F_h^{(i_0)}(g_h) = \begin{pmatrix} f^1 + \frac{y_0(a_{1/2})}{h} \\ f^2 \\ \vdots \\ f^{i_0-1} + \frac{g_h}{h^2} \end{pmatrix} \in I\!R^{(i_0-1)(p-1)}.$$

Then we can decompose $y_h^{(i_0)}(g_h) = \gamma^{(i_0)}(g_h) + \beta^{(i_0)}$ into its part linear in g_h and independent of g_h respectively. We define the operator $Q_{i_0} \in \mathcal{L}(I\!R^{p-1}, I\!R^{p-1})$ by

$$Q_{i_0} g_h = \frac{\gamma^{i_0} - \gamma^{i_0-1}}{h} = \frac{g_h - \gamma^{i_0-1}(g_h)}{h} \quad \forall g_h \in I\!R^{p-1}, \tag{28}$$

where γ^i is the (i)-component of the vector $\gamma^{(i_0)}(g_h)$, and the vector $w^{i_0-1/2} \in I\!R^{p-1}$ is defined by $w^{i_0-1/2} = \frac{-\beta^{i_0-1}}{h}$. Then taking $g_h = y^i$ one has a discrete counter part to (1)

$$\xi^{i-1/2} = Q_i y^i + w^{i-1/2} \quad \forall i \in \{2, ..., N\}. \tag{29}$$

Now, since y_h is solution of $(\mathcal{P}_h)$, using (29), we have that

$$\begin{aligned} -\nabla_{h,2}^2 y^i - f^i &= \frac{y^{i-1} - 2y^i + y^{i+1}}{h^2} \\ &= Q_{i+1}\frac{(y^{i+1} - y^i)}{h} + \frac{(Q_{i+1} - Q_i)}{h} y^i + \\ &\quad + \frac{(w^{i+1/2} - w^{i-1/2})}{h}. \end{aligned} \tag{30}$$

Further, from (29), we obtain

$$(I - hQ_{i+1})(y^{i+1} - y^i) = hQ_{i+1}y^i + hw^{i+1/2}.$$

Now, for every $i \in \{2, ..., N\}$, it is very easy to check that $det(I - hQ_i) \neq 0$ and therefore the matrix $(I - hQ_i)^{-1}$ is defined. Thus,

$$\begin{aligned} -\nabla_{h,2}^2 y^i - f^i = \; & Q_{i+1}(I - hQ_{i+1})^{-1}(Q_{i+1}y^i + w^{i+1/2}) \\ & + \frac{(Q_{i+1} - Q_i)}{h} y^i + \frac{(w^{i+1/2} - w^{i-1/2})}{h}, \end{aligned} \tag{31}$$

for $i \in \{2, ..., N-1\}$ with y^i "arbitrary". Then, extending formula (29) by $Q_1 = 0, w^{1/2} = -y_0(a_{1/2})$ at $i = 1$ we obtain from the independent terms that for $i \in \{1, ..., N-1\}$

$$\begin{cases} \dfrac{w^{i+1/2} - w^{i-1/2}}{h} = -Q_{i+1}(I - hQ_{i+1})^{-1}w^{i+1/2} - f^i, \\ w^{1/2} = -y_0(a_{1/2}), \end{cases} \tag{32}$$

which can be rewritten as $w^{i-1/2} - L_i w^{i+1/2} = hf^i$, for every $i \in \{1, ..., N-1\}$, if we put $L_i = I + hQ_{i+1}(I - hQ_{i+1})^{-1} = (I - hQ_{i+1})^{-1}$. Further, from the terms depending on y_i in (31) we obtain the discrete Riccati equation for $i \in \{1, ..., N-1\}$

$$\begin{cases} -\dfrac{Q_{i+1} - Q_i}{h} = Q_{i+1}(I - hQ_{i+1})^{-1}Q_{i+1} + \nabla_{h,2}^2, \\ Q_1 = 0. \end{cases} \tag{33}$$

Now, equation (29) is equivalent to

$$\begin{cases} \dfrac{y^{i+1}-y^i}{h} = Q_{i+1}y^{i+1} + w^{i+1/2} \quad \forall i \in \{1,...,N-1\}, \\ y^N = y_1(a_N), \end{cases} \tag{34}$$

which can be rewritten as $-y^i + U_{i+1}y^{i+1} = hw^{i+1/2}$ for every $i \in \{1,...,N-1\}$, if we set $U_i = I - hQ_i = L_{i-1}^{-1}$ for every $i \in \{2,...,N\}$. Then, if we write these equations of y_h and w in matrix form we obtain

$$\frac{1}{h}\begin{pmatrix} L_1 & & & 0 \\ -I & L_2 & & \\ 0 & \ddots & \ddots & \\ & & -I & L_{N-1} \end{pmatrix} \frac{1}{h}\begin{pmatrix} I & -U_2 & & \\ & \ddots & \ddots & \\ & & I & -U_{N-1} \\ 0 & & & I \end{pmatrix} y_h = F_h. \tag{35}$$

In the linear system of equation (35) the first matrix is block lower triangular and the second one is block upper triangular with diagonal unity so it is the well known LU block factorization of the block tridiagonal matrix A_h, due to the uniqueness of this factorization. One can check this fact directly by expanding the product and using the Riccati equation (33). Reciprocally, if one considers the block LU factorization (35) of A_h the upper blocks U_i can be interpreted as $I - hQ_i$ where Q_i is the discrete DtN operator defined by (28).

Remark 5.1 *The equation for Q_i looks like a discretization of the first equation of system (2) up to the modification $O(h)$ due to the term $(I - hQ_{i+1})^{-1}$. Such terms are classical for discrete time Riccati equations (see for example [3]). Using different definitions for the operator Q_i (for example downstream or centered finite difference), leads to other factorizations of A_h (the block LU factoriaztion with blocks I in the diagonal of the lower triangular part or the block Cholesky factorization respectively).*

6 Some advantages of the factorized form

As shown in section 3 the factorized form (2) (or (5)) is equivalent to the original form of problem $(\mathcal{P}_0)$. In this section we present some situations where the factorization gives some advantages.

¿From section 5 the factorization can be viewed as a continuous counterpart to the Gauss block factorization of the discrete problem, it inherits the well known properties of the Gauss method : the boundary value problem $(\mathcal{P}_0)$ is transformed in uncoupled first order initial value problems for functions and operators. Furthermore if $(\mathcal{P}_0)$ has to be solved for various data $f_i, y_{0,i}, y_{1,i}$ for $i = 1, \ldots, K$, the Riccati equation for Q in (2) (or for P in (5)) has to be solved only once. The corresponding solutions y_i to $(\mathcal{P}_0)$ are then given by the uncoupled system

$$\begin{cases} \dfrac{dw_i}{dx_1} + Qw_i = -f_i, & w_i(0) = -y_{0,i}, \\ -\dfrac{dy_i}{dx_1} + Qy_i = -w_i, & y_i(a) = y_{1,i}. \end{cases} \tag{36}$$

If only the Dirichlet data y_1 is changed, it is sufficient to compute the solution of the equation for y_i accordingly. In particular this is interesting when one wants to solve a control problem whose state satisfies a problem of the type $(\mathcal{P}_0)$ by a minimization algorithm. In [7], for such a problem, it was shown how the factorization of the state equation and of the optimality system can be done at the same time.

Another interest of the method is that it allows to exhibit the Riccati equation satisfied by the DtN operator Q in (2) (or NtD P in (5)). These operators are of great interest when the boundary interaction is the main concern. For example [14] introduce the Steklov-Poincaré operator S to study the domain decomposition method. Given the value of y on an internal boundary, it provides the sum of normal derivatives of the solutions in the adjacent subdomains. It can be easily expressed on Γ_s as

$$S_s y = Q(s)y - P^{-1}(s)y,$$

and the Steklov-Poincaré equation for the matching of Ω_s and $\widetilde{\Omega}_s$ becomes

$$(P(s)Q(s) - I)y = -P(s)w(s) - r(s).$$

This equation is the basis of several domain decomposition methods presented in [14]. Its data can be obtained from (2) and (5), which also allows an easy transformation of this equation if one wants to move the internal boundary Γ_s.

The operators P and Q can also be used to compute transparent boundary conditions in subdomains. For example consider the problem $(\mathcal{P}_0)$ which has to be solved many times for various f differing only in $\widetilde{\Omega}_s$. Then it is sufficient to solve the following problem in $\widetilde{\Omega}_s$:

$$\begin{cases} -\Delta y = f \qquad \text{in } \widetilde{\Omega}_s, \\ y|_\Sigma = 0, \\ -\dfrac{\partial y}{\partial x_1}|_{\Gamma_s} = -Q(s)y|_{\Gamma_s} - w(s), \quad y|_{\Gamma_a} = y_1. \end{cases}$$

for the various f, where Q and w have been computed once from 0 to s by (2). This problem with a new non-local condition on Γ_s can be shown to be well posed and its solution is the restriction of the solution of $(\mathcal{P}_0)$ to $\widetilde{\Omega}_s$. In the same way $Q_\infty = \lim_{x_1 \to \infty} Q(x_1)$ can be used to define transparent condition on semi-infinite cylinders.

¿From the numerical methods viewpoint, new methods can be derived from a direct discretization of the factorized forms (2) or (5). For example classical methods for stepsize adaption of ordinary differential equations applied to these forms will lead to automatic gridsize adaption in the x_1 direction within the integration process.

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