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Identification of a Heat Transfer Coefficient Depending on Pressure and Temperature

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Abstract

This paper deals with an inverse problem concerning the identification of the heat exchange coefficient H (assumed depending on the temperature and/or pressure) between a certain material with the external environment, when only experimental measurements of the temperature are supposed to be known. The main difficulties are that we consider the case of functions H depending on the solution of the state equation and we use experimental data that may have errors. We develop rigorous mathematical strategies for this identification. We consider separately pressure and temperature dependence and, in both cases, we set several scenarios for the inverse problem. For each scenario, we know the initial and ambient temperatures, we identify function H through different methods and we obtain error bounds in adequate norms (uniform and square integrable). Finally, we perform numerical tests in order to compare the results obtained with these algorithms and with some classical regularization algorithms.

1 Introduction

In this work, we focus our attention on an inverse problem concerning the identification of the heat transfer coefficient H (assuming it depends on pressure or temperature) between a certain material with the external environment. Some practical applications in which this coefficient appears can be seen in [7], [10], [11] and [16].

The physical problem modeled in the references mentioned above is the evolution of the temperature in a homogeneous sample of a material placed on a high pressure equipment which is, also, able to warm or cool it. To describe the temperature distribution within the sample complex models based on partial differential equations are often used (see, for example, [11]). These equations involve functions and parameters that must be known before computing the solution. These functions and parameters are determined usually either by experimentation based protocols ([11]) or by solving inverse problems posed in an appropriate mathematical framework (see, e.g., [3], [4] and [6]). Other works regarding numerical approaches for inverse problems can be seen in [8] and [9].

In some contexts, and under certain conditions, it can be assumed that H has a known expression (e.g., H is a function with a few real parameters to identify). In these cases, the least squares method may provide a good tool for identifying those parameters (see, for instance, [7]). However, when the goal is to identify a function, the problem becomes more complicated, especially if the function depends on the solution of the state and the experimental data can be given with measurement errors. The challenge that we face in this work is to identify function H when continuity and positivity are the only information available about H.

For simplicity, let us consider an homogeneous sample and let us assume that the temperature gradient inside it is negligible. The Newton Cooling Law and the relation describing the change in temperature due to the pressure variation, when isentropic changes of temperature are assumed (see [16]), provide a simple mathematical model for this phenomenon through the following initial value problem (*direct problem*):

$$\begin{cases} T'(t) = H(T(t), P(t))(T^e - T(t)) + \alpha P'(t)T(t), \ t \geq t_0 \\ T(t_0) = T_0. \end{cases} \tag{1}$$

Here $T(t)$ (K) is the temperature of the sample at time t; $P(t)$ is the pressure of the equipment at time t; T^e is the temperature of the external environment; T_0 is the temperature at the initial time t_0; $\alpha \geq 0$ is the thermal diffusivity (involving thermal expansion, density and specific heat capacity); and H is the pressure/temperature dependent heat transfer coefficient. In order to solve problem (1), constants T_0, $T^e \in \mathbb{R}$, pressure curve P and function $H : [T_a, T_b] \times [P_{\min}, P_{\max}] \to \mathbb{R}$ are needed ($[T_a, T_b]$ and $[P_{\min}, P_{\max}]$ are suitable ranges for temperature and pressure, respectively).

The values of T_0 and T^e can be obtained by measuring devices (*thermocouples*), the coefficient α is assumed to be known and the pressure is provided by the equipment. However, function H cannot be obtained easily. We design strategies to identify function H, from experimental measurements (*inverse problem*); by doing that, we are able to approximate the solution of model (1) for other data T_0, T^e and P (provided they are kept in the initial ranges of temperature and pressure $[T_a, T_b]$ and $[P_{\min}, P_{\max}]$, respectively) without requiring new measurements.

There are two main difficulties:

- Function H may depend on the solution of the state equation T.
- Temperature and pressure measurements can be given with errors, due to measurement equipment accuracy limitations. This can be a serious drawback because of the ill–possedness of the problem (see [13]).

We note that the value of H is not relevant when T is close to T^e. So, we set a *threshold* μ to separate it from T^e ($H(T, P)$ is not identified for values of T too close to T^e).

The identification of heat transfer coefficients has already been considered in previous works. An experimental procedure was proposed in [20] based on a genetic algorithm for determining a heat transfer coefficient. In [1] and [14] some methods based on inverse analysis are developed in order to identify the heat transfer on a machine tool surface. A method for the determination of the heat transfer coefficient was proposed in [17] for the first falling drying period of potato cubes where heat and mass transfer were considered as coupled phenomena. In [5] an identification problem for the heat transfer coefficient in foods during freezing using cooling curves obtained from an industrial survey is solved. The coefficient to be identified is supposed to be constant in all the works cited in this paragraph, that do not use regularizing algorithms able to compensate the sensitivity of the identification process to experimental measurement errors, as it is done in this work.

The paper is organized as follows:

In Section 2 we consider the simplest case, in which the coefficient H depends only on the (known) pressure. We design an *ad hoc* experiment for the determination of this coefficient. If this experiment cannot be performed, we propose a numerical algorithm that allows to approximate function H (sometimes with a greater accuracy than the previous method). Finally we present some numerical test comparing both methods.

Section 3 deals with the case when H only depends on temperature. Then, we can assume that the pressure remains constant, and so the second term on the right hand side of the equation of problem (1) vanishes. As in Section 2, we also propose a numerical algorithm that allows to approximate function H and we present some numerical test comparing the results of this iterative method with those obtained by the classical regularization theory.

2 Pressure Dependent Coefficient

This section is devoted to identify the heat transfer H of problem (1) when only pressure dependence is assumed. Assuming that pressure increases from initial time t_0 until instant $t_{\rm f}$, problem (1) becomes

$$\begin{cases} T'(t) = H(P(t))(T^{\rm e} - T(t)) + \alpha P'(t)T(t),\ t \in [t_0, t_{\rm f}] \\ T(t_0) = T_0. \end{cases} \tag{2}$$

In order to define a suitable framework to carry out this identification, we suppose that:

- The external temperature T^e (K) is a constant.
- The initial temperature T_0 (K) is higher than T^e.
- P is a known, non–decreasing, continuous and piecewise $\mathcal{C}^1$ function on the temporal interval $[t_0, t_{\rm f}]$.
- H is a positive and continuous function on the pressure range $[P_0, P_{\rm f}] = [P(t_0), P(t_{\rm f})]$.

The temperature measurements are assumed to have been taken during an experiment in which the entire range of pressures has been covered. In practice, a linear pressure can be used.

The following result collect the main qualitative properties of the solution of direct problem (2):

Proposition 2.1 *Under assumptions above, the problem* (2) *has a unique solution* T*. This solution is the continuous and piecewise* $\mathcal{C}^1$ *function*

$$\begin{aligned} T(t) = {} & (T_0 - T^e)e^{\alpha(P(t)-P_0)}e^{-\int_{t_0}^t H(P(s))\,ds} \\ & + T^e\left(1 + \int_{t_0}^t \alpha P'(s)e^{\alpha(P(t)-P(s))}e^{-\int_s^t H(P(r))\,dr}\,ds\right). \end{aligned}$$

Moreover, denoting

$$H_m = \min_{s\in[P_0,P_f]} H(s),\ H_M = \max_{s\in[P_0,P_f]} H(s),$$

inequalities

$$\begin{aligned} & (T_0 - T^e)e^{\alpha(P(t)-P_0)}e^{-H_M(t-t_0)} + T^e\left(1 + \int_{t_0}^t \alpha P'(s)e^{\alpha(P(t)-P(s))}e^{-H_M(t-s)}ds\right) \\ & \le T(t) \\ & \le (T_0 - T^e)e^{\alpha(P(t)-P_0)}e^{-H_m(t-t_0)} + T^e\left(1 + \int_{t_0}^t \alpha P'(s)e^{\alpha(P(t)-P(s))}e^{-H_m(t-s)}ds\right), \end{aligned}$$

hold for all $t \in [t_0, t_{\rm f}]$*. In particular,*

$$(T_0 - T^e)\,e^{-H_M(t_{\rm f}-t_0)} + T^e \le T(t) \le T_0e^{\alpha(P_{\rm f}-P_0)},\ t \in [t_0, t_{\rm f}]. \quad \square \tag{3}$$

2.1 Scenarios of the inverse problems

Depending on the knowledge one has on the solution T in $[t_0, t_{\rm f}]$, we consider the inverse problem immersed in various scenarios:

- In the best case (and, in practice, unrealistic) that function T is known throughout the interval $[t_0, t_{\rm f}]$ (and also its derivative) under the assumption that $H \in \mathcal{C}([P_0, P_{\rm f}])$ is positive, identification of H is given by the equality

$$H(P(t)) = \frac{T'(t) - \alpha P'(t)T(t)}{T^e - T(t)},$$

due to the growth of P. Where P is strictly increasing, P is invertible and we can write

$$H(s) = \frac{T'(P^{-1}(s)) - \alpha P'(P^{-1}(s))T(P^{-1}(s))}{T^e - T(P^{-1}(s))}.$$

If the pressure takes the constant value P_c in an interval, the quotient

$$\frac{T'(t) - \alpha P'(t)T(t)}{T^e - T(t)} = \frac{T'(t)}{T^e - T(t)}$$

is a constant function in this interval, so that we can take there any instant t^* to determine (uniquely) the value

$$H(P_\mathrm{c}) = \frac{T'(t^*)}{T^e - T(t^*)}.$$

- In a second scenario we assume that T can be evaluated exactly in a finite number of arbitrary instants in $[t_0, t_\mathrm{f}]$. The identification of H in $[P_0, P_\mathrm{f}]$ can be treated as an approximate derivation standard problem (in particular for the evaluation of T', which is unknown, unlike what happened in the previous scenario).

- The following scenario arises when a function $\widetilde{T}$ that represents the approximate value of the temperature at any instant of time is supposed to be known.

- However, the usual situation is that only a discrete amount of values $\widehat{T}_k$ approximating the values of T at the corresponding instants is known.

For the last three scenarios, we will develop a "stable" method to approximate T' from the data and thereby obtain a discrete number of approximate values of H in points of interval $[P_0, P_\mathrm{f}]$. To this end, we must ensure that the temperature values are sufficiently far from T^e, otherwise the coefficient H have a negligible influence in the equation, and its identification cannot be performed. For the latter two cases we will set a certain "threshold" as follows:

a) In the third scenario, we suppose that $\widetilde{T}$ satisfies

$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([t_0,t_\mathrm{f}])} < \delta.$$

We consider the threshold $\mu = \widetilde{m} - T^e$, where

$$\widetilde{m} = \min_{t\in[t_0,t_\mathrm{f}]} \widetilde{T}(t). \tag{4}$$

We always assume that $\mu > \delta$.

b) In the fourth scenario, we assume available a set of measurements $\widehat{T}_k$ such that $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, with $\widehat{\delta} > 0$, where $\{\tau_0 = t_0, \tau_1, \tau_2, \ldots, \tau_p = t_\mathrm{f}\}$ is a sequence of instants. We will denote by $\widetilde{T}$ a function that interpolates the values $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ in points $\{\tau_0, \tau_1, \ldots, \tau_p\}$ and consider $\delta > 0$, a bound of the norm of the difference between T and $\widetilde{T}$ in the interval $[t_0, t_\mathrm{f}]$, i.e.,

$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([t_0,t_\mathrm{f}])} < \delta.$$

The threshold μ is defined from $\widetilde{T}$ as in the previous scenario.

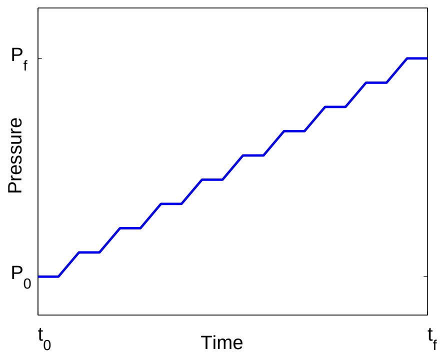

Figure 1: *Ad hoc* experiment.

2.2 *Ad hoc* experiment

Let us suppose we are in the fourth (more general) scenario and we can perform an *ad hoc* experiment, which is designed as follows: in order to identify function H, we assume we know measurements of temperature in an even number of instants $\{t_k\}_{k=0}^n$, which form an equally spaced partition of $[t_0, t_\mathrm{f}]$ with step h. We choose the pressure applied by the equipment as a continuous function that increases linearly with the same slope in the intervals

$$[t_{2k-1}, t_{2k}],\ k = 1, 2, \dots, \frac{n-1}{2}$$

and remains constant in the rest of the intervals

$$[t_{2k}, t_{2k+1}],\ k = 0, 1, \dots, \frac{n-1}{2}$$

(see Figure 1). Thus $P(t_{2k}) = P(t_{2k+1}),\ k = 0, 1, \dots, \frac{n-1}{2}$ and the values $\{P(t_{2k})\}_{k=0}^{\frac{n-1}{2}}$ form a partition of the range of pressures $[P_0, P_\mathrm{f}]$.

Denoting by $\{\widehat{T}_k\}_{k=0}^n$ the temperature measurements, we can find the approaches

$$\widetilde{H}_k \simeq H(P(t_{2k})),\ k = 0, 1, \dots, \frac{n-1}{2}$$

through the following methodology: for each $k \in \left\{0, 1, \dots, \frac{n-1}{2}\right\}$ we consider the interval $[t_{2k}, t_{2k+1}]$. Here, since pressure is constant, the solution of problem (2) verifies

$$T'(t) = H(P(t_{2k}))(T^e - T(t)),\ t \in (t_{2k}, t_{2k+1}).$$

Then, $T(t) = T^e + (T(t_{2k}) - T^e)e^{-H(P(t_{2k}))(t-t_{2k})},\ t \in [t_{2k}, t_{2k+1}]$. In particular, for $t = t_{2k+1}$,

$$H(P(t_{2k})) = \frac{1}{t_{2k+1} - t_{2k}} \ln\left(\frac{T(t_{2k}) - T^e}{T(t_{2k+1}) - T^e}\right) = \frac{1}{h} \ln\left(\frac{T(t_{2k}) - T^e}{T(t_{2k+1}) - T^e}\right). \tag{5}$$

This suggests taking as an approximation of the value of H at $P(t_{2k})$ the value

$$\widetilde{H}_k = \frac{1}{h} \ln\left(\frac{\widehat{T}_{2k} - T^e}{\widehat{T}_{2k+1} - T^e}\right). \tag{6}$$

Denoting by $T_k = T(t_k)$ and $\sigma_k = \dfrac{\widehat{T}_k - T_k}{T_k - T^e}$, the following result holds:

Proposition 2.2 *The error made by approximating the values of H by (6) is*

$$\widetilde{H}_k - H(P(t_{2k})) = \frac{1}{h} \ln\left(\frac{1 + \sigma_{2k}}{1 + \sigma_{2k+1}}\right)$$

for $k = 0, 1, \dots, \dfrac{n-1}{2}$.

DEMOSTRACIÓN. The definition of σ_k implies

$$\widehat{T}_k - T^e = (1+\sigma_k)(T_k - T^e) \tag{7}$$

for $k = 0, 1, \ldots, \dfrac{n-1}{2}$. Then, from (5) and (6),

$$\begin{aligned}
\widetilde{H}_k - H(P(t_{2k})) &= \frac{1}{h}\ln\left(\frac{\widehat{T}_{2k} - T^e}{\widehat{T}_{2k+1} - T^e}\right) - \frac{1}{h}\ln\left(\frac{T_{2k} - T^e}{T_{2k+1} - T^e}\right) \\
&= \frac{1}{h}\ln\left(\frac{\widehat{T}_{2k} - T^e}{\widehat{T}_{2k+1} - T^e}\,\frac{T_{2k+1} - T^e}{T_{2k} - T^e}\right) \\
&= \frac{1}{h}\ln\left(\frac{(1+\sigma_{2k})(T_{2k} - T^e)}{(1+\sigma_{2k+1})(T_{2k+1} - T^e)}\,\frac{T_{2k+1} - T^e}{T_{2k} - T^e}\right) \\
&= \frac{1}{h}\ln\left(\frac{1+\sigma_{2k}}{1+\sigma_{2k+1}}\right). \quad \square
\end{aligned}$$

Remark Note that if, as in the second scenario, the temperature measurements are exact (and, consequently, all σ_k vanish) then this method provides the exact values of H. $\square$

Remark The previous proposition also shows that the error made when approaching function H using this method only depends on the values of $1+\sigma_k$, i.e., errors between $\widehat{T}_k - T^e$ and $T_k - T^e$ (see (7)). As the errors are a characteristic of the equipment, there is the unusual fact that the error in the approximation of function H by this methodology is independent of the function. $\square$

2.3 Iterative algorithm

It may happen that the equipment does not allow an experiment as described in Section 2.2. Therefore, different strategies are developed here, in order to identify coefficient H depending on the scenario we are dealing with. First, for the scenario where we know a set of exact values of temperature, we propose to use an approximate derivation operator of second order.

2.3.1 Identifying from a finite amount of exact values of temperature.

In this scenario we know n values of temperature T corresponding to instants of time $t_k = t_0 + kh$ for $k = 0, 1, \ldots, n$, where $h = \dfrac{t_\mathrm{f} - t_0}{n}$. We denote

$$T_k = T(t_k) \text{ and } P_k = P(t_k),\ k = 0, 1, \ldots, n.$$

From (2) we have

$$H(P(t)) = \frac{T'(t) - \alpha P'(t) T(t)}{T^e - T(t)},\ t_0 < t < t_\mathrm{f}. \tag{8}$$

Thus, our goal is to find, for each $k = 0, 1, \ldots, n$, an expression $\widetilde{H}_k$ that approximates the quotient

$$\frac{T'(t_k) - \alpha P'(t_k) T_k}{T^e - T_k}$$

i.e., an approximation of $H(P_k)$. To this end, we consider the approximate differentiation operator $R_h : \mathcal{C}([t_0, t_\mathrm{f}]) \to \mathcal{C}([t_0, t_\mathrm{f}])$ given by

$$R_h(v)(t) = \begin{cases}
\dfrac{-3v(t) + 4v(t+h) - v(t+2h)}{2h} + \Psi_h(v)(t_0), & t \in [t_0, t_0 + h] \\[2ex]
\dfrac{v(t+h) - v(t-h)}{2h}, & t \in [t_0 + h, t_\mathrm{f} - h] \\[2ex]
\dfrac{3v(t) - 4v(t-h) + v(t-2h)}{2h} + \Psi_h(v)(t_\mathrm{f} - 3h), & t \in [t_\mathrm{f} - h, t_\mathrm{f}],
\end{cases}$$

where

$$\Psi_h(v)(t) = \frac{v(t+3h) - 3v(t+2h) + 3v(t+h) - v(t)}{2h}.$$

Remark This approximate derivation operator provides an error bound slightly worse than that provided by the standard operator of order 2 (without $\Psi_h(v)$). However, with this definition, R_h is an operator from the Banach space of continuous functions with the L^∞–norm to itself; this will allow us to establish, in Section 3.2, an analogy with the classical methods there presented. □

Let us denote by $||\cdot||$ the norm in $\mathcal{C}([t_0, t_\mathrm{f}])$. The following result (whose proof can be seen in [10, pag. 50]) shows that R_h has order two:

Lemma 2.3 *If* $v \in \mathcal{C}^3([t_0, t_\mathrm{f}])$ *then* $||v' - R_h(v)|| \leq \frac{29}{6} h^2 \, ||v'''||$. □

The relation (8) and the operator R_h suggest to approximate the values $H(P_k)$ by means of

$$\widetilde{H}_k = \frac{R_h(T)(t_k) - \alpha P'(t_k) T_k}{T^e - T_k},$$

for $k = 0, 1, \ldots, n$. With this approximation, the following error estimate holds:

Proposition 2.4 *Let* $T \in \mathcal{C}^3([t_0, t_\mathrm{f}])$. *Then*

$$\max_{k=0,1,\ldots,n} \left| H(P_k) - \widetilde{H}_k \right| \leq \frac{29 M_3}{6(m - T^e)} \, h^2,$$

where $M_3 = ||T'''||$ *and* $m = \min_{k=0,1,\ldots,n} T_k$. □

Remark Inequality (3) shows that $m > T^e$ and therefore the above estimate does not blow–up. □

2.3.2 Identifying from a function that approximates the temperature.

In this context, an approximation $\widetilde{T} \in \mathcal{C}([t_0, t_\mathrm{f}])$ of T is assumed to be known. More precisely,

$$\left|\left| T - \widetilde{T} \right|\right| < \delta \tag{9}$$

with $0 < \delta < \mu = \widetilde{m} - T^e$, where $\widetilde{m}$ is given in (4).

Again using expression (8), we define the function

$$u(t) = \frac{T'(t) - \alpha P'(t) T(t)}{T^e - T(t)}, \; t_0 < t < t_\mathrm{f}$$

and its approximation

$$\widetilde{u}_h(t) = \frac{R_h(\widetilde{T})(t) - \alpha P'(t) \widetilde{T}(t)}{T^e - \widetilde{T}(t)}, \; t_0 < t < t_\mathrm{f}.$$

Let us estimate the error made in this approach:

Proposition 2.5 *Let* $T \in \mathcal{C}^3([t_0, t_\mathrm{f}])$ *and* $\widetilde{T} \in \mathcal{C}([t_0, t_\mathrm{f}])$ *verifying* (9) *with* $0 < \delta < \mu$. *Then*

$$||u - \widetilde{u}_h|| \leq \frac{1}{\mu - \delta} \left(\frac{29 M_3}{6} \, h^2 + \frac{4\delta}{\mu h} \big(\widetilde{M} - \widetilde{m} + 2\mu \big) \right) + \frac{\alpha P'_M T^e \delta}{\mu(\mu - \delta)}, \tag{10}$$

where

$$\widetilde{m} = \min_{t \in [t_0, t_\mathrm{f}]} \widetilde{T}(t), \; \widetilde{M} = \max_{t \in [t_0, t_\mathrm{f}]} \widetilde{T}(t) \; \text{ and } \; P'_M = \max_{s \in [t_0, t_\mathrm{f}]} P'(s).$$

DEMOSTRACIÓN. First of all, let us note that

$$\begin{cases} |T^e - \widetilde{T}(t)| = \widetilde{T}(t) - T^e \geq \widetilde{m} - T^e = \mu \\ |T^e - T(t)| = T(t) - T^e \geq \widetilde{T}(t) - \delta - T^e \geq \mu - \delta. \end{cases}$$

Next, for each $t \in [t_0, t_{\mathrm{f}}]$ we can write

$$|u(t) - \widetilde{u}_h(t)| \leq \left| \frac{T'(t)}{T^e - T(t)} - \frac{R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)} \right| + \alpha P'_M \left| \frac{T(t)}{T^e - T(t)} - \frac{\widetilde{T}(t)}{T^e - \widetilde{T}(t)} \right|. \tag{11}$$

i) The first term of the right hand side of (11) can be estimated as

$$\left| \frac{T'(t)}{T^e - T(t)} - \frac{R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)} \right| \leq \left| \frac{T'(t) - R_h(T)(t)}{T^e - T(t)} \right|$$
$$+ |R_h(T)(t)| \left| \frac{T(t) - \widetilde{T}(t)}{(T^e - T(t))(T^e - \widetilde{T}(t))} \right| + \left| \frac{R_h(T)(t) - R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)} \right|.$$

Now, the first term of this inequality can be bound (see Lemma 2.3) by

$$\frac{29h^2}{6\,|T^e - T(t)|}\,||T'''|| \leq \frac{29M_3}{6(\mu - \delta)}\,h^2.$$

For the second and third terms, we consider three cases, according to the definition of the operator R_h:

a) For $t \in [t_0 + h, t_{\mathrm{f}} - h]$

$$|R_h(T)(t)| \leq \frac{(\widetilde{M} + \delta) - (\widetilde{m} - \delta)}{2h} = \frac{\widetilde{M} - \widetilde{m} + 2\delta}{2h}.$$

Then

$$|R_h(T)(t)| \left| \frac{T(t) - \widetilde{T}(t)}{(T^e - T(t))(T^e - \widetilde{T}(t))} \right| \leq \frac{\widetilde{M} - \widetilde{m} + 2\delta}{2h}\,\frac{\delta}{\mu(\mu - \delta)}.$$

For the third term,

$$\left| \frac{R_h(T)(t) - R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)} \right| \leq \frac{\left|T(t+h) - \widetilde{T}(t+h)\right| + \left|T(t-h) - \widetilde{T}(t-h)\right|}{2h\mu} \leq \frac{\delta}{\mu h}.$$

b) For $t \in [t_0, t_0 + h]$

$$\begin{aligned} |R_h(T)(t)| &\leq \left| \frac{-3T(t) + 4T(t+h) - T(t+2h)}{2h} \right| + |\Psi_h(T)(t_0)| \\ &\leq \frac{4((\widetilde{M} + \delta) - (\widetilde{m} - \delta))}{2h} + \frac{4((\widetilde{M} + \delta) - (\widetilde{m} - \delta))}{2h} \\ &= \frac{4(\widetilde{M} - \widetilde{m} + 2\delta)}{h}, \end{aligned}$$

which implies

$$|R_h(T)(t)| \left| \frac{T(t) - \widetilde{T}(t)}{(T^e - T(t))(T^e - \widetilde{T}(t))} \right| \leq \frac{4(\widetilde{M} - \widetilde{m} + 2\delta)}{h}\,\frac{\delta}{\mu(\mu - \delta)}.$$

For the third term, we note that

$$\left|\frac{R_h(T)(t)-R_h(\widetilde{T})(t)}{T^e-\widetilde{T}(t)}\right| \le \frac{3\left|T(t)-\widetilde{T}(t)\right|}{2h\mu} + \frac{4\left|T(t+h)-\widetilde{T}(t+h)\right|}{2h\mu}$$

$$+\frac{\left|T(t+2h)-\widetilde{T}(t+2h)\right|}{2h\mu} + \frac{\left|\Psi_h(T)(t_0)-\Psi_h(\widetilde{T})(t_0)\right|}{\mu} \le \frac{8\delta}{h}.$$

c) In the interval $[t_f - h, t_f]$, the same bounds as in $[t_0, t_0 + h]$ can be obtained.

Collecting these three estimates, for $t \in [t_0, t_f]$ we have

$$\left|\frac{T'(t)}{T^e-T(t)} - \frac{R_h(\widetilde{T})(t)}{T^e-\widetilde{T}(t)}\right| \le \frac{29M_3}{6(\mu-\delta)}\,h^2 + \frac{4(\widetilde{M}-\widetilde{m}+2\delta)}{h}\,\frac{\delta}{\mu(\mu-\delta)} + \frac{8\delta}{\mu h}$$

$$= \frac{1}{\mu-\delta}\left(\frac{29M_3}{6}\,h^2 + \frac{4\delta}{\mu h}(\widetilde{M}-\widetilde{m}+2\mu)\right).$$

ii) For the second term of the right hand side of (11) we have

$$\left|\frac{T(t)}{T^e-T(t)} - \frac{\widetilde{T}(t)}{T^e-\widetilde{T}(t)}\right| = \left|\frac{T^e(T(t)-\widetilde{T}(t))}{(T^e-T(t))(T^e-\widetilde{T}(t))}\right| \le T^e\frac{\delta}{\mu(\mu-\delta)}. \quad \square$$

In (10) the time step h appears multiplying (squared) in a term and dividing in another one. Consequently, the optimal estimate is obtained when choosing a value of h that allows to balance both terms to get the minimum value. Next result (whose prove is straightforward by using Proposition 2.5) indicates how to choose such a value of h and its corresponding estimate:

Proposition 2.6 *Let $T \in \mathcal{C}^3([t_0, t_f])$ and $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ verifying (9) with $0 < \delta < \mu$. Then, the smallest value of the bound in (10) is reached when taken as a time step*

$$h^* = \left(\frac{12(\widetilde{M}-\widetilde{m}+2\mu)}{29\mu M_3}\,\delta\right)^{\frac{1}{3}}. \tag{12}$$

For this optimal value of time step, the following error estimate holds

$$||u-\widetilde{u}_{h^*}|| \le \frac{1}{\mu-\delta}\left(522M_3\frac{(\widetilde{M}-\widetilde{m}+2\mu)^2}{\mu^2}\,\delta^2\right)^{\frac{1}{3}} + \frac{\alpha P'_M T^e}{\mu(\mu-\delta)}\,\delta. \quad \square$$

Let h^* be as in (12). Let us denote n the integer part of $\dfrac{t_f - t_0}{h^*}$, $t_k = t_0 + kh^*$ and $\widetilde{T}_k = \widetilde{T}(t_k)$. If we approximate $H(P_k)$ by

$$\widetilde{H}_k = \widetilde{u}_{h^*}(t_k) = \frac{R_{h^*}(\widetilde{T})(t_k) - \alpha P'(t_k)\widetilde{T}_k}{T^e - \widetilde{T}_k}, \tag{13}$$

for $k = 0, 1, \ldots, n$, Proposition 2.6 leads to the main result of this section:

Theorem 2.7 *Under the assumptions in Proposition 2.6 one has*

$$\max_{k=0,1,\ldots,n}\left|H(P_k)-\widetilde{H}_k\right| \le \frac{1}{\mu-\delta}\left(522M_3\frac{(\widetilde{M}-\widetilde{m}+2\mu)^2}{\mu^2}\,\delta^2\right)^{\frac{1}{3}} + \frac{\alpha P'_M T^e}{\mu(\mu-\delta)}\,\delta,$$

where $\widetilde{H}_k$, $k = 0, 1, \ldots, n$, are given in (13). $\square$

Remark Theorem 2.7 provides a bound on the error made when we approximate H taking as time step the optimum value h^*. The difficulty is that this value is unknown, since it depends on M_3. In Section 2.3.3 we introduce an iterative algorithm in order to compute the values given in (13), from measurements of temperature, and successive approximations of h^*. $\square$

2.3.3 Identifying from a finite number of approximated values of the temperature.

The starting point now is that we have measurements of temperature $\{\widehat{T}_0, \widehat{T}_1, \dots, \widehat{T}_p\}$ corresponding to the instants $\{\tau_0 = t_0, \tau_1, \tau_2, \dots, \tau_p = t_{\mathrm{f}}\}$ and we assume that the error is of order $\widehat{\delta}$. We consider a function $\widetilde{T}$ that interpolates the previous values and assume that the interpolation method used is such that the error δ with respect to T is of the order of measurement error $\widehat{\delta}$, i.e.,

$$\delta = C\widehat{\delta}.$$

To do this, eventually, it could be necessary to increase the number of measurements.

Once function $\widetilde{T}$ is defined in this way, we are in the situation of the previous section; hence it suffices to consider the threshold $\mu = \widetilde{m} - T^e$, take the time step h as in (12) and n as the integer part of $\dfrac{t_{\mathrm{f}} - t_0}{h}$. Thus, the values $\widetilde{H}_k$ in (13) provide an approximation of H with an error estimate given in Theorem 2.7.

Next, we describe an algorithm to approximate the values of H at points $P_k = P(t_k) \in [P_0, P_{\mathrm{f}}]$, for instants t_k in equally spaced partitions of $[t_0, t_{\mathrm{f}}]$. The time step of these partitions should be defined, in an iterative way, approaching the value of h^*.

The input data are: $\{\widehat{T}_k\}_{k=0}^p$ and $\widehat{\delta} > 0$. First of all, we construct a function $\widetilde{T}(t)$ interpolating $\{\widehat{T}_k\}_{k=0}^p$. Next, we estimate the error $\delta > 0$ due to the interpolation. Then, the admissible threshold $\mu = \widetilde{m} - T^e$, under the constraint $\mu > \delta$, is obtained.

The algorithm is based on an iterative process beginning from a guess value of h for the optimal time step h^*. From this value, we consider the instants $t_k = t_0 + kh$, $k = 0, 1, \dots, n$, where n is the integer part of $\dfrac{t_{\mathrm{f}} - t_0}{h}$. Therefore, the values $\widetilde{T}_k = \widetilde{T}(t_k)$ are obtained. From these values, an approximation Λ_3 of M_3 is computed as the maximum absolute value of

$$\begin{cases} \dfrac{-5\widetilde{T}_k + 18\widetilde{T}_{k+1} - 24\widetilde{T}_{k+2} + 14\widetilde{T}_{k+3} - 3\widetilde{T}_{k+4}}{2h^3}, & k = 0, 1 \\ \dfrac{\widetilde{T}_{k+2} - 2\widetilde{T}_{k+1} + 2\widetilde{T}_{k-1} - \widetilde{T}_{k-2}}{2h^3}, & k = 2, 3, \dots, n-2 \\ \dfrac{3\widetilde{T}_{k-4} - 14\widetilde{T}_{k-3} + 24\widetilde{T}_{k-2} - 18\widetilde{T}_{k-1} + 5\widetilde{T}_k}{2h^3}, & k = n-1, n. \end{cases} \tag{14}$$

These formulas are based, respectively, on standard order two progressive, central and backward approximate derivative schemes of a regular function.

From Λ_3, the next value of the time step is computed (following (12)) as

$$h = \left(\frac{12(\widetilde{M} - \widetilde{m} + 2\mu)}{29\mu\Lambda_3}\,\delta \right)^{\frac{1}{3}}, \tag{15}$$

and so on.

The process stops when two consecutive values of h are close. From the final value of h, the corresponding instants t_k, interpolation $\widetilde{T}$ and the quotients

$$\widetilde{H}_k = \widetilde{u}_h(t_k) = \frac{R_h(\widetilde{T})(t_k) - \alpha P'(t_k)\widetilde{T}_k}{T^e - \widetilde{T}_k}, \tag{16}$$

are computed. These quantities approach the values of H in the pressures $P_k = P(t_k)$, for $k = 0, 1, \dots, n$.

Algorithm

DATA	$\{\widehat{T}_k\}_{k=0}^{p}$: Temperature measurements at times $\{\tau_k\}_{k=0}^{p}$.
	$\widehat{\delta} > 0$: bound of measurements errors.
	ε: stopping test precision.
	h: guess value of h^*.
Step 1:	Determine $\widetilde{T}$ and δ according to $\widehat{\delta}$ so that $\mu = \widetilde{m} - T^e > \delta$.
Step 2:	While the relative error in h is greater than ε:
	a) Determine the new discrete instants $\{t_k\}$ and compute $\{\widetilde{T}_k\}$.
	b) Compute Λ_3 as the maximum absolute value of (14).
	c) Compute the new value of h as in (15).
Step 3:	Obtain the final discrete instants $\{t_k\}$ and the values $\{\widetilde{T}_k\}$.
Step 4:	Compute the approximations $\widetilde{H}_k$ according to (16).

2.4 Nondimensionalization of the problem

Before performing the numerical experiments with different sets of data illustrating the behavior of the methods developed, it is convenient to nondimensionalize the problem. We want the model to involve as few dimensionless parameters as possible. Here, it suffices to consider two parameters: the pressure and a relationship between the initial and ambient temperature, as discussed below. We consider the new dimensionless variables

$$t^* = \frac{t - t_0}{t_{\mathrm{f}} - t_0}, \quad T^*(t^*) = \frac{T(t) - T^e}{T_0 - T^e} \quad \text{and} \quad P^*(t^*) = \big(P(t) - P_0\big)\alpha.$$

Problem (2) can be written in these new variables (see [10, pag. 57]) as

$$\begin{cases} \dfrac{dT^*}{dt^*}(t^*) = -H^*(P^*(t^*))T^*(t^*) + \dfrac{dP^*}{dt^*}(t^*)\left(T^*(t^*) + T^{ea}\right), \ t^* \in (0,1) \\ T^*(0) = 1, \end{cases} \tag{17}$$

where

$$\begin{cases} H^*(s) = (t_{\mathrm{f}} - t_0)H\left(\dfrac{s}{\alpha} + P_0\right) \quad \big(\Rightarrow \ H^*(P^*(t^*)) = (t_{\mathrm{f}} - t_0)H(P(t))\big) \\ T^{ea} = \dfrac{T^e}{T_0 - T^e}. \end{cases}$$

We use this approach to identify coefficient H^* and to find the temperature distribution for several functions P^* and several values of T^{ea}.

Remark After identifying function H^*, H can be obtained by

$$H(s) = \frac{1}{t_{\mathrm{f}} - t_0} H^*\left(\alpha(s - P_0)\right), \ s \in [P_0, P_{\mathrm{f}}]. \tag{18}$$

From T^* we can express temperature T as

$$T(t) = T^e + (T_0 - T^e)T^*\left(\frac{t - t_0}{t_{\mathrm{f}} - t_0}\right), \ t \in [t_0, t_{\mathrm{f}}]. \quad \square$$

Remark If the order of magnitude of function H^* is small compared with

$$\frac{dP^*}{dt^*}(t^*)\left(T^*(t^*) + T^{ea}\right),$$

this term will be dominant. Hence any function H^* of that order of magnitude would provide values of temperature with few differences. To avoid this problem we can modify the original experiment so that the new one results in a function H^* of a higher order of magnitude. If pressure in the original experiment is given by

$$P(t) = a(t - t_0) + P_0, \; t \in [t_0, t_{\rm f}],$$

a slower increase pressure (for a longer time in order to cover the same range of pressures $[P_0, P_{\rm f}]$) could be considered. That is, we can take

$$P_1(t) = ac(t - t_0) + P_0, \; t \in \left[t_0, t_0 + \frac{t_{\rm f} - t_0}{c}\right],$$

with $0 < c \leq 1$. If T_1 is the temperature obtained with this pressure, the changes of variable

$$t_1^* = \frac{c(t - t_0)}{t_{\rm f} - t_0}, \quad T_1^*(t_1^*) = \frac{T_1(t) - T^e}{T_0 - T^e} \quad \text{and} \quad P_1^*(t_1^*) = \big(P_1(t) - P_0\big)\alpha$$

lead to

$$\begin{cases} \dfrac{dT_1^*}{dt_1^*}(t_1^*) = -\dfrac{1}{c} H^*\big(P_1^*(t_1^*)\big) T_1^*(t_1^*) + \dfrac{dP_1^*}{dt_1^*}(t_1^*)\left(T_1^*(t_1^*) + T^{ea}\right), \; t_1^* \in (0, 1) \\ T_1^*(0) = 1. \end{cases} \tag{19}$$

Since

$$\frac{dP_1^*}{dt_1^*}(t_1^*) = \alpha a(t_{\rm f} - t_0) = \frac{dP^*}{dt^*}(t^*),$$

the equations of problems (17) and (19) are identical, except that the new function H^* is amplified by the factor $\dfrac{1}{c} \geq 1$. □

Remark The dimensionless problem is governed by an equation different from the original, and this will be taken into account in the methods we will use:

a) For the method based on an *ad hoc* experiment in Section 2.2 it suffices to note that, in each interval when the pressure is constant, the temperature satisfies the same equation but with $T^e = 0$. We must therefore consider the approaches

$$\widetilde{H}_k = \frac{1}{h} \ln\left(\frac{\widehat{T}_{2k}}{\widehat{T}_{2k+1}}\right)$$

instead of (6).

b) Concerning the iterative algorithm of Section 2.3, we can say that the optimal step expression (15) and the quantities (14) that are used for the calculation of Λ_3 remain the same (changing, of course, the roles of $\widetilde{T}$ and $\widetilde{T^*}$), while the approximation (16) of $\widetilde{H}_k$ becomes

$$\widetilde{H}_k = -\frac{R_h(\widetilde{T^*})(t_k^*) - \dfrac{dP^*}{dt^*}(t_k^*)(\widetilde{T_k^*} + T^{ea})}{\widetilde{T}_k}. \quad \square$$

2.5 Numerical Results

In this section we perform a comparative study of the results obtained when using the methods considered for the identification of function H. While working on the nondimensional problem, the value of T^{ea} and the range of pressures are linked to a real situation. Specifically, we use the P2 treatment data considered in [11], i.e., $T_0 = 313\,^{\rm o}$K, $T^e = 295\,^{\rm o}$K and $\alpha = 4.5045 \times 10^{-5}$ MPa^{-1}. The pressure increases from atmospheric pressure up to 360 MPa.

For the iterative algorithm, we suppose that the experiment has been done with a linear pressure curve with slope a. For the *ad hoc* experiment, we suppose the same slope a for the pressure curve when it is non constant. Therefore, the time interval for the *ad hoc* experiment, is twice as long as that of the iterative one. Then, according to (18), the function H^* corresponding to the *ad hoc* experiment is twice the function H^* corresponding to the iterative algorithm.

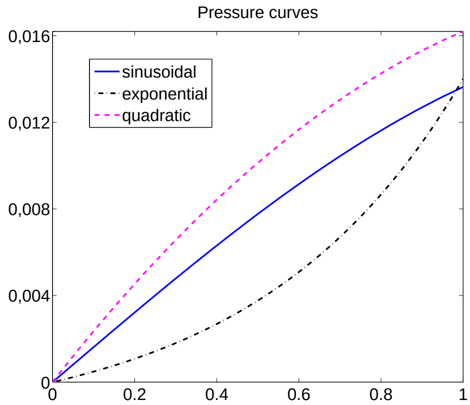

Figure 2: Considered pressures for the numerical tests after identification of H.

The maximum value of dimensionless pressure is $a^* = 360\alpha = 0.0162$ in both approaches. We point out that since $t^* \in [0,1]$, the slope of the nondimensional pressure for the *ad hoc* experiment is $2a^*$ when it is not constant, while it is a^* in the equation considered for the iterative algorithm.

For the sake of simplicity, from now on, we omit the superscript *.

The data for numerical tests have been obtained as follows: with a given function H, we solve direct nondimensional problem (17), obtaining the temperature T. Then T is evaluated on an equally spaced partition of instants of time. We assume that in both experiments, the measurements have been carried out with the same step, so that we work in the first method with twice the values in the second (in particular, we take 200 values $T_k = T(t_k)$ in the *ad hoc* experiment and 100 values in the iterative algorithm). The error measurements $\widehat{T}_k$ are built by perturbing T_k by means of random oscillations of order 1% of T_k. More precisely,

$$\widehat{T}_k = T_k \left(1 + \frac{r(t_k)}{75}\right),$$

where $r(t) = \sin(q\pi t)$ and q is a random integer between 1 and 99. Function $\widetilde{T}$ is taken as the piecewise linear interpolation of values $\widehat{T}_k$.

To allow an easy comparison, the same seven perturbations of temperature values have been generated, corresponding to the values $q = 3, 14, 27, 42, 65, 84$ and 97. Among them we selected the two which produces the smallest and the largest error in L^∞–norm in H, respectively.

After identifying an approximation of function H, we compute the temperature T solving problem (17) and we compare it with the known solution of the direct problem. Also, different values for dimensionless parameters of the problem (the pressure curve and T^{ea}) are prescribed and the corresponding solutions are calculated. In order to analyze the quality of the identification, these solutions are compared to the exact temperature.

The different values of the parameters are generated by multiplying by the factors $d = 2, d = 1$ and $d = \frac{1}{2}$ the original value of T^{ea} and choosing as pressure curves (see Figure 2)

$$P(t) = a \sin t,\ P(t) = a(e^{2t-2} - e^{-2}) \ \text{ and } \ P(t) = \frac{a}{2}\, t(3-t).$$

In all figures and tables “Error” denotes the L^∞–norm error in H and “% Error” denotes the percentage relative error in maximum norm of T, i.e.,

$$\frac{\max_k |\widetilde{T}_k - T_k|}{\max_k |T_k|} \times 100.$$

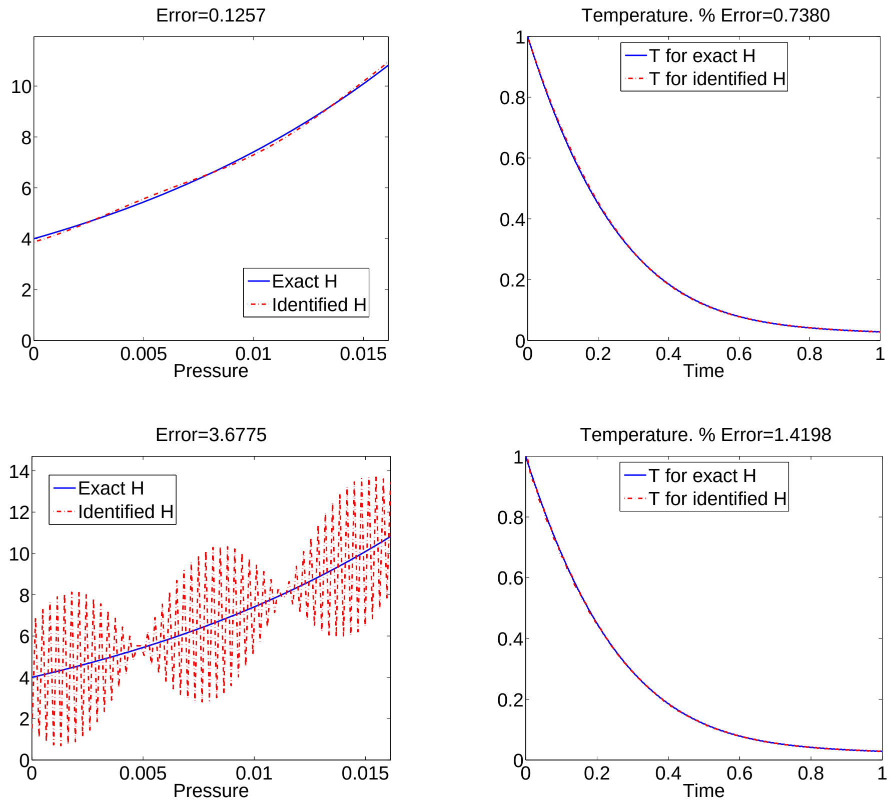

Figure 3: *Ad hoc* algorithm (Top: smallest error in H. Bottom: largest error in H)

2.5.1 First method: *ad hoc* experiment

For this method, the value for the first parameter of the nondimensional problem is $T^{ea} = \dfrac{295}{18}$, while the slope of the pressure (where not constant) is 0.0324 $(= 2a)$; what causes the pressure to take all values in the pressure range $[0, a]$ when time lies in $[0, 1]$. We consider the function

$$H(s) = 4 \exp\left(\frac{s}{a}\right).$$

Figure 3 shows the identified function H (and corresponding computed temperature) for the smallest and largest error in H.

Table 1 shows the percentage relative error in maximum norm of T (% Error) for each of the nine data sets considered, both for the smallest and largest error in H.

Table 1: % Error for the *ad hoc* experiment and considered pressures: smallest (above) and largest (below) error in H.

Pressure	Factor over parameter T^{ea}		
	$d = 2$	$d = 1$	$d = 0.5$
sinusoidal	0.48	0.48	0.47
exponential	0.87	0.86	0.85
quadratic	0.36	0.36	0.36
sinusoidal	1.37	1.36	3.84
exponential	5.11	5.10	5.09
quadratic	1.06	1.97	2.45

The error in this method (we remind that it provides exact values when there are not measurement errors) increases with the frequency of the oscillatory perturbation: the error in H grows with the value of q, being smaller for $q = 3$ (the smoother perturbation) and larger for $q = 97$ (more oscillatory perturbation). The interested reader can see the details in [10].

2.5.2 Second method: iterative algorithm

Now, the value $T^{ea} = \dfrac{295}{18}$ is the same as before, but the pressure increase changes (since now there is no constant steps); in fact, $P(t) = 0.0162t$.

As already mentioned, the corresponding function to the original H must be half of the chosen in the previous method, i.e.,

$$H(s) = 2\exp\left(\frac{s}{a}\right).$$

Figure 4 shows the identified function H (and corresponding computed temperature) for the smallest and largest error in H.

Table 2 shows the percentage relative error in temperature (in maximum norm) for each of the nine data sets considered, both for the smallest and largest error in H.

Table 2: % Error for the iterative algorithm and considered pressures: smallest (above) and largest (below) error in H.

Pressure	Factor over parameter T^{ea}		
	$d = 2$	$d = 1$	$d = 0.5$
sinusoidal	0.17	0.16	0.15
exponential	0.17	0.17	0.17
quadratic	0.14	0.13	0.13
sinusoidal	1.49	1.49	1.49
exponential	4.46	4.44	4.44
quadratic	0.94	0.94	0.94

This algorithm uses interpolation of approximate values of T at instants that are not the same as those used for the temperature measurements. Therefore, in contrast with what happens in the above method (see, Section 2.5.1), their behavior is not directly linked to the frequency of oscillatory perturbations.

In conclusion, although the size of the error in H is moderate for both methods, temperatures calculated from approximate identifications are quite accurate (the error is always of the order of measurement error). The first method usually provides a better approximation of the temperature when solving for the initial parameters used to identify H. However, when the identified temperature for the nine data sets is considered, the second method is generally more accurate in the case of largest error in H and therefore it can be considered more robust.

3 Temperature Dependent Coefficient

This section deals with the case where the coefficient of heat transfer to be identified depends on the temperature (instead of pressure). The fact that H depends on the solution of the state equation complicates the resolution of the inverse problem. On the other hand, the fact that H does not depend on the pressure allows to work at constant pressure.

Therefore, the problem (1) becomes

$$\begin{cases} T'(t) = H(T(t))(T^e - T(t)),\ t \geq t_0 \\ T(t_0) = T_0 \end{cases} \tag{20}$$

and we work under the following hypothesis:

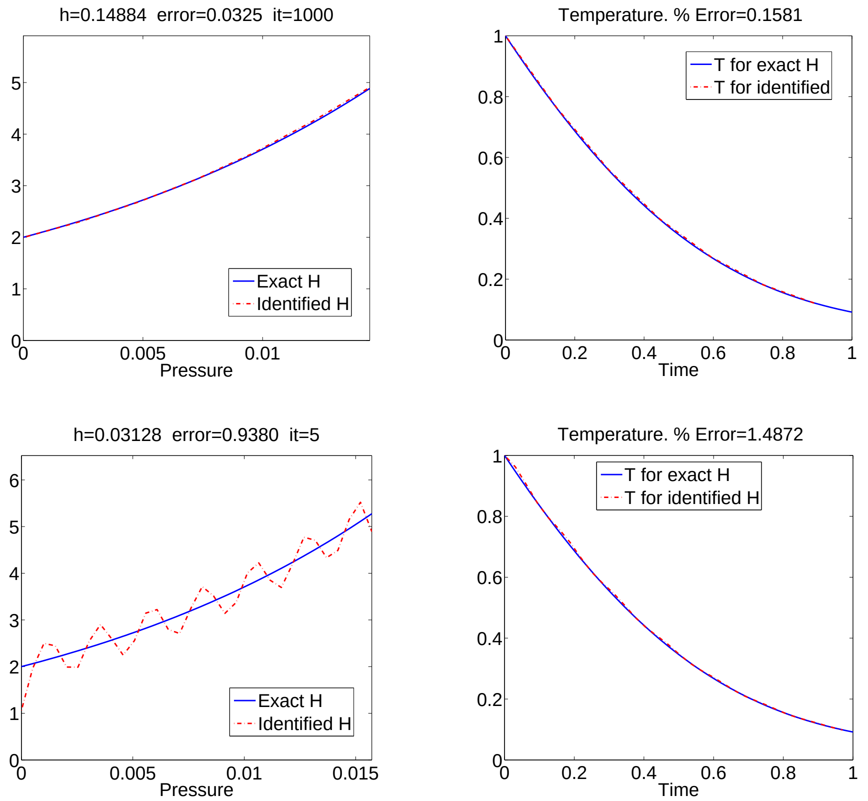

Figure 4: Iterative algorithm (Top: smallest error in H. Bottom: largest error in H)

- The external temperature T^e is constant.
- The initial temperature T_0 is lower than T^e (a quite similar study could be done if it were assumed that $T_0 > T^e$). As we will see below, This ensures that the solution T is an increasing function that takes its values in the range $[T_0, T^e]$.
- H is a continuous and positive function in the interval $[T_0, T^e]$.

The following result collect the main qualitative properties of the solution of the direct problem (20):

Proposition 3.1 *Under assumptions above, the problem* (20) *has a unique solution T satisfying:*

a) T is well defined for all $t \geq t_0$ and $T \in \mathcal{C}^1([t_0, +\infty))$.

b) $T'(t) > 0$ for all $t \geq t_0$ and so, T is an increasing function.

c) $T_0 \leq T(t) < T^e$ for all $t \geq t_0$.

d) For every $t \geq t_0$, inequalities

$$T^e - (T^e - T_0)e^{-H_m(t-t_0)} \leq T(t) \leq T^e - (T^e - T_0)e^{-H_M(t-t_0)},$$

hold, where $H_m = \min_{s\in[T_0,T^e]} H(s)$ and $H_M = \max_{s\in[T_0,T^e]} H(s)$.

e) $\lim_{t\to+\infty} T(t) = T^e$, i.e., all solutions go asymptotically to the unique equilibrium value T^e of the differential equation.

f) For every $t \geq t_0$,

$$H_m(T^e - T_0)e^{-H_M(t-t_0)} \leq T'(t) \leq H_M(T^e - T_0)e^{-H_m(t-t_0)}. \quad \square$$

We are interested in identifying H in a range of temperatures as wide as possible. To do this, it is advisable to take measurements during an experiment that starts with an initial temperature as low as possible and an ambient temperature as high as possible.

3.1 Scenarios of the inverse problem.

The model is not very sensitive to changes in $H(s)$ for s close to T^e. For this reason, it is unrealistic (and unnecessary) pretend to identify H near T^e. These considerations lead us to pose the problem of identifying function H as follows:

i) A *threshold* $\mu > 0$, depending on the admissible error in the approximation of the temperature, is fixed so that the identification of H in the interval $[T^e - \mu, T^e]$ is not part of our goal. From this threshold, a time $t_\mathrm{f} = t_\mathrm{f}(\mu, T_0, T^e, H)$ is determined (by arguments explained later) such that

$$|T^e - T(t)| \le \mu,\ t \ge t_\mathrm{f}. \tag{21}$$

Thus, the error in the temperature will be smaller than μ for $t \ge t_\mathrm{f}$.

ii) We use model (20) in $[t_0, t_\mathrm{f}]$ and identify H in $[T_0, T(t_\mathrm{f})] \supset [T_0, T^e - \mu]$.

As in Section 2, according to the available information about T in $[t_0, t_\mathrm{f}]$, we set the inverse problem in several scenarios:

- The trivial (and unrealistic) case is to suppose that functions T and T' are known in $[t_0, t_\mathrm{f}]$. Then, assuming $H \in \mathcal{C}([T_0, T(t_\mathrm{f})])$ and positive, we can identify H in a direct way from
$$H(s) = \frac{T'(T^{-1}(s))}{T^e - s}.$$
- If function T can be evaluated without error in a finite number of arbitrary instants $t \in [t_0, t_\mathrm{f}]$, the identification of H in $[T_0, T(t_\mathrm{f})]$ becomes, as in Section 2, a standard problem of numerical differentiation (in order to approximate T' from data).
- Next scenario arises when a function $\widetilde{T}$, representing an approximate value of the temperature in every instant, is supposed to be known.
- However, in a realistic context, only discrete values $\widehat{T}_k$ approximating the temperature at some instants are available.

For the last three scenarios we use a method to approach T' from data. Let us see how to determine t_f satisfying (21) in the situations described before:

a) For the first scenario, the value of t_f (more precisely, the minimum value of t_f verifying (21)) is the solution of the equation $T(t) = T^e - \mu$, i.e.
$$t_\mathrm{f} = T^{-1}(T^e - \mu).$$

b) In the second scenario, we will consider only the values of T lower than $T^e - \mu$. So, given $p+1$ exact values $\{T_0, T_1, \dots, T_p\}$ of the temperature at instants $\{\tau_0 = t_0 < \tau_1 < \cdots < \tau_p\}$, we consider $\mu_k = T^e - T_k$. Then μ is changed by one of the values μ_k. If the threshold μ is lower than all quantities μ_k, we take $\mu = \mu_p$ and $t_\mathrm{f} = \tau_p$; otherwise, we define $m = \max\{k :\ \mu \le \mu_k\}$ and we take $\mu = \mu_m$ and $t_\mathrm{f} = \tau_m$.

c) The assumptions in the third scenario are that function $\widetilde{T}$ is known in some interval $[t_0, t^*]$ and
$$\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([t_0, t^*])} < \delta,$$
where $\delta < \mu$ (if $\mu \le \delta$ we would need to increase the value of μ). Then, we choose t_f as follows:

- If $\widetilde{T}(t) < T^e - \mu + \delta$ for all $t \leq t^*$ we take $t_{\rm f} = t^*$, and we must increase the value of μ by taking $\mu = T^e - \widetilde{T}(t^*) + \delta$.
- Otherwise, we consider $t_{\rm f} = \min\{t : \widetilde{T}(t) = T^e - \mu + \delta\}$.

To sum up, we choose

$$t_{\rm f} = \begin{cases} t^*, & \text{if } \widetilde{T}(t) < T^e - \mu + \delta \text{ for all } t \leq t^* \\ \min\{t : \widetilde{T}(t) = T^e - \mu + \delta\}, & \text{otherwise,} \end{cases} \tag{22}$$

with the appropriate value of μ. Note that (22) ensures

$$T(t) \geq T(t_{\rm f}) \geq \widetilde{T}(t_{\rm f}) - \delta = T^e - \mu,\ t \geq t_{\rm f},$$

i.e., the approximate temperature values that are not used correspond to instants at which the exact temperature is outside the admissible range. Moreover, the choice of $t_{\rm f}$ and the monotonicity of T, imply

$$\begin{cases} \widetilde{T}(t) \leq \widetilde{T}(t_{\rm f}) \\ T^e - T(t) \geq T^e - T(t_{\rm f}) \geq T^e - \widetilde{T}(t_{\rm f}) - \delta = \mu - 2\delta \\ T^e - \widetilde{T}(t) \geq T^e - \widetilde{T}(t_{\rm f}) = \mu - \delta \end{cases} \tag{23}$$

for $t \leq t_{\rm f}$. To ensure that these lower bounds are positive, we will impose to the threshold the restriction $\mu > 2\delta$.

d) Finally, in the fourth scenario, measurements $\{\widehat{T}_k\}_{k=0}^p$ such that $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, with $\widehat{\delta} > 0$, are available. Let $\widetilde{T}$ be an interpolation function of values $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ in $\{\tau_0, \tau_1, \ldots, \tau_p\}$ such that

$$\left|\left| T - \widetilde{T} \right|\right|_{\mathcal{C}([\tau_0,\tau_p])} < \delta$$

for some $\delta > 0$. From this function $\widetilde{T}$ the choice of $t_{\rm f}$ is made as in the previous scenario.

3.2 Regularization theory. Classical algorithms.

Let us suppose the fourth scenario (the more general one) exposed in Section 3.1. Once $t_{\rm f}$ is determined, we consider the initial value problem

$$\begin{cases} T'(t) = H(T(t))(T^e - T(t)),\ t \in [t_0, t_{\rm f}] \\ T(t_0) = T_0. \end{cases} \tag{24}$$

By denoting $u(t) = H(T(t)),\ t \in [t_0, t_{\rm f}]$, we have that

$$\int_{t_0}^{t} u(s)\, ds = \int_{t_0}^{t} \frac{T'(s)}{T^e - T(s)}\, ds = -\ln\left(\frac{T^e - T(t)}{T^e - T_0}\right).$$

Thus, for suitable functional spaces X and Y, by defining the operator $K : X \to Y$ as

$$Kx(t) = \int_{t_0}^{t} x(s)\, ds,$$

we have that $Ku = y$, where

$$y(t) = -\ln\left(\frac{T^e - T(t)}{T^e - T_0}\right),\ t \in [t_0, t_{\rm f}]. \tag{25}$$

Note that function y is well defined and it is positive (see Proposition 3.1). In order to apply the classical regularization theory for inverse problems in Hilbert spaces (see, e.g., [2], [12], [13]), we choose $X = Y = L^2(t_0, t_{\rm f})$.

Next result shows some well–known properties (see, for example, [10]) of operator K:

Proposition 3.2 *$K : L^2(t_0, t_\mathrm{f}) \to L^2(t_0, t_\mathrm{f})$ is a linear and compact operator. Moreover:*

a) $||K|| \leq \dfrac{t_\mathrm{f} - t_0}{\sqrt{2}}$ *(here $||\cdot||$ denotes the $\mathcal{L}\left(L^2(t_0, t_\mathrm{f}), L^2(t_0, t_\mathrm{f})\right)$ norm).*

b) For every $x \in L^2(t_0, t_\mathrm{f})$, $Kx \in H^1(t_0, t_\mathrm{f})$ and $(Kx)' = x$.

c) K is an injective operator and $\mathrm{range}(K) = \{y \in H^1(t_0, t_\mathrm{f}) : \, y(t_0) = 0\}$ is a dense set in $L^2(t_0, t_\mathrm{f})$.

d) The adjoint operator $K^* : L^2(t_0, t_\mathrm{f}) \to L^2(t_0, t_\mathrm{f})$ is given by

$$K^* y(t) = \int_t^{t_\mathrm{f}} y(s)\, ds. \quad \square$$

In our problem we have measurements $\widehat{T}_k$ verifying $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, and an interpolation function $\widetilde{T}$ such that $\left|\left|T - \widetilde{T}\right|\right|_{\mathcal{C}([\tau_0, \tau_p])} < \delta$. This provides a right hand term

$$y_\delta(t) = -\ln\left(\frac{T^e - \widetilde{T}(t)}{T^e - T_0}\right) \tag{26}$$

and the approximate problem

$$K u_\delta = y_\delta.$$

Remark The third estimate of (23) ensures that $y_\delta \in H^1(t_0, t_\mathrm{f})$ if the chosen interpolation $\widetilde{T}$ is regular (for example, piecewise linear). The absence of error in the initial temperature implies $y_\delta(t_0) = 0$ and then y_δ belongs to $\mathrm{range}(K)$. $\square$

Next proposition estimates the error between y_δ and y in terms of error between $\widetilde{T}$ and T (given by δ).

Proposition 3.3 *Let $y(t)$ and $y_\delta(t)$ given by (25) and (26), respectively. By denoting*

$$e(\delta) = \frac{\sqrt{t_\mathrm{f} - t_0}}{\mu - 2\delta}\,\delta, \tag{27}$$

the estimate $||y - y_\delta||_{L^2(t_0, t_f)} \leq e(\delta)$ holds.

DEMOSTRACIÓN. A first order Taylor expansion of function $s \mapsto \ln(T^e - s)$ around $s = T(t)$, provides

$$|y(t) - y_\delta(t)| = \left|\ln(T^e - \widetilde{T}(t)) - \ln(T^e - T(t))\right| = \left|\frac{T(t) - \widetilde{T}(t)}{T^e - T_\theta}\right|,$$

where T_θ is a value between $T(t)$ and $\widetilde{T}(t)$ which can be written as

$$T_\theta = \theta T(t) + (1 - \theta)\widetilde{T}(t)$$

for some $0 < \theta < 1$. Estimates (23) imply

$$\begin{aligned} T^e - T_\theta &= \theta(T^e - T(t)) + (1 - \theta)(T^e - \widetilde{T}(t)) \\ &\geq \theta(\mu - 2\delta) + (1 - \theta)(\mu - \delta) \\ &= \mu - (1 + \theta)\delta \geq \mu - 2\delta. \end{aligned}$$

Thus,

$$|y(t) - y_\delta(t)| \leq \frac{|T(t) - \widetilde{T}(t)|}{\mu - 2\delta} \leq \frac{\delta}{\mu - 2\delta},$$

which allows to conclude the result easily. $\square$

Next, we describe two of the classic strategies of regularization: Tikhonov and Landweber methods. These methods are implemented in Section 3.4 for several test problems.

3.2.1 Tikhonov regularization. Discrepancy Principle of Morozov

The Tikhonov strategy to solve $Ku_\delta = y_\delta$, (see, for instance, [18], [19]) consists of minimizing the Tikhonov functional

$$J_\alpha(x) = ||Kx - y_\delta||^2_{L^2(t_0,t_\mathrm{f})} + \alpha\, ||x||^2_{L^2(t_0,t_\mathrm{f})}\,,$$

where $\alpha = \alpha(\delta) > 0$ is suitable chosen. J_α has a unique minimum $u_{\alpha,\delta}$ (see, for instance, Theorem 2.11 of [13]), which is also the unique solution of the *normal equation*

$$(\alpha + K^*K)x = K^*y_\delta. \tag{28}$$

The regularization strategy is given for the linear operators $R_\alpha : L^2(t_0,t_\mathrm{f}) \to L^2(t_0,t_\mathrm{f})$ defined by

$$R_\alpha y = (\alpha + K^*K)^{-1}K^*y.$$

For $\alpha = 0$ this becomes the normal equation associated to operator K. Since this operator is injective and compact in an infinite–dimensional space, K is not surjective; then, minimizing operator J_0 is an ill–posed problem (see [13], Lemma 2.10). For this reason, a penalty term $\alpha\, ||x||^2_{L^2(t_0,t_\mathrm{f})}$ is added.

Let us see that the solution of (28) also solves a boundary value problem of second order.

Proposition 3.4 *The solution $u_{\alpha,\delta}$ of (28) is the solution of the boundary problem*

$$\left\{\begin{array}{l} -\alpha x''(t) + x(t) = y'_\delta(t),\ t \in (t_0,t_\mathrm{f}) \\ x'(t_0) = 0,\ x(t_\mathrm{f}) = 0. \end{array}\right. \tag{29}$$

Moreover, denoting $\gamma(r) = \dfrac{t_\mathrm{f} - r}{\sqrt{\alpha}}$, the solution is

$$u_{\alpha,\delta}(t) = \frac{1}{\sqrt{\alpha}}\left(\varphi_{\alpha,\delta}(t)\cosh\gamma(t) + \psi_{\alpha,\delta}(t)\sinh\gamma(t)\right),$$

where

$$\varphi_{\alpha,\delta}(t) = \int_t^{t_\mathrm{f}} y'_\delta(s)\sinh\gamma(s)\,ds$$

and

$$\psi_{\alpha,\delta}(t) = \int_{t_0}^{t} y'_\delta(s)\cosh\gamma(s)\,ds - \tanh\gamma(t_0)\varphi_{\alpha,\delta}(t_0).$$

DEMOSTRACIÓN. Proposition 3.2 allows to write equation (28) as

$$\alpha x(t) + \int_t^{t_\mathrm{f}}\left(\int_{t_0}^{s} x(\tau)\,d\tau\right)ds = \int_t^{t_\mathrm{f}} y_\delta(s)ds.$$

Thus $x(t_\mathrm{f}) = 0$. Further, since $y_\delta \in R(K)$, we have $y_\delta \in H^1(t_0,t_\mathrm{f})$ and $y_\delta(t_0) = 0$. Therefore, by differentiating the above expression, we obtain

$$\alpha x'(t) - \int_{t_0}^{t} x(s)\,ds = -y_\delta(t),$$

and, in particular, $x'(t_0) = -y_\delta(t_0) = 0$. By differentiating again, we get to

$$\alpha x''(t) - x(t) = -y'_\delta(t).$$

Finally, standard calculations for solving the boundary value problem (29) lead to the above expression for $u_{\alpha,\delta}$. □

Remark Since $||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta)$ (see Proposition 3.3), by choosing $\alpha(\delta)$ satisfying

$$\lim_{\delta\to 0} \alpha(\delta) = 0 \text{ and } \lim_{\delta\to 0} \frac{(e(\delta))^2}{\alpha(\delta)} = 0,$$

the Theorem 2.12 of [13] states that the Tikhonov regularization strategy for our problem is admissible, i.e.,

$$\lim_{\delta\to 0} \left|\left|u_{\alpha(\delta),\delta} - u\right|\right|_{L^2(t_0,t_f)} = 0. \quad \square$$

Remark Note that, for every $\alpha \neq 0$, the solution of problem (29) vanishes in t_f. This is because the boundary condition is, in fact, $\alpha x(t_f) = 0$. This constraint appears in the approaches based on Tikhonov's strategy when working in $L^2(t_0, t_f)$. Therefore, although the error in $L^2(t_0, t_f)$ is small, the error in the maximum norm may be large. In order to avoid this problem we develop an alternative methodology in Section 3.3. $\square$

The Morozov's discrepancy principle (see [15]) provides a way to choose the parameter $\alpha = \alpha(\delta)$ for the Tikhonov regularization strategy: it is chosen so that the solution $u_{\alpha(\delta),\delta}$ of (28) satisfies

$$\left|\left|Ku_{\alpha(\delta),\delta} - y_\delta\right|\right|_{L^2(t_0,t_f)} = e(\delta), \tag{30}$$

supposing that

$$||y - y_\delta||_{L^2(t_0,t_f)} \le e(\delta) < ||y_\delta||_{L^2(t_0,t_f)}.$$

The regularization strategy associated to this choice of $\alpha(\delta)$ is admissible (see Theorem 2.17 of [13]).

3.2.2 Landweber's iterative method.

Landweber's iterative method is defined as

$$\begin{cases} x_0 = 0 \\ x_m = (I - \lambda K^* K)x_{m-1} + \lambda K^* y_\delta, \; m = 1, 2, \ldots, \end{cases} \tag{31}$$

where $\lambda > 0$. Using Theorem 2.19 of [13], we choose λ such that

$$0 < \lambda < \frac{1}{||K||^2} \tag{32}$$

and we consider the stopping test

$$||Kx_m - y_\delta||^2_{L^2(t_0,t_f)} \le r(e(\delta))^2,$$

for some $r > 0$ satisfying $||y_\delta|| \ge re(\delta)$ for all $\delta \in (0, \delta_0)$.

Remark The bound of $||K||$ in Proposition 3.2 states that if we choose λ satisfying

$$0 < \lambda < \frac{2}{(t_f - t_0)^2},$$

the condition (32) is fulfilled. $\square$

Remark Since $x_0(t_f) = 0$ and $K^*(x(t_f)) = 0$ for any function x, every iteration of Landweber method also satisfies $x_m(t_f) = 0$.

Again, the approximations obtained by this method will have this bad property. $\square$

3.3 Iterative algorithm

We present in this section the adaptation of the arguments in Section 2.3 to the current problem. Again, we collect the different ways to address the problem of identifying the coefficient H according to the information about the temperature, i.e., the corresponding scenario of the inverse problem. In all cases, we assume that the value of t_f is set as prescribed in Section 3.1 and, therefore, the direct problem is (24).

3.3.1 Identifying from a finite amount of exact values of temperature.

Given $n \in \mathbb{N}$, the values of the temperature T at $t_k = t_0 + kh$ for $k = 0, 1, \ldots, n$, are supposed to be known, where $h = \dfrac{t_{\mathrm{f}} - t_0}{n}$. Let us denote $T_k = T(t_k)$, $k = 0, 1, \ldots, n$. The differential equation of problem (24) can be rewritten as

$$\frac{T'(t)}{T^e - T(t)} = H(T(t)),\ t_0 < t < t_{\mathrm{f}}. \tag{33}$$

Therefore, our goal is to find, for $k = 0, 1, \ldots, n$, an approximation $\widetilde{H}_k$ of

$$\frac{T'(t_k)}{T^e - T(t_k)},$$

which is also an approximation of $H(T_k)$. We consider again the continuous approximate differentiation operator $R_h : \mathcal{C}([t_0, t_{\mathrm{f}}]) \to \mathcal{C}([t_0, t_{\mathrm{f}}])$ given in Section 2.3.1. In order to approach $H(T_k)$ we take

$$\widetilde{H}_k = \frac{R_h(T)(t_k)}{T^e - T_k},$$

for $k = 0, 1, \ldots, n$. Thus, Lemma 2.3, leads to the following estimate of error:

Proposition 3.5 *If $T \in \mathcal{C}^3([t_0, t_{\mathrm{f}}])$ then*

$$\max_{k=0,1,\ldots,n} \left| H(T_k) - \widetilde{H}_k \right| \leq \frac{29 M_3}{6\mu}\, h^2,$$

where $M_3 = ||T'''||$. $\square$

3.3.2 Identifying from a function that approximates the temperature.

In this context, we suppose to have a function $\widetilde{T} \in \mathcal{C}([t_0, t_{\mathrm{f}}])$, where t_{f} is chosen according to (22) and

$$\left|\left| T - \widetilde{T} \right|\right| < \delta \tag{34}$$

for some $\delta \in (0, \mu)$. For the sake of simplicity and consistency with the properties of T, we assume that $\widetilde{T}(t) \geq T_0,\ t \in [t_0, t_{\mathrm{f}}]$. From (33), we define

$$u(t) = \frac{T'(t)}{T^e - T(t)},\ t_0 < t < t_{\mathrm{f}}$$

and the approximation

$$\widetilde{u}_h(t) = \frac{R_h(\widetilde{T})(t)}{T^e - \widetilde{T}(t)},\ t_0 < t < t_{\mathrm{f}}.$$

Next, an error estimate is obtained:

Proposition 3.6 *If $T \in \mathcal{C}^3([t_0, t_{\mathrm{f}}])$ and $\widetilde{T} \in \mathcal{C}([t_0, t_{\mathrm{f}}])$ satisfies (34) with $0 < \delta < \dfrac{\mu}{2}$, then*

$$||u - \widetilde{u}_h|| \leq \frac{1}{\mu - 2\delta} \left(\frac{29 M_3}{6}\, h^2 + \frac{4\delta}{h}\, \frac{T^e - T_0 + \mu - 2\delta}{\mu - \delta} \right). \tag{35}$$

DEMOSTRACIÓN. It suffices to argue as in Proposition 2.5, taking into account estimates (23). $\square$

Remark It is interesting to analyze the estimate (35) in the context of regularization strategies, although the functional framework is different, since the regularization theory followed above has been posed in Hilbert spaces.

We consider the operator

$$K : \mathcal{C}([t_0, t_{\mathrm{f}}]) \to \mathcal{C}([t_0, t_{\mathrm{f}}])$$

defined as follows: for each $u \in \mathcal{C}([t_0, t_{\mathrm{f}}])$ we take $Ku = T$ where T is the solution of problem

$$\begin{cases} T'(t) = u(t)(T^e - T(t)), \ t \in [t_0, t_{\mathrm{f}}] \\ T(t_0) = T_0, \end{cases}$$

i.e., $T(t) = T^e - (T^e - T_0)e^{-\int_{t_0}^t u(s)\,ds}$. Now, we define the operator's family

$$\mathcal{R}_h : \mathcal{C}([t_0, t_{\mathrm{f}}]) \to \mathcal{C}([t_0, t_{\mathrm{f}}])$$

as

$$\mathcal{R}_h y = \frac{R_h(y)}{T^e - y}.$$

This family is a regularization strategy for the operator K; here h is the regularization parameter. Thus, the methods in this section can be considered in the framework of regularization theory but in the space of continuous function.

The term

$$\frac{1}{\mu - 2\delta}\left(\frac{29M_3}{6}h^2\right)$$

in (35) is the one corresponding to $||\mathcal{R}_h K u - u||$ in the standard inequality

$$||x_{\alpha,\delta} - x||_X \leq \delta\,||R_\alpha|| + ||R_\alpha \mathcal{K} x - x||_X$$

for the Hilbert framework (see [13, pag. 26]). The second term of the right hand side of (35) corresponds to the term $\delta\,||\mathcal{R}_h||$ in the above inequality.

Consequently, we choose the parameter h in terms of δ in order to minimice the bound in (35). □

The following result determines how to carry out this minimization:

Proposition 3.7 *Under the assumptions of Proposition 3.6, the minimum value for the right hand side of* (35) *is obtained for*

$$h^* = \left(\frac{12(T^e - T_0 + \mu - 2\delta)}{29(\mu - \delta)M_3}\,\delta\right)^{\frac{1}{3}}. \tag{36}$$

In this case, estimate (35) *becomes*

$$||u - \widetilde{u}_{h^*}|| \leq \frac{1}{\mu - 2\delta}\left(522M_3\frac{(T^e - T_0 + \mu - 2\delta)^2}{(\mu - \delta)^2}\,\delta^2\right)^{\frac{1}{3}}.$$

DEMOSTRACIÓN. It suffices to argue as in Proposition 2.6. □

Remark Again in the context of regularization theory, the above result proves that the family of operators $\mathcal{R}_h$ defined in Remark 3.3.2 is an admissible regularization strategy when h is taken according to (36). □

From Proposition 3.7, choosing h^* as in (36), taking n as the integer part of $\dfrac{t_{\mathrm{f}} - t_0}{h^*}$, denoting $t_k = t_0 + kh^*$, $\widetilde{T}_k = \widetilde{T}(t_k)$ and

$$\widetilde{H}_k = \widetilde{u}_{h^*}(t_k) = \frac{R_{h^*}(\widetilde{T})(t_k)}{T^e - \widetilde{T}_k}$$

for $k = 0, 1, \ldots, n$, we obtain the main result of this section:

Theorem 3.8 *If $H \in \mathcal{C}^1([T_0, T^e])$ and $\widetilde{T} \in \mathcal{C}([t_0, t_{\mathrm{f}}])$ satisfies (34) with $0 < \delta < \frac{\mu}{2}$, then*

$$\max_{k=0,1,\ldots,n} \left| H(\widetilde{T}_k) - \widetilde{H}_k \right| \leq \delta \, ||H'||_{\mathcal{C}([T_0,T^e])}$$
$$+ \frac{1}{\mu - 2\delta} \left(522 M_3 \frac{(T^e - T_0 + \mu - 2\delta)^2}{(\mu - \delta)^2} \, \delta^2 \right)^{\frac{1}{3}}. \quad \square$$

3.3.3 Identifying from a finite number of approximated values of the temperature.

We assume that the interpolation method used is such that the error δ between T and $\widetilde{T}$ is of the same order of magnitude as the measurement error $\widehat{\delta}$.

For example, if $\widetilde{T}$ is the piecewise linear interpolation of measurements $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ and we denote T_{int} the piecewise linear interpolation of values of T at points τ_k, the monotonicity of T provides

$$\left|\left| T - \widetilde{T} \right|\right| \leq ||T - T_{\text{int}}|| + \left|\left| T_{\text{int}} - \widetilde{T} \right|\right| \leq \max_{1 \leq k \leq p} |T(\tau_k) - T(\tau_{k-1})| + \widehat{\delta}$$
$$\leq \max_{1 \leq k \leq p} \left(|\widetilde{T}(\tau_k) - \widetilde{T}(\tau_{k-1})| + 2\widehat{\delta} \right) + \widehat{\delta} = \max_{1 \leq k \leq p} |\widehat{T}_k - \widehat{T}_{k-1}| + 3\widehat{\delta}.$$

Therefore, when the interpolation considered is the piecewise linear interpolation, if the difference between consecutive measurements is of order $\widehat{\delta}$, then δ and $\widehat{\delta}$ are of the same order of magnitude. The number of measurements will be increased if needed.

Remark In order to adapt to this scenario the analysis in Remark 3.3.2, it suffices to choose the operator's family $\mathcal{R}_h : \mathcal{C}([t_0, t_{\mathrm{f}}]) \to \mathcal{C}([t_0, t_{\mathrm{f}}])$ defined as

$$\mathcal{R}_h y = \frac{R_h(I_h(y))}{T^e - I_h(y)},$$

where I_h is the interpolation operator used for $\widetilde{T}$. If

$$\lim_{h \to 0} ||I_h(y) - y||_{\mathcal{C}([t_0, t_{\mathrm{f}}])} = 0 \ \text{ for all } y \in \mathcal{C}([t_0, t_{\mathrm{f}}])$$

(for example, if I_h is the piecewise linear interpolaction operator) then $\mathcal{R}_h$ is a regularization strategy for the operator K. $\square$

The algorithm proposed is an adaptation of the introduced in Section 2.3.3 with suitable modifications.

The input data are the temperature measurements $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$, the measurement error bound $\widehat{\delta}$ and the admissible threshold $\mu > 0$. The algorithm starts defining the function $\widetilde{T}$, interpolation of values $\{\widehat{T}_k\}_{k=0}^p$. Then, a bound of the interpolation error $\delta > 0$ is determined. Next, the final instant t_{f} is computed by using (22) with the appropriate value of μ.

After that, an iterative process starts from a guess value of time step h. Through it, we define the instants $t_k = t_0 + kh, k = 0, 1, \ldots, n$, where n is the integer part of $\frac{t_{\mathrm{f}} - t_0}{h}$. Then, the values $\widetilde{T}_k = \widetilde{T}(t_k)$ are obtained.

Next, an approximation Λ_3 of the maximum norm of the third derivative of temperature is computed as the maximum absolute value of quantities

$$\begin{cases} \dfrac{-5\widetilde{T}_k + 18\widetilde{T}_{k+1} - 24\widetilde{T}_{k+2} + 14\widetilde{T}_{k+3} - 3\widetilde{T}_{k+4}}{2h^3}, & k = 0, 1 \\ \dfrac{\widetilde{T}_{k+2} - 2\widetilde{T}_{k+1} + 2\widetilde{T}_{k-1} - \widetilde{T}_{k-2}}{2h^3}, & k = 2, 3, \ldots, n. \end{cases} \tag{37}$$

We note that the values $\widetilde{T}_{n+1} = \widetilde{T}(t_{n+1})$ and $\widetilde{T}_{n+2} = \widetilde{T}(t_{n+2})$ are well defined if the choice of t_{f} has enough measurements left out. Otherwise, we will use the same regressive formula as in (14).

From this value Λ_3 a new time step

$$h = \left(\frac{12(T^e - T_0 + \mu - 2\delta)}{29(\mu - \delta)\Lambda_3} \delta \right)^{\frac{1}{3}}, \tag{38}$$

is computed, and so on.

Again, the process stops when two consecutive values of h are close. From the final value of h, the corresponding instants t_k, interpolation $\widetilde{T}$ and the quotients

$$\widetilde{H}_k = \widetilde{u}_h(t_k) = \frac{R_h(\widetilde{T})(t_k)}{T^e - \widetilde{T}_k}, \tag{39}$$

are computed. These quantities approach the values of H in the temperatures $\widetilde{T}_k$, for $k = 0, 1, \ldots, n$.

Algorithm

DATA	$\{\widehat{T}_k\}_{k=0}^p$: Temperature measurements at times $\{\tau_k\}_{k=0}^p$.
	$\widehat{\delta} > 0$: bound of measurements errors.
	$\mu > 0$: threshold considered.
	ε: stopping test precision.
	h: guess value of h^*.
Step 1:	Determine $\widetilde{T}$ and δ according to $\widehat{\delta}$.
Step 2:	Compute t_{f} as in (22) adapting, if needed, the value of μ.
Step 3:	While the relative error in h is greater than ε:
	a) Determine the new discrete instants $\{t_k\}$ and compute $\{\widetilde{T}_k\}$.
	b) Compute Λ_3 as the maximum absolute value of (37).
	c) Compute the new value of h as in (38).
Step 4:	Obtain the final discrete instants $\{t_k\}$ and the values $\{\widetilde{T}_k\}$.
Step 5:	Compute the approximations $\widetilde{H}_k$ according to (39).

3.4 Numerical results. Comparison between the methods

Before describing the example on which the three algorithms studied have been tested, we make some considerations about the nondimensionalization of the problem.

3.4.1 On the nondimensional problem

By mimicking what was done in Section 2.4, we take the dimensionless temperature vanishing at the lower value of the original temperature (there T^e, here T_0). Now, we consider the dimensionless variables

$$t^* = \frac{t - t_0}{t_{\mathrm{f}} - t_0} \quad \text{and} \quad T^*(t^*) = \frac{T(t) - T_0}{T^e - T_0}.$$

By denoting

$$H^*(s) = (t_{\mathrm{f}} - t_0) H\big((T^e - T_0)s + T_0\big) \quad \big(\Rightarrow H^*(T^*(t^*)) = (t_{\mathrm{f}} - t_0) H(T(t))\big),$$

the equivalent nondimensional problem is given by

$$\begin{cases} \dfrac{dT^*}{dt^*}(t^*) = H^*(T^*(t^*))(1 - T^*(t^*)),\ t^* \in (0, 1) \\ T^*(0) = 0. \end{cases} \tag{40}$$

Note that problem (40), at a first glance, does not depend of any parameter; however, further analysis shows that this is not true. The identification of H should be used to find the temperature corresponding to different values of the initial (higher T_0) and ambient (lower T^e) temperatures. Without loss of generality, we can assume fixed the initial temperature; in addition, we take an external temperature $\mathcal{T}^e$ with $T_0 < \mathcal{T}^e < T^e$, and we assume that the evolution time reaches the value t_{f}. The nondimensional problem for this situation is

$$\left\{ \begin{array}{l} \dfrac{dT^*}{dt^*}(t^*) = H^*(T^*(t^*))(T^{ea} - T^*(t^*)),\ t^* \in (0,1) \\ T^*(0) = 0, \end{array} \right. \tag{41}$$

where

$$T^{ea} = \frac{\mathcal{T}^e - T_0}{T^e - T_0}.$$

So, the different situations can be described by solving the problem (41) for the values of parameter T^{ea} between zero and one.

3.4.2 Comparative study of the numerical results

Here we present the results obtained when solving an example, using the three methods described in this section applied to the nondimensional problem (40). The data for numerical tests have been obtained as in Section 2.5 by considering

$$H(s) = 2\exp(s).$$

Again, for ease of comparison, we have chosen the same seven perturbations of temperature and we have selected the two that produce the smallest and largest error in H. This error is measured in standard L^2 norm for the Morozov's principle and Landweber methods and in maximum norm for the iterative algorithm. We identify H with each of the three methods and we compute the corresponding temperature by taking $T^{ea} = 1$. Also, we solve the problem using this identification, shrinking the parameter T^{ea} by factors $d = 0.75, 0.50$ and $d = 0.25$. The solution computed this way is compared with the corresponding exact temperature (the error in temperature is always measured in maximum norm). The label "% Error" in the titles of figures, means the same as in Section 2.5.

We use piecewise linear interpolation for function $\widetilde{T}$. Every definite integral has been computed by the trapezoidal rule based on the measurements instants. Therefore, this calculations do not depend on the interpolation method used.

The initial value of threshold is chosen as $\mu = 0.2$. Once the function $\widetilde{T}$ is constructed, we correct this value as described in Section 3.1 in order to determine the final time t_{f} according to (22).

First method: Discrepancy principle of Morozov

For the Morozov discrepancy principle, in order to obtain the value of parameter α, we approximate the solution of (30) by applying the secant method to the function

$$F(\alpha) = ||Ku_{\alpha,\delta} - y_\delta||^2_{L^2(t_0,t_{\mathrm{f}})} - (e(\delta))^2,$$

where $u_{\alpha,\delta}$ is stated in Proposition 3.4 and y_δ and $e(\delta)$ are given in (26) and (27), respectively.

As we can seen in Figure 5, this method provides a large error in the identification of H (when comparing to the results showed in Figures 6 and 7). Moreover, this identification is not very good since the approximate temperatures are relatively far from the exacts with high relative errors (compare again to Figures 6 and 7).

Second method: Landweber iterative method

We have chosen $\lambda = \dfrac{0.9}{t_{\mathrm{f}}^2}$ in the iteration (31), according to Remark 3.2.2. For other values of this parameter (within the allowable range) the results are very similar. Figure 6 shows the results when applying Landweber method. This method exhibits a behavior slightly better than the previous one in the identification of H but much better in T, although they still appear not very small errors in the approximation of the temperature (compare to Figure 7).

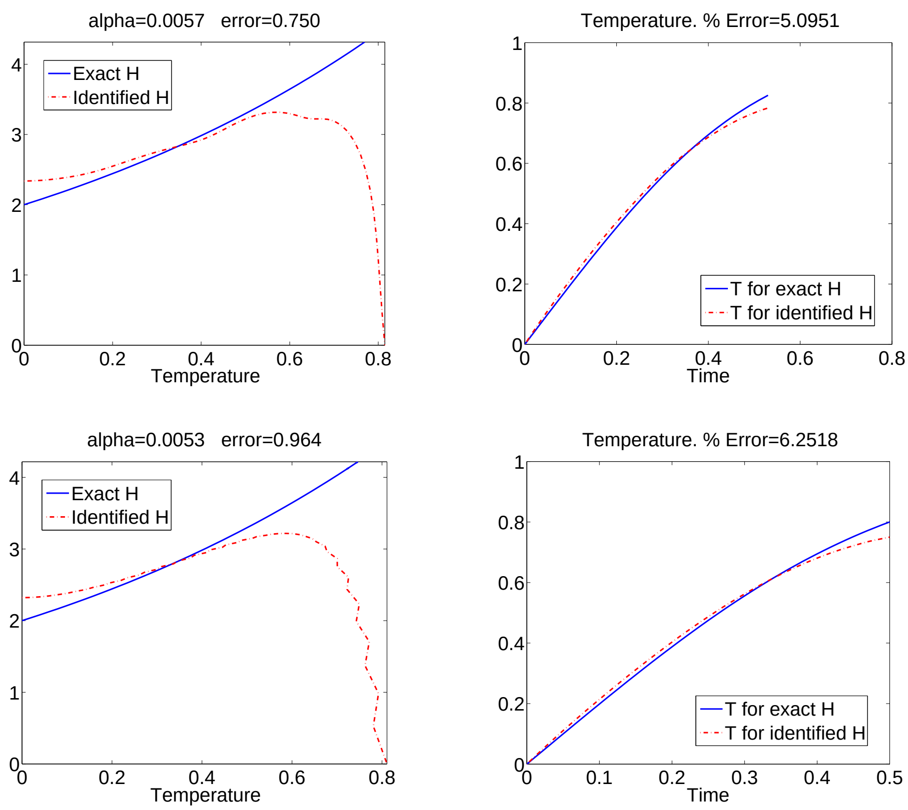

Figure 5: Morozov's discrepancy principle. (Top: smallest error in H. Bottom: largest error in H)

Third method: Iterative algorithm

Figure 7 suggests that this method provides very good results: the error in the identification of H is the same order of the error in Landweber method (each one measured in the corresponding norm) and the relative error in temperature is lower than 1% (size of the perturbations). Also, we remind that this method avoids the vanishing at t_{f} property of the previous ones (see Remark 3.2.1). Therefore, we can conclude that this algorithm is well adapted to the problem considered and improves the presented algorithms based on the Classical Theory, from a qualitative and quantitative point of view.

Table 3: % Error for the considered methods and the cases of smallest (above) and largest (below) error in H.

Method	Factor over parameter T^{ea}		
	$d = 0.75$	$d = 0.50$	$d = 0.25$
Morozov	2.89	3.74	5.27
Landweber	0.52	0.65	0.56
Iterative algorithm	0.25	0.31	0.56
Morozov	2.72	3.61	5.19
Landweber	0.72	0.88	0.71
Iterative algorithm	0.68	0.91	1.77

Table 3 shows the results obtained when the value of T^{ea} is reduced by a factor d. Clearly, the Morozov's discrepancy principle method provides much worse results than those obtained by the other two methods.

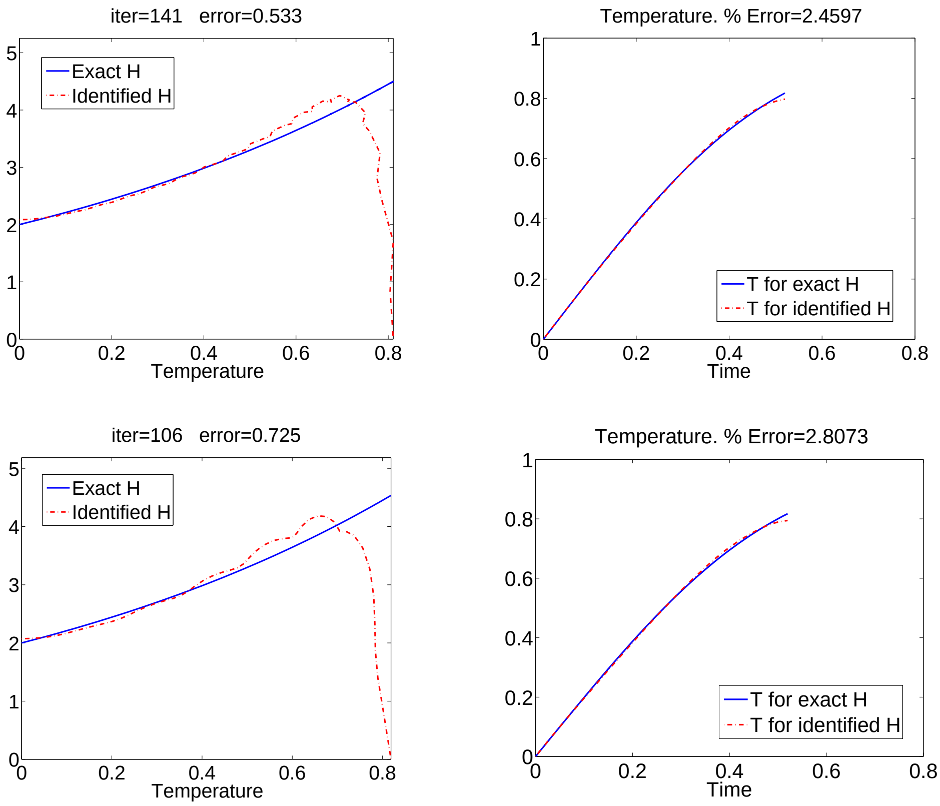

Figure 6: Landweber iterative method. (Top: smallest error in H. Bottom: largest error in H)

The errors in Landweber method and the iterative algorithm are of the same order of magnitude.

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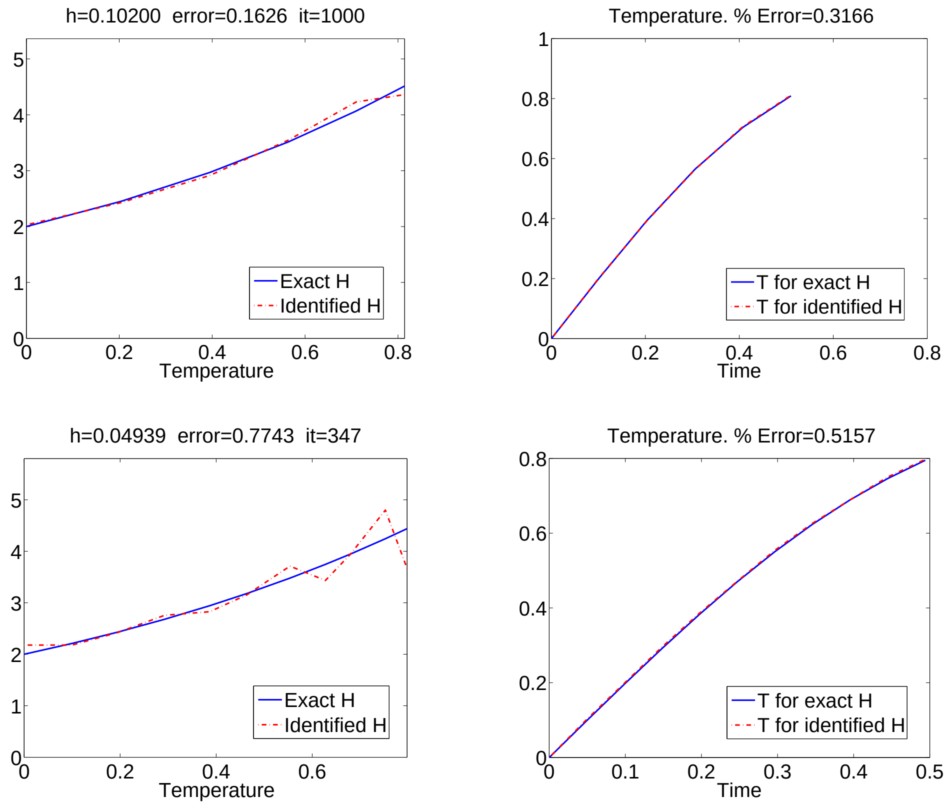

Figure 7: Iterative algorithm. (Top: smallest error in H. Bottom: largest error in H)

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