On an optimal control problem involving the location of a free boundary

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Abstract

We study an optimal control problem for a semilinear elliptic boundary value problem giving rise to a free boundary due the fact that the nonlinear reaction term of the state equation is not differentiable. The new aspect, with respect to other control problems involving free boundaries, is that here the cost functional explicitly depends on the location of the free boundary.

The problem is related to a simplified version of a model for the discharge of brine in the sea by desalination plants.

The main difficulty is to show the continuous dependence (in measure) of the free boundary with respect to the control function.

1 Introduction

We consider here the optimal control problem

$$\min_{u \in U_{\mathrm{ad}}} J(u)$$

where

$$J(u) = \int_\Omega \chi_{S(y(x;u))\cap B}(x)\, dx + \int_\Omega \frac{1}{G(y(x;u))}\, dx, \tag{1}$$

where χ_A is the characteristic function of a subset $A \subset \Omega$ ($\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in \Omega - A$). The state function $y(x;u)$ is the solution of the boundary value problem

$$\begin{cases} -Ly(x) + f(y(x)) = u(x)\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where L is an elliptic linear operator of the form

$$Ly = \sum_{ij=1}^{N} a_{ij}(x)\frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial y}{\partial x_i} \tag{3}$$

with $a_{ij}, b_i \in L^\infty(\Omega)$ such that there exist $\Lambda, \lambda \geq 0$, for which

$$\lambda|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{4}$$

The set $\Omega \subset \mathbb{R}^N$ is a bounded domain, ω (an open set) and B (a compact set) are two subset contained in Ω. The reaction nonlinearity is given by

$$f(t) = |t|^{q-1}t, \text{ for some } q \in (0,1) \tag{5}$$

which is a crucial fact in our study. The function G is a given real continuous increasing function such that $G(0) > 0$.

We shall use the following notation: if $y : \Omega \to \mathbb{R}$, we define $N(y) = \{x \in \Omega : y(x) = 0\}$ the null set of y, $S(y) = \{x \in \Omega : y(x) \neq 0\}$, and if $y(x)$ is a solution of (2) we set $\mathcal{F} = \partial S(u) \cap \Omega$, the free boundary of y.

Before introducing the set of admissible controls we need to present an important growth condition property which we shall impose over all the possible controls. We denote the set

$$S_u(R) = \{x \in S(u) : \operatorname{dist}(x, \partial S(u)) = R\}$$

for any $R > 0$, $R < \operatorname{diam}(\omega)$. As we shall show later (see Example 1), if the control $u(x)$ is very flat near the boundary of its support then the continuous dependence of the free boundary of the state solution may fails. This is the reason why we shall suppose that any admissible control satisfies the following condition: there exist $R, C > 0$ and $\gamma = \frac{2q}{1-q}$ such that for any $x_1 \in S_u(R)$

$$u(x) \geq C(R - |x - x_1|)^\gamma \quad \text{if } x \in B_R(x_1). \tag{6}$$

Note that the above condition implicitly implies some kind of constraint on the weak derivatives of u near the boundary of its support $\partial S(u)$. Indeed, if we take for instance $n = 1$, $x_0 \in \partial S(u)$ and $x_m \in B_R(x_1)$ (i.e. $|x_m - x_1| < R$) with $|x_0 - x_1| = R$, then (since $u^{1/\gamma}(x_0) = 0$) (6) implies that

$$\frac{u^{1/\gamma}(x_m) - u^{1/\gamma}(x_0)}{|x_m - x_0|} \geq C.$$

Passing to the limit, when $x_m \to x_0$, we get that necessarily $\frac{d}{dx}(u^{1/\gamma}(x_0)) \geq C$. This explains why we shall use, in what follows, some requirements on the derivatives of the controls in order to be able to ensure that condition (6) remains true for a control u which is the limit (in some suitable sense) of a sequence of admissible controls u_n satisfying each of them the associated property (6).

Given M, M^*, R_0, C_0 (C_0 given by (11)), the set of admissible controls we shall consider in this paper is defined by

$$\begin{aligned} U_{\text{ad}} = \{ & u \in H^1(\omega) \cap L^\infty(\omega) \,|\, 0 \leq u(x) \leq M, \, ||u||_{H^1(\omega)} \leq M^* \text{ and} \\ & \text{and } u \text{ satisfies (6) for some } R > R_0 \text{ and } C > C_0 \}. \end{aligned} \tag{7}$$

Theorem 1. *Under the above assumptions there exists at least one minimum of J in U_{ad}.*

We have to point out that the first term of the functional J is non trivial. In fact due to the presence of the non differentiable nonlinearity, it is well known that a dead core can be formed, and so the intersection $S(y) \cap B$ is not always equal to B (according to the properties of the control u). The existence of a non-empty null set of y in Ω, and so of the associated free boundary $\mathcal{F}$, is discussed in the monograph [12] where the reader can find many references dealing also with the existence and uniqueness of the state solution $y(x)$.

As mentioned in the abstract, the problem under consideration is related to desalination plants (see, for instance, Niepelt [18], Bleninger-Jirka [8], and Díaz, Sánchez, N. Sánchez, Veneros and Zarzo [14] as well as [7] and [16] for other discharges problems). The modelling of the state equation usually concerns with parabolic equations of the type

$$\begin{cases} \rho(\frac{\partial y}{\partial t} + \mathbf{v}\cdot\nabla y) - \Delta y + \lambda \left|y\right|^{p-1} y = u\chi_\omega & \text{on } \Omega \times (0,T), \\ y = 0 & \text{in } \partial\Omega \times (0,T), \\ y(0,x) = y_0(x) & \text{on } \Omega, \end{cases}$$

where $T > 0$ is arbitrary and $\mathbf{v}$ is assumed to be given (for instance, as solution of an uncoupled Navier-Stokes system). Here $p \in [0,1)$ is the order of the chemical reaction produced in the brine discharges to seawater and we suppose that $y_0 \in L^\infty(\Omega)$, $y_0 \geq 0$ and that $u \in L^\infty(0,T : L^\infty(\omega))$, $u(t,x) \geq 0$. Since the order of the chemical reaction is less than one, it is well known that some free boundary is formed corresponding to the boundary of the support of $y(t,. : u)$ (which we shall denote by $S(u(t,.))$ and corresponds to the plume discharges). Very often, the brine discharges (depending on the control of the brine flux $u(t,.)$ on a small open subregion of the spatial domain Ω) must obey to some regulations protecting some given subregion of Ω (corresponding, for instances to some beach or protected zones in the sea), reason why we include a cost functionals as the above mentioned. Notice that the study carried out in the present communication concerns the associated stationary problem.

Before giving the proof of our main theorem we have to introduce a central tool in our study: the continuous dependence of the free boundary with respect to the control. We end this communication with some numerical experiences concerning two different types of controls.

2 On the continuous dependence of the free boundary

The aim of this part is to prove the continuous dependence (in some sense to be determined later) of the support of the solution of (2) with respect to the data u. Let us recall first some properties of the solutions of (2). According to the theory of quasilinear elliptic equations, whenever u is bounded and non negative, the solution y belongs also to $L^\infty(\Omega)$, and due to the comparison principle it is also a non negative function. Concerning some additional regularity, we recall that in this case the solution belongs at least to $C^{1,\alpha}(\bar{\Omega})$ for some $0 \leq \alpha < 1$. For details we refer, for instance, to [12].

A curious property, studied in different papers, is the so called "non-diffusion" property of the support (see [12], [3], [2] and [4]), which under suitable hypothesis on u guarantees that the support of the solution coincides with that of the datum u. So, in these special cases, it is clear that we can control exactly the support of our solution just by considering the support of the data.

But here we are not interested in this very special case. Our aim is to control the state function under suitable hypothesis ensuring that the support of the solution is strictly larger than the one of the control.

The strict propagation of the support was studied initially in Álvarez-Díaz [3] and Álvarez [2] for the case $L = \Delta$ and later generalized in Álvarez-Díaz [4] to the case in which the second order linear operator L is replaced by the quasilinear operator Δ_p. We point out that in [4] the case of the generic linear elliptic operator L satisfying (4) was considered in the first part of the paper (the non-diffusion of the support). We present here the extension for the general operator L (see Theorem 3).

Our main idea in order to control the behavior of $\mathcal{F}(y)$ relies on the use of some non-degeneracy property of the solution near its free boundary in a way very close to the one followed by Álvarez and Díaz in [3]. To be more precise we want to prove the following result:

Theorem 2. *Let $u, u_n \in U_{ad}$, with $u_n \to u$ strongly in $L^2(\omega)$ and weakly star in $L^\infty(\omega)$, and let y_n and y be the solutions of the associated problems* (2)*. Then there exists a subsequence (still labeled as y_n) such that there exist $\varepsilon_0 > 0$ and $h_\infty : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $h_\infty(0) = 0$ such that for all $\varepsilon < \varepsilon_0$ and for any element of this subsequence*

$$|\{x : 0 < y_n(x) < \varepsilon\}| \leq h_\infty(\varepsilon). \tag{8}$$

To prove this result we need to divide the analysis on two different subsets, $S(u)$ and $\Omega \setminus S(u)$.
On $S(u)$ we use the family of subsolutions built in [4] to show the strict diffusion of the support of y and the non-degeneracy property (8) for y on this part of the domain. To do that we introduce the quantity

$$S = ess \sup_{\Omega} \left[\sum_i a_{ij}(x) - \sum_{i,j} a_{ij}(x) \frac{x_i x_j}{r^2} + \sum_i b_i(x) x_i \right], \tag{9}$$

which we know to be finite because the coefficients of L are bounded.

Theorem 3. *Let $u \in L^1_{loc}(\omega)$, $u \geq 0$, $x_0 \in \partial S(u) \cap \omega$ and $y \geq 0$ such that*

$$-L(y) + y^q \geq u \quad in\, \omega. \tag{10}$$

Let $1 \leq \delta \leq 1 + \lambda(\beta q + 1)/S$, then there exist $C, K_1, K_2, K_3 > 0$ such that if $\varepsilon > 0, x_1 \in \omega$ satisfy $\delta\varepsilon > |x_1 - x_0| \geq ((\delta + 1)/2)\varepsilon$, $B_\varepsilon(x_1) \subset \omega$ and

$$u(x) \geq C_0 |x - x_0|^{\beta q} \quad a.e.\ x \in B_\varepsilon(x_1). \tag{11}$$

Then

$$y(x) \geq \begin{cases} K_1\varepsilon^\beta - K_2|x-x_1|^\beta & if 0 \leq |x-x_1| \leq \varepsilon, \\ K_3(\delta\varepsilon - |x-x_1|)^\beta & if \varepsilon \leq ||x-x_1| \leq \delta\varepsilon, \end{cases}$$

in particular, $y > 0$ in $B_{(\delta\varepsilon - |x_1 - x_0|)}(x_0)$.

Idea of the Proof of Theorem 3. As in Álvarez-Díaz [3], [4] (and Álvarez [2]), it is enough to construct a local subsolution near the boundary of S(u). In particular, it can be checked that, by choosing suitable constants, such a subsolution can be built in the form

$$\theta(r) = \begin{cases} \theta_1(r) = K_1\varepsilon^\beta - K_2 r^\beta & 0 \leq r \leq \varepsilon, \\ \theta_2(r) = K_3(\delta - r)^\beta & \varepsilon \leq r \leq \delta\varepsilon. \end{cases}$$

The details will be given in a next publication by the authors.

Remark 1. A careful study of the influence of the convection velocity $\mathbf{v}(\mathbf{x})$ in the formation of the free boundary for the stationary problem (but with $y = 1$ at $\partial\Omega$ and without any control $u \equiv 0$) was carried out in Pinsky [22] and [23]. His results, when particularized to a ball, show the important difference between inward and outward pointing convection vector fields.

With Theorem 3 we obtain property (8) on $S(u)$ for all $u \in U_{\mathrm{ad}}$, which is the statement of the following theorem.

Theorem 4. *Let y be the solution of* (2)*, and assume that u satisfies* (6) *for suitable $R, C > 0$. Then for any compact $K \subset \Omega$ there exist $\varepsilon_0, k > 0$ such that*

$$|\{x \in K \cap S(u) \,:\, 0 < y(x) < \varepsilon^\beta\}| \leq k\varepsilon \tag{12}$$

for any $\varepsilon < \varepsilon_0$, with $\beta = \frac{2}{1-q}$.

Proof. It is similar to Lemma 2.2 of [2] and so we drop it. □

Now we pass to analyze the behavior of the solution on $\Omega \setminus S(u)$.

Theorem 5. *Let D be an open subset of Ω, $y \in W^{1,p}(D)$, for some $p \geq 1$, $y \geq 0$, such that $y(x)$ satisfies $-Ly + y^q = 0$, with $q \in (0,1)$, in a weak sense on D. Then there exist ε_0 and $h : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $h(0) = 0$ such that for all $\varepsilon < \varepsilon_0$*

$$|\{x \in D : 0 < y(x) < \varepsilon\}| \leq h(\varepsilon). \tag{13}$$

Proof. By the Fleming-Rishel-Federer formula (see, e.g., Rakotoson [24] Proposition 6.2.2) we know that if we define the function of distribution of y by

$$m_y(t) := |\{x \in D : t < y(x)\}|$$

and if we define

$$m_{o,y}(t) := |\{x \in D : t < y(x), \nabla y(x) = 0\}|$$

then the function

$$m_{1,y}(t) := m_y(t) - m_{o,y}(t) \tag{14}$$

is absolutely continuous on $\mathbb{R}$. But, thanks to the assumptions on the coefficients of L, and since $q \in (0,1)$, we know (by the Agmon-Douglas -Nirenberg regularity result) that $y \in W^{2,p}_{loc}(D)$ and so, by Lemma A.4 of Kinderlehrer-Stampacchia [17], if the subset $\{x \in D : \nabla y(x) = 0\}$ has a positive measure then $Lu = 0$ a.e. on this set. Thus, since $Ly = y^q$ a.e. on D, we deduce that necessarily $\{x \in D : \nabla y(x) = 0\} \subset \{x \in D : y(x) = 0\}$. In other words, $m_{o,y}(t) := |\{x \in D : y(x) < t, \nabla y(x) = 0\}| = 0$ for any $t \geq 0$. Thus, (14) implies that $m_y(t)$ is absolutely continuous on $[0, +\infty)$ and so

$$m_{1,y}(t+\varepsilon) - m_{1,y}(t) = \int_t^{t+\varepsilon} (\int_{\{u=s, \nabla y(x) \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y(x)|}) ds, \text{ for any } t \in [0, +\infty).$$

Finally, it suffices to note that

$$|\{x \in D : 0 < y(x) < \varepsilon\}| = m_{1,y}(\varepsilon) - m_{1,y}(0)$$

and to take

$$h(\varepsilon) := \int_0^{\varepsilon} (\int_{\{u=s, \nabla y(x) \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y(x)|}) ds.$$

□

Remark 2. Notice that the conclusion (13) is ensured merely on the subset where the control vanishes.

Remark 3. The above theorem extends (in different senses) many previous results in the literature. For instance, in the special case of $L = \Delta$, a stronger property was obtained firstly in Caffarelli [10] for the obstacle problem (q=0) and then in Phillips [21] and Alt-Phillips [5] for $0 < q < 1$: it was shown there that the property holds with $h(\varepsilon) = \varepsilon^{2/(1-q)}$. Notice that this is equivalent to say that the function of distribution of y, $m_y(t)$, is Hölder continuous near $t = 0$. Our results is weaker in this sense (although it is enough for our purposes) but it is more general since it applies to the general operators L under the assumptions indicated above. We point out that the deep local study made in Phillips [21] and Alt-Phillips [5] leads to many other qualitative information on y, but it requires sharp properties on the elliptic operator and so it seems difficult to extend this approach to the case of a general operator L.

Theorem 6. *Let $u_n \to u$ in $L^2(\Omega)$ and weakly star in $L^\infty(\omega), u_n \geq 0$, and let y_n and y the solutions of the associated problems (2). Then there exists a subsequence (still labeled as y_n) such that $y_n \to y$ in $W^{2,p}(\Omega)$ for any $p \in [1, +\infty)$, y is a co-area regular function in $N(u) \cap S(y)$ (in the sense of [1]) and, in particular, there exist ε_0 and $\bar{h} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $\bar{h}(0) = 0$ such that for all $\varepsilon < \varepsilon_0$ and for any n of this subsequence*

$$|\{x \in N(u_n) : 0 < y_n(x) < \varepsilon\}| \leq \bar{h}(\varepsilon). \tag{15}$$

Proof. Since $u_n(x)\chi_\omega - f(y_n(x))$ are uniformly bounded, by the Agmon-Douglas-Nirenberg regularity result for linear operators (as L) we know that $y_n \to y$ in $W^{2,p}(\Omega)$ (the inverse of the operator L, with zero Dirichlet boundary conditions, is a compact operator from $L^p(\Omega)$ into $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$). Moreover, from the monotonicity of the nonlinear term y^q and Sobolev inequalities we know that we know $y_n \to y$ in $L^\infty(\Omega)$ and so y is the solution corresponding to the limit control u. As in the previous Theorem, y_n and y are co-area regular functions in $N(u_n) \cap S(y_n)$ and $N(u) \cap S(y)$ respectively (since, e.g. $|\{x \in N(u) \cap S(y) : y(x) = t \text{ and } \nabla y(x) = 0\}| = 0$). Then we know (see [1] and the presentation made in [24]) that *length of* $\{y_n(x) = t\} \to$*length of* $\{y(x) = t\}$, once $t \in (0, \varepsilon_0)$, and since $|\nabla y_n(x)| \to |\nabla y(x)|$ uniformly in any compact of Ω and $\int_0^\varepsilon (\int_{\{u=s,\nabla y(x)\neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y(x)|})ds < \infty$, for any $\varepsilon < \varepsilon_0$, we deduce that $\exists\, \bar{h} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $\bar{h}(0) = 0$ such that

$$h_n(\varepsilon) = \int_0^\varepsilon (\int_{\{u_n=s,\nabla y_n(x)\neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y_n(x)|})ds \leq \bar{h}(\varepsilon).$$

for all $\varepsilon < \varepsilon_0$ and for any n of this subsequence. □

We want to give now a simple example showing how the non-degeneracy condition (6) is, in some sense, optimal if we want to be able to have the continuous dependence of the support with respect to the data. In other words, we cannot expect it without a condition of the type (6).

Example 1. Let us consider the one dimensional case

$$\begin{cases} -\varphi''(r) + \varphi^q(r) = u(r) \quad r \in (-2,2), \\ \varphi(-2) = \varphi(2) = 0. \\ \end{cases}$$

We set

$$\varphi_\varepsilon(r) = \begin{cases} 0 & r \in (1,2) \\ e^{-\frac{1}{1-r}} & r \in (1-\varepsilon, 1) \\ C_2 - C_1 & r \in (0, 1-\varepsilon) \\ \end{cases}$$

and define $\varphi_\varepsilon(r)$ by reflection on the interval $(-2,0)$. The constants C_1 and C_2 have to be chosen so as to make $\varphi_\varepsilon \in C^1(-2,2)$, which means

$$C_1 = e^{-\frac{1}{\varepsilon}} \frac{1}{2\varepsilon^2(1-\varepsilon)}, \quad C_2 = e^{-\frac{1}{\varepsilon}} + e^{-\frac{1}{\varepsilon}} \frac{1-\varepsilon}{2\varepsilon^2}.$$

We want to check now that these functions satisfy

$$-\varphi_\varepsilon'' + \varphi_\varepsilon^q \geq 0. \tag{16}$$

On the interval $[0, 1-\varepsilon)$ the functions are concave and positive and the result follows. On $(1-\varepsilon, 1)$ the behavior is the same of $e^{-\frac{1}{1-r}}$. In this case

$$-\varphi_\varepsilon''(r) + \varphi_\varepsilon^q(r) = e^{-\frac{1}{1-r}} \left[\frac{2}{(1-r)^3} - \frac{1}{(1-r)^4} \right] \geq 0$$

for $1-r < 1-r_0$ for some $r_0 > 0$. So if we take $\varepsilon < r_0$ we obtain (16) on the whole interval. Now it easy to check that $u_\varepsilon := -\varphi_\varepsilon'' + \varphi_\varepsilon^q \to 0$ uniformly as $\varepsilon \downarrow 0$, and that $\varphi_\varepsilon \to 0$. Nevertheless $S(\varphi_\varepsilon) = (-1,1)$, for any $\varepsilon < r_0$, and so there is not continuous dependence of the free boundary. We point out that condition (12) is not satisfied by the family φ_ε.

Let us show now how we can prove that the support depends continuously (in measure) on u by using this kind of non-degeneracy property.

Lemma 1. *Let $\{y_n\}$ converging in $L^\infty(\Omega)$ to y. Suppose that the following non-degeneracy property holds uniformly for all $n \in \mathbb{N}$: there exist $\varepsilon_0 > 0$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $lim_{t\to 0} h(t) = 0$ and*

$$|\{x \in \Omega \,:\, |y_n(x)| < \varepsilon\}| \leq h(\varepsilon) \quad \forall \varepsilon < \varepsilon_0. \tag{17}$$

Then $|N(y_n) \div N(y)| \to 0$, where $\div$ stands for the symmetric difference of two sets, i.e.

$$N(y_n) \div N(y) = (N(y_n) \setminus N(y)) \cup (N(y) \setminus N(y_n)) \,.$$

Proof. Let us consider the case of $N(y) \setminus N(y_n) = N(y) \cap S(y_n)$, and let $\varepsilon < \varepsilon_0$. For n sufficiently large we know that $|y_n(x) - y(x)| < \varepsilon$, for a.e. $x \in \Omega$, and hence $|y_n(x)| < \varepsilon$ a.e on $N(y)$. But due to the non-degeneracy property (17) we have that $|\{|y_n < \varepsilon|\}| \leq h(\varepsilon)$, and so, we conclude that

$$|N(y) \cap S(y_n)| \leq |\{|y_n| < \varepsilon\}| < h(\varepsilon) \quad \forall n \geq n(\varepsilon).$$

Hence, letting $\varepsilon \to 0$ and using the convergence to zero of h we obtain that $|N(y) \cap S(y_n)| \to 0$ as $n \to \infty$.

The proof that $|N(y_n) \setminus N(y)|$ goes to zero follows similar arguments. □

It is clear that we can divide the study of the continuous dependence in measure of the support in two different cases: we are just interested in the support of the solution restricted to a compact subset of Ω; we are interested to its behavior on the whole Ω. The first case is the simplest one and we have already all the instruments to state a result. The second one is much more difficult. We need some further hypothesis on the data.

Proof of Theorem 2. By well known results on the continuous dependence of the solution with respect to the data we obtain that $y_n \to y$ in $L^\infty(D)$. Combining Theorem 6 and Theorem 4 and applying Lemma 1 with $h_\infty(t) = \sup(\bar{h}(t), k\varepsilon^{1/\beta})$, we obtain for a subsequence of y_n (still denoted with y_n) that $(N(y_n) \div N(y)) \cap D \to 0$ in measure. □

To handle the case with the whole Ω, we know that all the solutions of problem (2) related to the family of control $U_{\rm ad}$ satisfy $||y||_{L^\infty(\Omega)} \leq Y$ for some $Y > 0$ (take, for instance, $Y = M^{1/q}$). If we assume Y to be sufficiently small we can suppose that all the supports are contained in the same compact in Ω. In fact from Díaz [12] we have that if $||y||_{L^\infty(\Omega)} \leq Y$, then

$$N(y) \supset \{x \in N(u) \ : \ \mathrm{dist}\,(x, S(u)) \geq \varepsilon + W(\varepsilon) : \text{for some} : \varepsilon > 0\} \tag{18}$$

where the constante $W(\varepsilon)$ depends on the L^∞ norm of y. Of course this condition makes sense when

$$\mathrm{dist}\,(\partial S(u), \partial\Omega) > \varepsilon + W(\varepsilon). \tag{19}$$

At this point we apply Theorem 2 with the set D given by the one which contains all the supports of the sequence of solutions and we obtain, in this way, the global continuity in measure of the support.

Now we are ready to give the proof of Theorem 1.

Proof of Theorem 1. Let $\{u_n\} \subset U_{\mathrm{ad}}$ be a minimizing sequence for J. As U_{ad} is bounded in $H^1(\omega) \cap L^\infty(\omega)$ there exists a subsequence (which we still denote with $\{u_n\}$) which converges weakly in $H^1(\omega)$ and weakly star in $L^\infty(\omega)$ to a function u, and hence (passing to another subsequence) also strongly in $L^2(\omega)$. Hence, from the convergence a.e. we obtain that u satisfies the condition (6). Hence u belongs to U_{ad}.

We will check now the continuity of J with respect to the $L^2(\omega)$ norm. The function $\frac{1}{G(y)}$ is uniformly bounded for all $0 \leq y \in L^2(\Omega)$ because $G(0) > 0$ and it is increasing. So, using the Lebesgue dominate convergence theorem, the functional

$$u \mapsto \int_\Omega \frac{1}{G(y(x;u))}\, dx$$

is continuous in $L^2(\omega)$. From the previous results we already know that

$$\int_\Omega \chi_{S(y(x;u)) \cap B}(x)\, dx$$

is continuous in $L^2(\omega)$ (again passing to a subsequence) and so, finally, we obtain that

$$J(u_n) \to J(u) = \min .$$

□

We point out that if we let $q = 0$ we end up with a variational inequality of the type of the obstacle problem. In this case the nonlinearity must be understood in the sense of multivalued maximal monotone operators i.e.

$$f(x) = \begin{cases} 1 & x > 0, \\ & x = 0, \\ 0 & x < 0. \end{cases}$$

The associated problem is relevant in many applications and have been intensively studied in the literature (see, e.g. [10]). Also many associated optimal control problems have been considered by many authors but in most of them for other kinds of cost functionals which do not involve explicitly the location of the free boundary (see, for instance, [6], [26], [25] and their references).

3 On some numerical experiences

We present here some numerical experiences: a research direction which is under the present consideration by the authors. Here we shall limit ourselves to a starting step of our study in which we get the comparison of the cost functional to different indicative controls. A deeper numerical approach of the optimal control requires more sophisticated tools due to the presence of the location of the free boundary at the cost functional (for some alternative techniques on the study of the approximate controllability of some related problems see [13]).

We start by simplifying the formulation of the problem in several ways. As spacial domains we shall take $\Omega = (0,4)$ and. $B = (3,4)$. As usual in the numerical approximation of free boundary problems (see, e.g., [9], [20] and, specially [19]) we approximate the reaction term $f(y)$ by a C^1 function $f_\delta(y)$, with $f'_\delta(y) \nearrow +\infty$ if $\delta \searrow 0$, and such that $f(y) = f_\delta(y)$ if $\delta \leq y$. In our case, we simply take $\delta = 1/1000$ and

$$f_\delta(y) = \arctan(1000y)$$

(a closer approximation of the function $f(t) = |t|^{q-1}t$, for some $q \in (0,1)$ does not modify strongly our qualitative commentary below).

We want to obtain some insights about the optimal control

$$\min_{u \in U_{\mathrm{ad}}} J_\lambda(u),$$

where

$$J_\lambda(u) = J_1(y(u)) + \frac{1}{\lambda} J_2(y(u)),$$

with

$$J_1(u) = \textstyle\int_\Omega \chi_{S(y(x;u)) \cap B}(x)\mathrm{d}x \quad \text{and} \quad J_2(u) = \int_\Omega \frac{1}{G(y(x;u))}\mathrm{d}x.$$

Note that the role of $\lambda > 0$ is to give some different relative weight to the penalty term (obeying to some regulations protecting subregion B of Ω) represented by $J_1(y(u))$ in comparison to the pure economic losses represented by $J_2(y(u))$.We shall consider the special case of

$$G(s) = s + 1.$$

Concerning the state equation we shall take $\omega = (1,2)$ and consider the equation associated to the numerical approximation on an original one (which is not relevant in our case):

$$\begin{cases} -y"(x) + f_\delta(y(x)) = u(x)\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

As in the above mentioned references, we shall formally identify the support of the solution $y(x;u)$ with the set of points in which it is sufficiently small: for instance

$$S(y(x;u)) \approx S_\delta(y(x;u)) := \{x \in \Omega : \delta \leq y(x;u)\},$$

(since we know that under suitable conditions $S_\delta(y(x;u)) \to S(y(x;u))$ as $\delta \searrow 0$ in some sense).

Before to study the cost functional for two different families of controls, let us mention that, $J_1(u)$ is increasing (as $S_\delta(y(x;u)) \cap B$ increases) and that

$$0 \leq J_1(u) \leq 1.$$

Analogously, at least formally, $J_2(u)$ is decreasing (as $y(x;u)$ increases) and that

$$0 \leq J_2(u) \leq 4.$$

Experience 1. *Controls very concentrated near its support (coinciding with* ω*).*

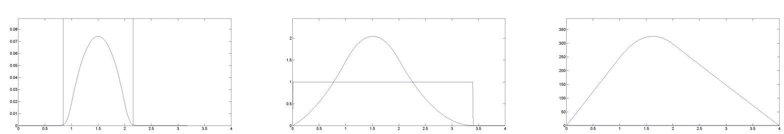
Let us consider

$$u(x)\chi_\omega(x) = \begin{cases} k & \text{if } x \in (1,2), \\ 0 & \text{if } x \notin (1,2), \end{cases}$$

for different values of $k > 0$. The numerical results have been obtained with COMSOL Multiphysics: in particular, for k very large the set $S_\delta(y(x;u))$ is practically the whole domain Ω which explains the approximate value $J_1 = 1$ in that case). The results are shown in the following table for $\lambda = 1$ (it is quite easy to play with different values of λ but we shall not do that here).

	J_1	J_2	J $(\lambda = 1)$
$k = 2$	0	1.978173132097462	1.978173132097462
$k = 6$	0.388888885000000	1.321929984523549	1.710818869523549
$k = 400$	1	0.039256384523821	1.039256384523821

The numerical approximations of the state solutions (for $k = 2, k = 6$, and $k = 400$) are presented in Figure 1 where we have pointed the dilatation of the support of the solution with respect the support of the control.



(a) k=2 (b) k=6 (c) k=400

Figure 1: Experience 1

We remark that, curiously, the cases in which $J_1 = 0$ (no penalty at all) does not correspond with slower values of the total cost functional J and that a larger invasion of the protected zone B may leave to a smaller value of J.
Experience 2. *Controls moderately concentrated near its support (strictly included in ω).*

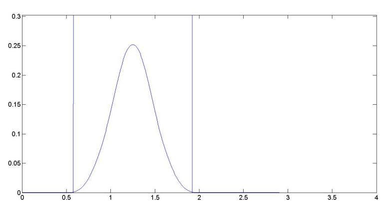
We consider now the case of controls of the form

$$u(x)\chi_\omega(x) = \begin{cases} K(1,5-x)(x-1) & \text{if } x \in (1, 1.5), \\ 0 & \text{if } x \notin (1, 1.5), \end{cases}$$

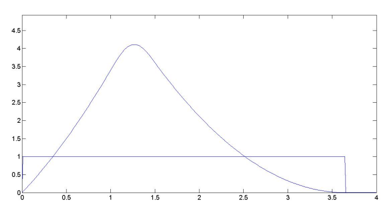
for different values of $K > 0$. The results are shown in the following table for $\lambda = 1$.

	J_1	J_2	$J\ (\lambda = 1)$
$K = 100$	0	1.943898475546399	1.943898475546399
$K = 400$	0.644444438000000	1.077663516211982	1.722107954211982
$K = 4000$	1	0.133968214033650	1.133968214033650

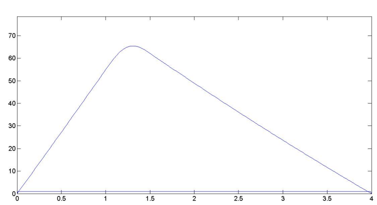
The numerical approximations of the state solutions (for $K = 100, K = 400$, and $K = 4000$) are presented in Figure 2, where we have pointed the dilatation of the support of the solution with respect the support of the control.



(a) K=100



(b) K=400



(c) K=4000

Figure 2: Experience 2

As before, the cases in which $J_1 = 0$ does not correspond with slower values of the total cost functional J . Moreover for $K = 4000$ the invasion of the pollution is total ($J_1 = 1$).

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