TEACHING AND LEARNING MATHEMATICS:

FROM SECONDARY SCHOOL TO UNIVERSITY[1]

Roberto Rodríguez del Río
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain
www.mat.ucm.es/~rrdelrio/
rrdelrio@ucm.es

IES San Mateo
Beneficencia, 4
28004 Madrid, Spain

Enrique Zuazua
BCAM – Basque Center for Applied Mathematics
Mazarredo, 14
48009 Bilbao, Basque Country, Spain
www.bcamath.org/zuazua
zuazua@bcamath.org

Ikerbasque – Basque Foundation for Science
Alameda Urquijo, 36-5, Plaza Bizkaia
48011 Bilbao, Basque Country, Spain

1. Introduction: key points for an important debate

Mathematics is not one of the main concerns in citizen lives, despite the will of many mathematicians and Mathematics teachers. However, most people have some contact with the discipline at some point in their lives, and, virtually, everybody agrees that some basic Mathematics knowledge is necessary to get by in day-to-day existence. Needless to say, Mathematics is, with no doubt, the one subject that creates at the same time enthusiasm and dissatisfaction in schools.

Mathematics is an ancient Science. It existed long before it acquired a name and its origins go back at least to the time when human beings learned to count. It is also worth mentioning, as Galileo once said, that the Universe is written in mathematical language[2],

[1] *Basically, this is a revised and updated version of the article by the same authors, "Teaching and Learning Mathematics", published in the MEC Journal of Education, n° 329 (2002), pp. 239-256. (http://goo.gl/GzT8xF)*

[2] *"Philosophy is written in this grand book – I mean the universe - which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it, without these, one is wandering around in a dark labyrinth."* Galileo Galilei, The Assayer.

thus we might say that Mathematics arose simultaneously with the emergence of our Universe. Without going back too far, at the beginning of the last century Albert Einstein asked himself the following question: "*How is it possible that Mathematics, a product of human thought, independent of experience, adapts itself so admirably to the objects of reality?*"

The debate about the role that Mathematics should play in our educational system and in society, is not new. The year 2000, declared The International Year of Mathematics by UNESCO, this debate was addressed again, and since then it has been the purpose of diverse initiatives in various fields. In Spain we could mention the following ones:

- The *Reales Decretos del MECD* [3] (Royal Decrees of the Spanish Ministry of Education, Culture and Sport), in which the reform of compulsory education in Secondary Schools and in the Baccalaureate was addressed.

- The Commission for Education, Culture and Sports a study group[4] was set up on the situation of scientific education in Secondary schools.

- The mathematical societies addressed this subject in processes of debate from which emerged some statements on the situation of mathematics in non-university education. See for example that of the *Real Sociedad de Matemática Española* (RSME – Royal Spanish Mathematical Society)[5].

- The *Ley Orgánica de la Calidad de la Enseñanza* (LOCE – Organic Law for Quality in Education) and the proposed *Ley Orgánica de Educación* (LOE - Organic Law for Education), whose aims among others are to achieve a substantial improvement in the teaching of Mathematics.

When addressing the subject of Mathematics in education, it is perhaps necessary to take into account the following aspects, which make Mathematics such a distinctive discipline:

1.- The omnipresence of Mathematics: we referred above to Galileo's remark about Mathematics being the language of the Universe. The history of humanity, and in particular the development of Science and Technology, have only underscored the accuracy of his

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[3] The R.D. 3473/2000, of December 29th on minimum requirements in Compulsory Secondary Education, and the R.D. 3474/2000, of December 29th on minimum requirements in the Baccalaureate. The *Ley Orgánica* 10/2002 of December 23rd on the Quality of Education.

[4] The work and provisional conclusions in the study report can be found at the website of the *Real Sociedad Matemática Española* (RSME), http://www.rsme.es

[5] A report on mathematical education drawn up by the Education Commission can also be found at the RSME website.

vision. In the other hand, the role that Mathematics plays in our society is becoming increasingly ubiquitous and multi-faceted: particularly in telecommunications, finance, computer science, medicine, biotechnology, etc, not to mention of the classical fields of Engineering. In addition, Mathematics forms part of our educational system, from the moment the child enters School until he or she concludes Secondary Education. Therefore we are in contact with Mathematics for over a period of at least ten years; and when we think we have left Mathematics behind for good, we frequently come into contact with it again because of our own children.

2.- *Is such a prolonged contact with Mathematics necessary?* One might be tempted to think that all this Mathematics presented within the early years of our education is largely the result of the perversity of mathematicians and their historical ability to influence those whose job is to lay down the guidelines of the educational system. But that is not the case. With language, Mathematics constitutes one of the two mainstays on which the entire educational process of children is based. In all civilizations, an understanding of the world and the nature of the processes and interactions that occur all pass through Mathematics. Indeed, as Roger Bacon said one day: “Without Mathematics, the Sciences cannot be understood, cannot be taught and cannot be learned”. Therefore, we have no choice but to learn numbers, operations, metric systems, the rule of three, solving simple geometric equations and many more, all of which require time and effort. Above all, these stages cannot be burned like bridges behind us; it is necessary to move forward by gradually increasing the degree of complexity of these concepts, returning to them time and again, thereby acquiring a deeper and more firmly understanding.

3.- *Is there any reason today for speaking about these things?* Mathematics is an ancient science that has been taught and learned over many centuries. Shouldn’t we have already resolved the problem of how it is taught? Unfortunately, that is not the case. One may go further and confidently state that the teaching of mathematics is suffering a crisis at every level. Crises (“moments at which a radical change in something occurs”, according to the María Moliner Dictionary) may also have their positive side, since they provide an exceptional occasion for improvement as well as causing us to redouble our efforts. There is no doubt that we are currently experiencing some intense and radical changes in the teaching of Mathematics.

4.- *How does this crisis affect us?* Secondary education has suffered multiple ~~many~~ changes. Its universalization and obligatory extension until the age of sixteen, which resulted on social achievement, have inevitably modified its outlook. Mathematics has undoubtedly been one of the subjects in which these changes have been most keenly felt. It is difficult, even in universities, to maintain the same programs followed until recently, and it is here where possibly these changes to which we refer affect us all; especially those teenagers who are studying in *Enseñanza Secundaria Obligatoria* (ESO – Compulsory Secondary Education).

5.- *Is it a bad idea to teach less Mathematics?* Quite a few might respond this way to the question posed above, and with justification. It may not seem important if students were to learn less Mathematics, due to the fact that as they now know more about computer science, and can handle a mobile phone with an ease that makes many adults envious. Although knowledge occupies no space, it does require time. so it is surprising that mathematics should come out losing. However, on this point, as professional mathematicians we should strive to ensure that everything essential should remain intact. The basic rules and concepts should be learned and inwardly digested. It does not make sense to resign oneself to a society in which individuals can handle sophisticated technological devices, but at the same time find it difficult to work out how much cake remains after cutting off a third or a quarter.

6.- *Do the most able and motivated students in Mathematics get everything they deserve out of our educational system?* On this issue the outlook is rather bleak, which in particular explains the symptoms that can be perceived in the Universities. However, it is not difficult to understand what is happening; if Mathematics must be within the reach of everyone, then it necessarily has to be simpler. But that does not mean that it should not therefore be easier, and it is a mistake to think that Mathematics can be conveyed without effort.

These questions provide an outline of some of the problems that arise when we speak of Mathematics and its teaching in Secondary Schools and Universities. When addressing this issue of the teaching of Mathematics in Secondary Schools and Universities, most people, if not everyone, all have an opinion on the matter, either divergent or in agreement (which does not mean that it is any easier to put these ideas into practice or make any tangible reforms or initiatives to change and improve reality). We should not forget that our educational system is extremely static, and while this inertia is indispensable for providing it with the necessary robustness and stability, it also absorbs much of the energy invested in debate and processes of change that find no outcome in terms of significant practical changes in the classrooms.

This discussion of ideas has brought to light the existence of at least two schools of thought which, at the risk of falling into an excessive simplification, may be described in the following way:

- One school of thought consisting of those who believe that, despite the complexity of the situation, the main point is still the maintenance and adaptation of the necessary contents in order to guarantee a satisfactory mathematical education; while at the same time by legislation enabling students to design their own itineraries in which both basic standards are maintained and the possibility of acquiring a deeper understanding of Mathematics they required is available. Obviously, this latter point will in particular be determined by the students' own inclination and ability in the discipline as well as the future career they wish to pursue.

- The other school of thought places greater emphasis on didactical aspects and believes that the main source of problems in the teaching of Mathematics resides in the excess and complexity of the curriculum, and therefore solutions must be based on their reduction; in addition to stepping up the didactical training of teachers. In other words, those who believe that the reason why students have problems with Mathematics is, on the one hand, because Mathematicians themselves have made the curriculum unnecessarily complicated, and on the other, because teachers have failed to acquire enough teaching skills to convey an understanding of Mathematical techniques clearly.

Before we proceed, and with the aim of leaving no doubt about our position, which provides the main thread of this article, we feel bound to say that we feel much closer to the first school of thought than the second. Indeed, while recognizing that any improvement in methodology is to be welcomed (for example, we remark below on the role played by computers in the classroom) and the necessity to avoid a curriculum too baroque in style (we will come back to this point), we believe that it is important to maintain clear targets in basic mathematics, which depart very little from the traditional approach. We also believe that the most effective way to reach these targets is the customary approach, which consists of clear explanations given by teachers and individual work performed by students. The two best formulas are: to ensure the best mathematical training of teachers, and to have students work on the different topics by means of solving problems, which has always been the approach in the past. On the other hand, it is difficult for us to accept that, on the basis of new didactical methodologies employed by different teaching bodies, students can be taught how to work with fractions in the same way as, for example, the different sections of a river are described, from where it rises to where it flows into the sea, or the water cycle, from the seas and oceans and back again.

In the previous legislature, and in parallel with this debate at the heart of the mathematical community, the Spanish Ministry of Education acted decisively to undertake a series of projects aimed to substantially improve the quality of our educational system. The last of these measures was the aforementioned Organic Law for Quality in Education (LOCE), which is currently in force, although only in virtual terms, since its application has been suspended in the present legislature, which has decided to continue with the previous legislation. But, we are presently in a period in which a new law for education is under debate, known as the Organic Law for Education (LOE), and which is currently subject to ratification by parliament.

Nevetheless, it is obvious that the problems of Education in a society such as ours, which is undergoing rapid and multiple changes, cannot be resolved merely by laws and regulations; what is called for, is a much deeper and wide-ranging approach in order to create a true awareness in our Society of the relevance of Education in general, but particularly in Mathematics. In addition, the dynamic in which we are now involved - new laws for education every time there is a change in government – is creating a climate of discouragement throughout the entire educational community.

In the same way that airports, roads and rail networks are planned 15 (or more) years in advance, education requires a calmer and more thoughtful planning that obliges everyone, politicians in particular, to tackle this problem in an open-minded way. Free of political considerations and placing at the forefront the interests of all, both parents and children should have with the common objective of achieving a better educational system. It is also necessary to leave behind once and for all the temptation to act as so often we have done in the past, sporadically and in haste without a clear vision of where we are going.

It is also obvious that too often in the past these mistakes and bad habits have not been too often corrected or avoided.

Bellow is first a brief summary of the situation of Mathematics at the present time, followed by some ideas about the current situation of our students and teachers, and at last our point of view about how Mathematics should be taught

2. The current situation of mathematics

The advances in Mathematics in the 20th century are truly remarkable. This fact is virtually clear when we analyze the current state of knowledge in the foremost disciplines of Mathematics; or when we consider the number of emerging disciplines and the enormous impact they are having. Actually, these advances have been so significant that a century is quite clearly a scale of measurement too rough to allow an initial comparative analysis to be drawn. Perhaps an analysis of the second half of the 20th century would enable us to achieve a clearer perspective of the rate at which Mathematics has progressed.

So when concentrating on this second half of the last century, we detect that indeed the progress of Mathematics has been spectacular. One of the most well-known popular breakthroughs is without doubt the definitive proof of Fermat's Theorem[6] by the British mathematician Andrew Wiles. However, it is perhaps in other areas where progress has been even more important, even though we have no landmark quite so outstanding. As a genuine representative of this second half of the 20th century, it is worth mentioning the world of Numerical Analysis, Numerical Simulation and Computation[7]. Forty years ago it was a field that was only beginning to emerge, while today it constitutes a highly complex and developed scientific discipline with an even brighter future for the century that has only just begun. This development has been possible thanks not only to the advances made in the field of Mathematical Analysis oriented towards numerical approximation[8], but also decisively to the spectacular progression of computers. Needless to say, in the last two decades, they have burgeoned from being strange objects difficult to operate, to take their place alongside many other domestic appliance.

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[6] WILES, A.: *Modular elliptic curves and Fermat's Last Theorem*. Ann. of Math, 141 (1995), pp. 443-551. This is the paper in which the definitive proof of Fermat' Theorem appears. An interesting summary of the history of this theorem can be found in the introduction.

[7] Fields of research that in our country are coordinated through the *Sociedad Española de Matemática Aplicada* (SEMA - Spanish Society of Applied Mathematics), the most noteworthy items of which are presented and discussed in its regular official Gazette. For the SEMA website, see: http://www.uca.es/sema.

[8] The field most commonly known as Numerical Analysis: readers interested in the exhaustive and systematic classification of the different mathematical disciplines can consult the Classifications of the American Mathematical Society (AMS) at the following website address http://www.ams.org/msc/.

In the last decade, Mathematics has also played an important role in different fields of technological and social development. It is worth mentioning, for example, the application of Fourier Analysis in Signal Theory and Partial Differential Equations in Image Processing, and emerging disciplines, such as Financial Mathematics.

All of this provides an extremely optimistic outlook in which the progress achieved is nothing if not a herald of what is still to come. In spite of this, however, it would be a serious mistake to overlook some of the most classical and important problems that still remain to be solved, among which we may mention the uniqueness and regularity of the solutions to the basic equations in Fluid Mechanics, the Navier-Stokes equations[9], which were constructed in the 1930s by the recently deceased Jean Leray.

There are also sociological aspects that should be taken into account when speaking about Mathematics today. For example, we should not forget the massive increase in the number of young people studying at university, which is a phenomenon that can without doubt be classified as something really "new" and is a distinguishing feature of the last quarter of the 20th century.

These two factors – the spectacular development of Mathematics, and the massive increase of young people studying at Universities – may appear to be in contradiction, or at least the source of considerable tension. How can a complex discipline like Mathematics, which is in a permanent state of growth and innovation, be accessible to an increasingly broader sector of our society?

There is a fact that makes it possible for the two above-mentioned phenomena – the spectacular development of Mathematics and the great rise in the number of university students – to coexist without tension. This is because the essential, basic and fundamental elements remain intact in the world of Mathematics, this also includes that minority that will pursue Mathematics as a profession as teachers and/or researchers and those who use Mathematics as a tool in their day-to-day work in the study or practice of other Sciences or technical disciplines.

From this perspective, and when it comes for example to drawing up a list of the knowledge, a student should have acquired before embarking on a scientific or technical course of study at one of our Universities, it is advisable, in our opinion, to adopt a somewhat "conservative" standpoint, since these skills and this basic knowledge are essentially the same as those required twenty years ago, when the efforts towards the resurgence of Mathematics in our country were just taking shape.

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[9] This is one of the seven problems included in the Millenium Prize Problems, an international competition organized by the Mathematics Clay Institute with an award of one million dollars for whoever is able to solve any of these problems. For more information, see the following website: http://www.claymath.org

Of course, we should point out that this persistence of the same contents must go hand in hand with adaptation to the ways of teaching Mathematics that make the subject attractive for the keenest and most able students, and at the same time becoming more accessible for the majority, a difficult task that we will address in the following sections.

When speaking about teaching methods, it is impossible not to mention computers. Indeed, the world of computers today, and computation in general, inevitably goes hand in hand with Mathematics. It is obvious that the personal computer will increasingly become a support tool for learning and understanding Mathematics. A bold use of the computer as a support tool in teaching should undoubtedly contribute to helping students towards a better understanding of processes such as limits and derivatives or bi- and tri-dimensional geometry. However, it is not a good idea to substitute the contents of a mathematics course by work on a computer, and this brings us up against a conflict that is difficult to solve, given the scarcity of time available to those whose job it is to teach this discipline. In any case, given also the progress we will see in the coming years, we must be aware that the fast spread of computers will result on perceptible medium and long-term changes in the teaching of Mathematics.

For instance, in the Universities, it is becoming quite usual for classes on some topics in Mathematics to include and be completed by practical classes or laboratory sessions in which computers are used to practice basic mathematical notions by means of programming, and with the use of specific mathematical software packages. This is an experience that should be promoted, rationalized, and also transferred to our Schools and Educational Institutes. However, we should not drop our guard when it comes to something (extremely controversial) like "mathematical rigour". It is necessary to be very clear about when and for what purpose the computer can act as a legitimate ally. With this precaution in mind, it would be a good idea to encourage and organize the exploration of this approach to the maximum, because if we do not make the effort now to incorporate the computer carefully into the world of Mathematics in our schools, the computer itself may eventually replace Mathematics in some areas. Even more serious is the possibility that students may end up by believing mistakenly that Mathematics can be reduced merely to what can be performed by and visualized on a computer.

We mentioned above this conflict between the universalization of education, including the Universities, and the maintenance of the contents and its depth. This is an issue that anyone could (and possibly would be appropriate) to discuss from different points of view. In particular, it would be interesting to analyze the fall in the standard of Mathematics in our Schools and Universities as well as the international impact of our mathematical research. A recent study[10] carried out by the Spanish mathematical societies places

[10] ANDRADAS, C. and ZUAZUA, E. (COORDINATORS): *La investigación matemática en España en el periodo 1990-1999.* Published by the Real Sociedad Matemática Española, Societat Catalana de Matemàtiques, Sociedad de Estadística e Investigación Operativa y Sociedad Española de Matemática Aplicada, Madrid, 2002.

Spain among the first ten most productive countries, with a production in research greater than 4%. In this sense, the International Mathematical Union (IMU)[11] had decided that one of International Congress of Mathematicians was going to be held in Madrid in 2006. The ICM (International Congress of Mathematicians) is coordinated by the IMU and has a history stretching back over more than one hundred years. It is held every four years and it is one of the most important events in the world of Mathematics. In particular, at the congress, the prizes known as the Fields Medals are awarded. They are the highest international distinctions to be conferred for research work in Mathematics. This decision is with no doubt an unmistakable recognition by the international community of the level that mathematical research has reached in our country. Nevertheless, we must remain on guard, because this current success in the field of research is to a large extent based on the teaching work that has been carried out for over more than two decades, and it is therefore necessary to take the greatest care of our educational system if we wish to maintain the level of which these indicators are a reflection. This recognition of our research work is not in the least incompatible with the deterioration of Mathematics in the different areas of education.

3. Who teaches, how and to whom is mathematics taught?

3.1 Diversity of students – and teachers too?

We believe that it is not possible to speak simultaneously about those students who show greater ability and keenness for Mathematics which later will probably go on to study Sciences or Engineering at the University, and those who at the age of 12 begin to drop this subject definitively and direct their interest towards others. In the same way, we also believe quite simply that those students of the second group should be treated differently. The recently approved Quality in Education bill spoke about different streaming in the 3rd and 4th years of Obligatory Secondary Education. The draft bill of the Organic law for Education states that in the fourth year, "...In order to guide students in their choices, classes in subjects with different options could be established."

We believe that it is within this framework that a different approach may be adopted, which until now has not been taken[12], towards those students who have a greater interest in subjects other than Mathematics. Nevertheless, we believe that this differentiation should be made beforehand. Let us look at an example that justifies and backs up this point of view: Many students would be able to solve the following problem: *evaluate the function* $y = x^2$ *at points x = 1,2,3,….*, but many would perhaps be unable to understand that this algebraic formula $y = x^2$ is a way of codifying innumerable relations.

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[11]Information about the ICM can be found at the IMU website: http://www.mathunion.org

[12]Although it is true that some schools had already set up schemes similar to streaming (which in some cases were known as *itineraries*), within a legal framework for which there was no such provision, in order to provide students with different educational choices.

Also, that working with this relation may be less difficult than managing a large number of data, and that this relation may have something to do with a parabolic path in the plane. Or, to use an example closer to everyday reality, many students know how to calculate how much 15% is of a given quantity, but perhaps may encounter difficulties when trying to determine what percentage is involved in a 15% rise in the cost of housing over four consecutive years.

On this point it is important not to confuse or be misled by terms: while Algebra has been passed down to us from the ancient Greeks and the East, to many people it may not be easy to understand. As well as possibly the records set by Greek athletes in the very first Olympic Games, which would not be easy to understand.

Obviously, the issue of training of our teachers must be added to the diversity of students.

Mathematical subjects have practically disappeared from Training Colleges for Primary School Teachers, to be replaced by the Didactics of Mathematics, which even in the best of cases scarcely account for 5% of their workload on these courses. This is precisely the area where the effects of the diagnosis we mentioned in the first section of this article can most clearly be perceived, and which to a large extent is opposed to the one we set out here. The problem faced by the teachers is not understanding or having a better knowledge of Mathematics in order to ensure a better teaching of the subject, but quite simply of acquiring the skills to enable them to share this knowledge with ease, not to mention with the desired success that the traditional methods have failed to achieve.

While recognizing the achievements and the efforts of those who defend these approaches, and being in favor, as we state above, of including in our educational system all the means and methodologies necessary to guarantee an enhanced quality in teaching, we sincerely believe that these types of diagnoses frequently fail to bring us closer to a solution. For example, proficiency in the rudiments of working with fractions requires well-guided individual effort, but especially a sound basis of practice and exercises, even if neither the method nor the strategic didactics that enable this skill to be taught in the way we all would like to are not known, such as an image on the Internet, just to draw an everyday comparison.

The day-to-day reality of a Mathematics teacher, whether it is in Primary School, Secondary School or University, often has little to do with the type of question occupying those who defend this stance. Indeed, any method ceases to be useful or effective for the teachers whose task is to explain the concept of a derivative without having understood its real and ultimate meaning.

Likewise we recognize the good will of those who believe that the difficulties that teachers face when it comes to conveying mathematical concepts are a consequence of the lack of didactical tools. Such tools are undoubtedly both useful and important, but no tool is as useful as a profound understanding of the mathematical concept to be taught.

At present, a large number of students who join Teacher Training courses, study Mathematics for the last time in the fourth year of E.S.O. (Compulsory Secondary Education). Therefore, if at the University they study practically no subjects that have any mathematical content, it is very difficult for them to know what to teach, even if they know how. We believe that this is a very serious problem which is imperative to tackle without delay. This does not mean that a Primary School Teacher must have knowledge of sophisticated aspects of Mathematical Analysis or Topology, not at all; but there is no one who has any doubts about the meaning of Pythagoras' Theorem, which is vital for understanding Euclidian Geometry.

We believe that a good way for Teachers to acquire greater teaching experience and skills is to increase practical training under appropriate guidance in schools where students will subsequently be taught. This would also apply to graduates who aspire to teaching posts in Mathematics in Secondary Education. To this end, the design of the new *Título de Especialización Didáctica* (Qualification in Didactic Specialization), as laid out in the LOCE and also referred to in the LOE draft bill, should also include this measure. We feel that this new legislation would thereby be substantially improved, since after reading the contents of the *Real Decreto*[13] (currently suspended) in which its application is set out, one wonders if the aim of the authors of the bill was to produce a useful piece of legislation or rather acquire clientele for certain courses, which have none at the moment.

3.2 Learning by calculations or by concept?

If many students for whatever reason are unable to perform formal reasoning immediately, will teachers of Mathematics have to confine themselves to teaching recipes and assess the performance of students in their application?

This is not a straightforward issue, because it frequently happens that by often repeating enough exercises human beings are able to comprehend the underlying logic, and thereby acquire a more abstract, comprehensive and long-lasting understanding that enables them to tackle more complex problems on their own. The mechanical and abstract aspects therefore go hand in hand and they are, in fact, difficult to separate.

What is the best way for students to understand the meaning and techniques of integral theory? Probably by solving a list of fifty carefully chosen integrals; in some cases this will allow students to manage drilling in exercises with bigger ease; while in other cases students will unconsciously reach the conclusion that the integral is a process of inversion of the derivative, and thus understand the motivation and meaning of formulas as simple and enigmatic as "integration by parts" and its relation with the derivation of the product of functions.

Therefore we believe that there is nothing to fear from learning by calculations when it is aimed at providing students with the skills required to move comfortably in the world of arithmetic.

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[13]Real Decreto 118/2004, B.O.E. (Official State Bulletin) February 4th, 2004

This is not the first time in the history of education that this issue has come up for debate. We were the ones who, in Basic General Education in the 1970s, studied Modern Mathematics for the first time. It was both saddening and ineffective to be removed from the Mathematics of numbers to enter prematurely into Set Theory.

Modern Mathematics was introduced into our educational system at that time on the basis of two principles: Evolutionary Psychology and a strong trend in Mathematics bent in burying classical Mathematics[14] and substituting them for a logical/abstract system.

This trend has dismissed in recent decades, both in the field of research and in teaching, and the failure of what was known as Modern Mathematics is obvious today. However, there are currently other trends of thought, such as “new" or “fuzzy" Mathematics, which involve the same or even worse risks than before, since now the intention is to make “Mathematics easy and fun”, with the aim of making it more popular. While it may be true that composing songs with a “putative mathematical content” is more fun than teaching how to solve equations, it is nevertheless necessary not to confuse a mathematics class with something else. Mathematics may indeed be more fun for some, but it may not be so for everyone, and this is a fact that is undeniable. As regards making Mathematics easier, every professional mathematician knows that when faced with a sufficiently difficult problem ~~ease~~ simplicity goes out of the window. Thus, the dividing line between ease and difficulty is something subjective that depends almost entirely on the individual.

In regard to this issue, the following question arises quite naturally: To what extent can students progress together if some of them will never go beyond learning how to work with the four elementary skills (addition, subtraction, multiplication and division), while others will go on to study Sciences at the University?

When addressing this question, it is a good idea not to lose sight of the fact that students have the right to receive from their teachers a sufficiently conceptual, abstract and motivating explanation in order to be able to progress beyond the simple ability to work with the given rule, if possible.

When José Antonio Marina was interviewed some years ago, when he had still to receive the recognition he enjoys today, he complained that Philosophy in our Secondary and High Schools was turning into a mere History of Philosophy, a collection of anecdotes consisting of names and dates, rather than what it really should be: the study of the great ideas in human thought and its evolution.

The same may be said of what is being done with Mathematics in some fields. It is therefore necessary to be extremely rigorous when drawing a dividing line between popularization and education, a task that first of all falls upon the professionals of Mathematics themselves and their teachings.

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[14] A school of thought symbolized by Jean Dieudonné’s famous phrase: *Down with Euclid*!

3.3 Minimum requirement curricula or curricula meeting all requirements?

Currently, Howard Gardner[15], for instance talks about multiple intelligences, and concepts such as emotional intelligence are acquiring even greater acceptance. This is an example of the recognition and respect accorded to the diversity of human nature, and is highly beneficial that characteristics belonging more to the natural world, sensibility and human relations acquire value along Science and Technology.

However, the only way to make these instances of diversity compatible with universal education is to provide a distinct and individualized attention to every student, while maintaining a basis of minimal requirements common to all. This fully justifies the need for the existence of minimum requirement curricula, especially in Obligatory Secondary Education.

Nevertheless, to the extent that individualized attention does not currently exist, such minimum requirement curricula automatically become curricula in which all requirements must be met.

The serious problem then arises that it is not possible to provide every student with the opportunity to develop his or her own potential to the full, and this in our opinion is an abdication of responsibility.

Thus, the authorities must decide on the following:

• Whether the personalized teaching set out in the current regulations can be implemented, and then adopt a course of action accordingly.

• Should the decision on the above point be in the negative, an alternative approach should be found, the possibilities of which are many:

∗ Whether to accept that minimum requirement curricula that do not fulfill their purpose automatically become curricula for meeting all requirements. Are we prepared for this? Do we resign ourselves to the fact that our children will not have close to home the education they might receive in Lisbon or Paris?

∗ Implement a differentiated model of attention, without it necessarily being individualized.

This latter possibility is quite likely to be the most realistic, although the ideas inspired by Evolutionary Psychology might conceivably be more attractive.

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[15] See GARDNER, H: Multiple Intelligences: The Theory in Practice. Basic Books, New York, 1993

Many are the professionals committed to the Teaching of Mathematics who believe that this hypothetical "individualized attention" is somewhat rhetorical, not unlike the refrain that "we all have a right to our own home". The calculation is easy to do; if all teaching were personalized, in a class of 20 students with three fifty-minute classes a week, each student would receive exactly seven and half minutes of individualized attention per week.

4. What to teach

When the LOGSE (General Law of the Educational System) was put into effect, and throughout its subsequent development, it was difficult to know exactly what to teach. First of all, it was hard to find in the official guidelines any precise description of the subjects that students ought to learn in any year and at any level. In addition, some were of the opinion that the important point was not what to teach but how. Essentially it seemed that everything revolved around procedures and attitudes: students were to develop an appreciation for the beauty of Mathematics and display a positive attitude towards the subject, and so on. We believe that all these intentions are good, but are they more than just intentions? However, we also believe that it is necessary to consider what should be taught, even if this may appear to be risky, and that is what we wish to discuss. We therefore indicate what in our opinion the basic skills that students of Mathematics at all levels in Secondary Education must be.

4.1 Compulsory Secondary Education

We believe that the skills that students should have acquired on completion of Compulsory Secondary Education are as follows:

- ✓ Proficiency in the four basic rules: addition, subtraction, multiplication and division.
- ✓ Possess a fairly precise knowledge of fractions and how to work with them.
- ✓ Knowledge of the use of the "rule of three" or cross-mutiplication. Unfortunately, no explicit reference is made to this point in the current curricula, perhaps because of the inertia to which we have grown accustomed. It is worth noting that the word "Geometry" was difficult to find in previous curricula.
- ✓ Likewise, students should be able to calculate the areas and volumes of elementary geometric objects and show a command of the decimal metric system.
- ✓ Special attention should be given to aspects of calculus, even though at times it might not be fun either to learn it or to teach it. In this regard, it is important not to confuse the learning or study of Mathematics with its popularization or even its vulgarization.

4.2 Baccalaureate. On the threshold of university

Here it is important to distinguish two types of students; those who choose Science and Technology subjects and those who opt for Humanities and Social Sciences.

For those who choose a course in Sciences, and thus may go on to pursue of scientific career, we believe that on conclusion of the Baccalaureate they should be able to:

- ✓ Know how to calculate elementary limits and derivatives with easiness.
- ✓ Sketch an outline of the most representative features of a function.
- ✓ Know the main principles of trigonometry.
- ✓ Know how to translate real situations into algebraic language.
- ✓ Be familiar with the fundamental concepts of Analytic Geometry: points, straight lines and vectors, with greater attention given to intuitive aspects rather than the algebraic structures underlying these concepts.
- ✓ Call upon some basic knowledge of descriptive Statistics and Probability.

In the case of those students who choose Humanities and Social Sciences, more attention should be given to the applications of the concepts. Perhaps a command of Analytic Geometry is not so important in this case, although maybe a little more in-depth study of Statistics would be important to consider. However, in spite of the fact that Discrete Mathematics should be given the importance it deserves, since it is a discipline increasingly present in the world of Science and Technology as well as in the business sector, which have often been left in the background in Secondary Education, we think that it is necessary to simplify as much as possible the elements taught in these disciplines. Perhaps the attention currently devoted to Statistical Inference (confidence intervals, hypothesis testing) is excessive for Mathematics Applied to Social Sciences in comparison to previous curricula.

In a different order of things, we believe that it is essential to simplify the avenues currently provided in the Baccalaureate and prevent students from starting a university degree in Mathematics or Chemistry without having studied the appropriate subjects in their last year of the Baccalaureate.

4.3 University entrance exam and/or general Baccalaureate exam

We are all aware that however much we define and redefine our curricula, what teachers are really going to teach, and above all what students are going to interested in, is completely determined by whatever course contents are to be evaluated.

In this regard, with reference to the Baccalaureate, the University Entrance Exam has so far been the touchstone that has determined in a single exam, in an order of 40%, the result that will also condition the future of our young people at university. .

The University Entrance Exam has thus traditionally acted as a barrier or “retaining dyke” to safeguard the minimum standards of requirement that our educational system has been unwilling to relinquish.

With the introduction of the LOCE, the University Entrance Exam was replaced by the so-called General Baccalaureate Exam. The new legislation provides for a similar exam to mark the passage between Baccalaureate and University. It is necessary to take a close look at the mechanisms by which this new exam will be determined immediately.

When performing this scrutiny, it is important to take into account the detectable defects in the current University Entrance Exam in recent years. In this sense, we have the impression that this exam and the board responsible for drawing it up have been immersed in a somewhat reiterative dynamic that has at times caused the exam to depart from what it really should have been: an exam for assessing students' basic knowledge and their readiness to adapt to university life in which study not only requires a sound educational basis but also command of written expression and a capacity for abstract reasoning and the association of ideas. In this sense, the University Entrance Exam, like many of the exams belonging to our educational system, has concentrated too much on the statement of artificially complex problems that have at times turned out to be confusing. This process is without doubt associated with the way in which the boards charged with drawing up these exams have been constituted, frequently without ensuring the necessary motivation and renewal of their members.

Basic mathematical ideas are simple, which does not mean that they are easy to understand. In the words of Albert Einstein: "*Everything should be made as simple as possible, but no simpler*". A good understanding of these mathematical ideas must necessarily be accompanied by a capacity for simplification in the way they are addressed. Whether exams are for university entrance or otherwise, they must be fundamentally infused with a clarity of ideas by those who draw them up, which is not always the case.

Furthermore, it is evident that where Mathematics is involved, it is impossible to make real learning progress without a good knowledge of how to calculate. It is therefore obvious that a rigorous assessment of skills in this topic is required.

However, our exams at whatever the level often confuse depth of ideas and complexity of calculation with an extravagant verbal style, which has two clear consequences: the exam frequently fails to act as an adequate means of assessment, and finishes by confusing even the most gifted students.

Experience shows that rarely do students enter University with a command of the subjects in the Baccalaureate curriculum. This is not necessarily a serious matter, providing that students have a sound knowledge of all the basic concepts and have likewise acquired an acceptable ability in calculus. In the field of Analysis, for instance, it is worth mentioning that students who know how to calculate limits, derivatives and integrals will be perfectly able to complete the first year of their degree without difficulty, even though they may not have studied sequences, cannot work confidently with Rolle's theorem and the mean value theorem, or have not inwardly digested the application of the integral to the calculation of areas. The maturity that leads to an understanding of these topics may very well be attained during the first months at University.

5. Some clichés and habits under review

As a general philosophy and the main theme of the different initiatives adopted in education, we believe that it is necessary to ensure that all students receive an education in Mathematics proportional to their ability and with which they are able to cope. This constitutes to a large extent a declaration of intent, and is in practice a complex and difficult problem to solve, and while we have devoted much of this article to the issue, we are aware that we are unable to provide a definitive response.

At any rate, when addressing this question, we believe that it is important to strip away some of the clichés that usually muddy the waters of the debate and which often lack any sound basis. Clichés to be remarked:

* The timetable wars. We believe that we must avoid being drawn into a "timetable war", "3 or 4 hours of mathematics?". Four hours may be too many or not enough, therefore ~~thus~~ we think that it is better to diversify the choices in terms of the number of hours and the depth to which Mathematics is going to be studied.

* The volume of information. Young people certainly do not need to be saturated in information, which is without doubt a growing problem in our society. However, they do need to develop their own criteria and the ability to make informed choices in an increasingly complex world.

* Computers. It is impossible to discuss the issue of information in modern society without including computers or the Internet. One of the most common clichés or slogans nowadays is the injunction that there should be "a computer in every classroom". Of course, we also believe that computers provide an extremely useful tool that may even exert an influence on the didactical methodologies to be employed in the near future. Nevertheless, it is necessary to demystify this issue and establish a rigorous list of priorities that is free from the influences of fashion. We venture to say, yes, a computer in every classroom, by all means, but before that a paintbrush and easel, a work of literature, a good textbook on basic Geometry, etc,. Computers are tools that are now inseparable from human beings in all scientific and technological endeavour; they are indeed an instrument of support and experimentation, but if we wish to teach a child to fish, are we to place a rod and reel in his or her hands on the very first day? Is the way to learn Chemistry simply to let a child run free in the classroom? Let us then use a little common sense and not adopt in the subject of Mathematics the type of approach that what we would by no means pursue in any other walk of life and education. Computers should therefore be introduced carefully in the teaching of Mathematics without them ending up by replacing it.

* Textbooks. The types of textbooks published today are excellent for teachers; they are packed with graphs, historical fact, applications and so on. But it is unlikely that fat textbooks of this type will attract the attention of students who are subject to a wide variety of subjects, the frequency of exams, and the lack of time that the pace of modern life imposes even on children and adolescents. Alongside, these undeniably excellent textbooks, might be worthwhile to publish shorter manuals that are more likely to attract the attention of students quicker. Unfortunately, at the present time the market in textbooks is dominated by only two or three publishers, while the rest are content only to try to imitate their output. In this field there is a clear lack of originality.

* The applications of Mathematics. We would like to draw attention to the way in which some Mathematical applications are presented in textbooks. For example, the applications of Linear Programming, to what nowadays is known as Social Sciences, date from the 1950s

in which there are some simple and nice Mathematical methods that can be taught. However, it is surprising to see that sometimes these applications are presented as if they belonged to the latest generation, and even more so when an attempt is made to use them as a justification of the need for changes to be made in the curricula or in methodology.

* The contents of competitive examinations. There are some very interesting areas in Mathematics, such as fractals, that some years ago became fashionable in our country, to the point where even appeared in competitive examinations for Secondary School Teachers[16]. Nevertheless, it is rather surprising that no reference is made in such topics to Differential Equations, even though these are historically related to the invention of calculus. There are many interesting branches of contemporary mathematical research, but we believe that one must exercise caution when it comes to making them part of secondary education, either directly in minimum requirements or indirectly by including them in exam questions for future teachers of Mathematics.

6. Conclusion

From all it has been mentioned, one may judge that the current state of the teaching and learning of Mathematics is far from optimum in our country, and we believe that no one would dispute this conclusion. However, although perhaps not everyone would share this opinion, the situation has gotten worse as a result of many factors. What is to be done about it? Everyone accepts that something has to change, but how can we make those changes?

The range of possible measures that are often considered is quite wide; we spoke about some of them above. But frequently they are reduced to the most obvious and immediate choices. The first measure is always an increase in the amount of classtime devoted to Mathematics. On this issue we would like to say that while regaining lost ground is desirable, it does not seem that we as professional mathematicians would be able to solve it by demanding more hours. It is a problem that is common to many other subjects, and in any case it is clear that the total number of teaching hours available is limited. Furthermore, the goal of Secondary Education is not to produce mini-researchers in all fields, but rather to educate and train people, and we would do well not to lose sight of the fact that it is not a question of simply reducing the problem of learning Mathematics to a dispute over hours. Moreover, many people believe that what is required is an increase in support personnel in order to make the recommendations in the LOGSE a reality through an individual attention to students in terms of their needs and their potential. Of course, an increase in teaching staff would provide a wider field for manoeuvre, but it would not be honest to reduce the problem to a dispute over economic issues.

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[16] For the topics that have appeared in the latest examinations for Secondary School Teachers, the reader may consult the following website: http:://www.mecd.es/inf/comoinfo/a-5-2-33.htm

Considering or not these types of initiatives, based on an increase of investment and the number of classroom hours, we believe that it is necessary to extend the analysis of the issue.

It should not be forgotten that, as mentioned above, the essential components of Mathematics remain practically the same, and therefore any changes in the way it is taught that new technologies will bring in the coming years should not alter the fundamental contents; it will still be necessary to retain calculation and memory as basic approaches, just as we and our forebears have done in the past. As stated above, in our opinion, a good standard of teaching in the phase of compulsory education must provide students with proficiency in the use of the four basic skills and simple fractions, the use of the rule of three, proportions, and the decimal system of measurement, finishing by basic notions of elementary Geometry.

The latest figures on school failure rates show that our current educational system is unable to guarantee these minimum requirements. What percentage of students who do not know how to use the rule of three and the decimal metric system are attending classes whose curriculum includes the calculation of derivatives?

In fact, it is important to observe new technologies, the information society, and the new teaching methodologies that promise to facilitate learning with less effort, with a view to incorporating the positive aspects they may provide, but in a critical light and without dropping our guard. The golfer Gary Player once said that: “The harder I work and practice, the luckier I seem to be”. One would do well to remember that there is no learning without effort. However much we turn it over in our minds, we still fail to understand what is the goal to be pursued in the draft bill of the LOE that foresees an increase in the minimum number of subjects that are entailing student failure. We fear that this will not spur students on to greater effort, although as we stated above, perhaps it is only we who fail to understand it.

Likewise, we ought to provide a means by which those students who wish to study Mathematics at a higher competitive level have the opportunity to do so, as is the case in neighbouring countries. However, it should consist of a robust solution extended to all students, without incurring the contradiction of making intransigence compatible with any initiative whose goal is to establish different streams or itineraries according to aptitude, and subsequently to promote private schemes only applicable to few students.

The range of possible solutions are many, and the responsibility for adopting them falls to both politicians and those with high positions in government and administration, with the introduction of the necessary changes in the regulations governing education, like all the professionals involved in Mathematics in our field of action.

So let us all work together to combat numeral illiteracy and to establish the conditions in our classrooms that will enable young people to enjoy the right climate in which to stimulate their interest in (and sometimes passion for) Mathematics, by developing serious and rigorous curricula and methods that ensure the access to the required knowledge in order to create favourable conditions for their young, questioning and budding intelligence.

In conclusion, we would like to quote the words with which Professor Soledad Rodríguez concluded her article “Numeral Illiteracy and the Year 2000”[17]:

“...We must insist once again that it is necessary to make our students exercise their brains in order to develop their intelligence. We must push repetition so that they develop their power of memory, and we must impose discipline in method to strengthen their will. This will undoubtedly help to erradicate numeral illiteracy and to ensure that in the near future we have students with greater self-esteem and more cultured as well as better educated citizens”.

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[17] *This article first appeared in the El País daily newspaper on July 12th, 1999. http://elpais.com/diario/1999/07/12/opinion/931730405_850215.html*