Isoperimetric inequality, p-parabolicity and doubling graphs

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Geodesic Metric Spaces

Definition

Given two points x, y in a metric space X, a geodesic from x to y is a path whose length equals the distance between them.
A metric space is called geodesic if for every $x, y \in X$ there exists a geodesic joining x and y.

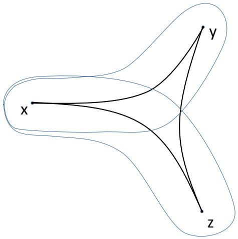
We view graphs as metric spaces by assigning length 1 to each edge and considering the shortest–path metric.

Example

A (geometric) graph endowed with the shortest–path metric is a geodesic metric space.

Gromov Hyperbolicity

Introduced by M. Gromov (1987) in the study of finitely generated groups.



A geodesic metric space X is δ-hyperbolic (in the sense of Gromov) if all geodesic triangles are δ-thin.

Examples

- Trees are 0-hyperbolic.
- The hyperbolic plane is also hyperbolic in the sense of Gromov.

This notion captures many large-scale properties of negatively curved spaces.

Quasi-isometries

A map between metric spaces, $f : \big(X, d_X\big) \to \big(Y, d_Y\big)$, is said to be quasi-isometric if there are constants $\alpha \geq 1$ and $\beta \geq 0$ such that $\forall x, x' \in X$,

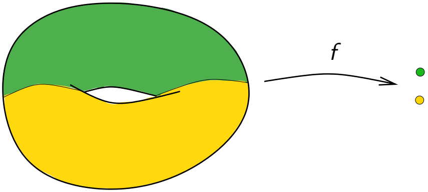
$$\frac{1}{\alpha} d_X(x, x') - \beta \leq d_Y\big(f(x), f(x')\big) \leq \alpha d_X(x, x') + \beta.$$

If $d_H(f(X), Y)$ is bounded, then f is a quasi-isometry and X, Y are *quasi-isometric.*

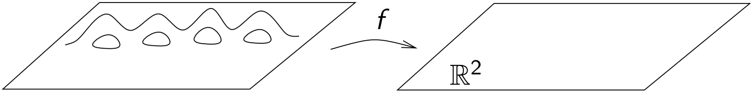
Being quasi-isometric is an equivalence relation.

Properties

Quasi-isometries need not be continuous...



...nor preserve the topology or the local geometric properties.



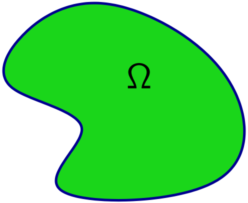
A bounded set is quasi-isometric to a point.

$\mathbb{Z}$ and $\mathbb{R}$, with the euclidean metric, are quasi-isometric.

The Classical Isoperimetric Inequality

Question

Among all closed curves in the plane with fixed length, which one maximizes the enclosed area?



The answer is the circle.

First proof by J. Steiner (1842), completed by K. Weierstrass (1870).

This leads to the isoperimetric inequality

$$A(\Omega) \leq \frac{1}{4\pi} L(\partial\Omega)^2$$

Linear Isoperimetric Inequality

An n-dimensional Riemannian manifold M satisfies the linear isoperimetric inequality if there exists $C > 0$ such that

$$\mathrm{Vol}_n(A) \leq C\,\mathrm{Vol}_{n-1}(\partial A)$$

for every bounded open set $A \subseteq M$.

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A graph X satisfies the linear isoperimetric inequality if there exists $C > 0$ such that

$$|A| \leq C\,|\partial A|$$

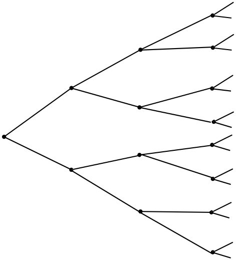
for every finite set $A \subseteq V(X)$.

The Linear Isoperimetric Problem

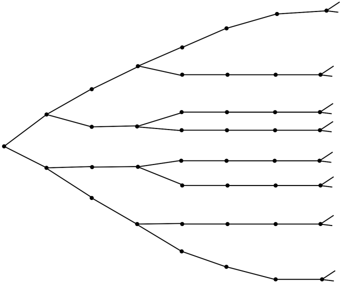
Problem

Which hyperbolic spaces satisfy the linear isoperimetric inequality?

Linear Isoperimetric Problem



Usual Cantor tree

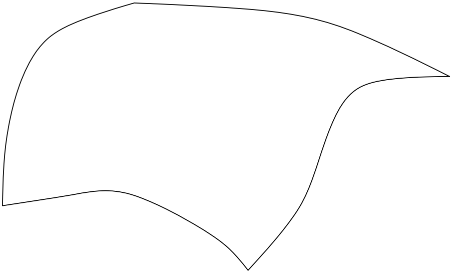


Cantor tree with successive branchings at distances 2^{k-1}

Has LII

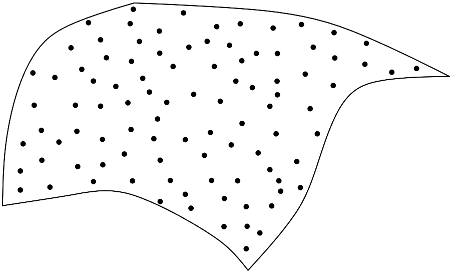
Does not have LII

Approximating a Manifold by a Graph



Approximating a Manifold by a Graph

V_M is a maximal ε-separated subset of M.

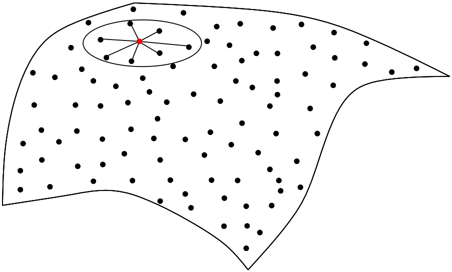


Approximating a Manifold by a Graph

V_M is a maximal ε-separated subset of M.

For $p, q \in V_M$, we connect p and q by an edge if

$$0 \leq d_M(p,q) \leq 2\varepsilon.$$



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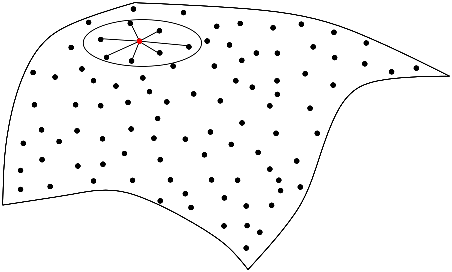

Theorem (Kanai, 1985)

If M is a manifold with bounded local geometry, then M satisfies the linear isoperimetric inequality if and only if the graph Γ_M does.

Linear Isoperimetric Problem

Problem

Which hyperbolic spaces satisfy the linear isoperimetric inequality?

Linear Isoperimetric Problem

Problem

Which hyperbolic spaces satisfy the linear isoperimetric inequality?

- Characterization for trees ✓
- Characterization for graphs ✓
- Characterization for Riemannian manifolds with bounded local geometry ✓
- Characterization for Riemann surfaces ✓
- Characterization for general Riemannian manifolds?

p-Parabolicity

A manifold M is p-parabolic if every p-superharmonic function on M is constant.

A graph $G = (V, E)$ is p-parabolic if every p-superharmonic function on V is constant.

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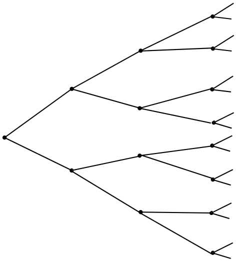
Parabolicity



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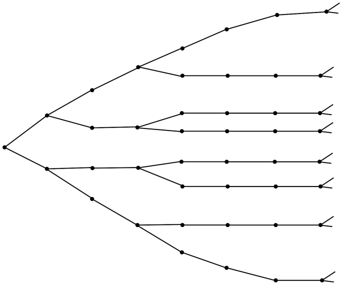

Cantor tree with successive branchings at distances 2^{k-1}

Has LII
Not parabolic

Does not have LII
Parabolic

Parabolicity

We managed to characterize p-parabolicity for a large family of trees.

Problem

- *Which trees are p-parabolic?*
- *Which Riemannian manifolds are p-parabolic? Necessary and sufficient conditions.*

Isoperimetric Inequality Implies Non-Parabolicity

Theorem (MP-R)

Let G be a uniform graph. If G has LII, then G is not p-parabolic for any $1 < p < \infty$.

Isoperimetric Inequality Implies Non-Parabolicity

Theorem (MP-R)

Let G be a uniform graph. If G has LII, then G is not p-parabolic for any $1 < p < \infty$.

For the proof:

1st We prove the same result for complete Riemannian manifolds.

2nd We construct a surface (quasi-isometric to the graph) that has LII or is p-parabolic if and only if the graph has the same property.

Doubling Spaces, Isoperimetric Inequality and Parabolicity

A metric space is doubling if there exists a constant m such that every ball can be covered by m balls of half the radius.

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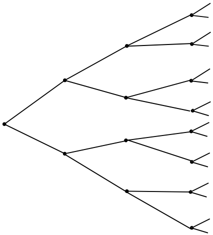
A metric space is doubling if there exists a constant m such that every ball can be covered by m balls of half the radius.

Theorem (MP-R)

For a uniform graph:

- *Linear isoperimetric inequality implies non-p-parabolicity*
- *Linear isoperimetric inequality implies non-doubling*
- *Doubling implies p-parabolicity for sufficiently large p*

Doubling Spaces, Isoperimetric Inequality and Parabolicity



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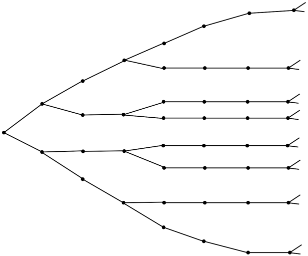

Cantor tree with successive branchings at distances 2^{k-1}

Has LII
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Not doubling

Does not have LII
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Linear Isoperimetric Inequality Implies Non-Doubling

Theorem (MP–R)

If a uniform graph has LII, then it is not doubling.

Idea of the proof.

Let G be a μ-uniform graph with linear isoperimetric constant C.

Using the linear isoperimetric inequality we obtain

$$|B(v_0, n)| \leq C|\partial B(v_0, n)| \leq \mu C|S(v_0, n)|.$$

In particular,

$$|B(v_0, n)| \leq |B(v_0, m)| \leq \mu \cdot C|S(v_0, m)| \quad \forall n \leq m < 2n.$$

From this we deduce the key estimate

$$\frac{n}{\mu C}|B(v_0, n)| \leq |B(v_0, 2n)|.$$

Linear Isoperimetric Inequality Implies Non-Doubling

Idea of the proof.

Assume that G is m-doubling.
Then

$$B(v_0, 2n) \subset \bigcup_{i=1}^{m} B(y_i, n).$$

Hence some v_1 satisfies

$$|B(v_1, n)| \geq \frac{n}{\mu C m} |B(v_0, n)|.$$

Iterating this argument,

$$|B(v_{i+1}, n)| \geq \left(\frac{n}{\mu C m}\right)^i |B(v_0, n)|.$$

But for large i this contradicts the μ-uniform bound on ball growth. □

Basic References

Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391–413.

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Questions?