Radius and diamater of weakly open subsets in Banach spaces

Abraham Rueda Zoca
Workshop on Geometric and Functional Analysis

Universidad de Granada
Departamento de Análisis Matemático

Support

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Colaborators

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The above examples satisfy that the unit ball contain non-empty relatively weakly open subsets of arbitrarily small diameter (they have the CPCP!).

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If possible, with the inf. of diameter of weakly open subsets equal to 1.

First approach

Theorem (G. López-Pérez, E. Martínez Vañó and A.R.Z. (2025))

Given $\varepsilon > 0$ there exists an equivalent norm on $X = c_0 \oplus \mathbb{R}$ with the rBWOP and such that

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Geometric idea: Find a suitable subset $K \subseteq B_{c_0}$ such that $\mathrm{diam}\,(K) = 1$ made of pointwise positive elements. Take the renorming $|\cdot|$ on $c_0 \oplus_\infty \mathbb{R}$ whose unit ball is

$$B = \overline{\mathrm{conv}}(K \times \{1\} \cup -K \times \{-1\} \cup ((1-\varepsilon)B_{c_0 \oplus_\infty \mathbb{R}} + \varepsilon B_{c_0 \times \{0\}})).$$

Killing big diameter slices

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Now $(1-\varepsilon)B_{c_0 \oplus_\infty \mathbb{R}} \subseteq B$ implies

$$\text{diam}_{|\cdot|}(S) \leq \frac{1}{1-\varepsilon} \text{diam}_{\Vert\cdot\Vert_\infty}(S) \approx \frac{1}{1-\varepsilon}$$

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To prove rBWOP we must show that, given $(x, \alpha) \in c_0 \oplus \mathbb{R}$ (x finitely supported), $W \subseteq B$ weak open and $\varepsilon > 0$, $\exists$ $(y, \beta) \in W$ such that $|(x, \alpha) - (y, \beta)| \geq 1 - \varepsilon$.

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Call $U := K \times \{1\}$, $V := ((1-\varepsilon)B_{c_0 \oplus_\infty \mathbb{R}} + \varepsilon B_{c_0 \times \{0\}})$. Observe that $\frac{U-U}{2} \subseteq V$.

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$\lambda(k, 1) + (1-\lambda)((1-\varepsilon)(y, \beta) + \varepsilon(z, 0)) \in W$, for y, z finitely supported.

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$(x, \alpha) \in c_0 \oplus \mathbb{R}$, x finitely supported, and
$v := \lambda(k, 1) + (1 - \lambda)((1 - \varepsilon)(y, \beta) + \varepsilon(z, 0)) \in W$, y, z finitely suppoted in B_{c_0}.

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1. $v + (e_{k_n}, 0) = \lambda(k + e_{k_n}) + (1-\lambda)((1-\varepsilon)(y + e_{k_n}, \beta) + \varepsilon(z + e_{k_n}, 0)) \in B$ for advanced n (k_n not in the support of y, z).

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Forcing big radius of weak open (II)

$(x,\alpha)\in c_0\oplus\mathbb{R}$, x finitely supported, and
$v:=\lambda(k,1)+(1-\lambda)((1-\varepsilon)(y,\beta)+\varepsilon(z,0))\in W$, y,z finitely suppoted in B_{c_0}.

Claim

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2. $v+(e_{k_n},0)\in W$ for advanced enough n since $(e_{k_n},0)\to(0,0)$ weakly and $v\in W$.
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$$\begin{aligned}|(x,\alpha)-(v+(e_{k_n},0))| &\geq \|(x,\alpha)-(v+(e_{k_n},0))\|_\infty\\ &\geq |(e_{k_n}^*,0)((x,\alpha)-(v+(e_{k_n},0))|=1.\end{aligned}$$

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Such monster exists, it is a P_0-simplex constructed by Argyros, Odell and Rosenthal in 1988.

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- If $m_n,\ l_n,\ K_n$, and $\{g_1, \ldots, g_{l_n}\}$ have been constructed, with $K_n \subseteq B_{\mathrm{span}\{e_1,\ldots,e_{m_n}\}}$ and $g_i \in K_n$ for every $1 \leq i \leq l_n$ such that $\{g_{l_{n-1}+1}, \cdots, g_{l_n}\}$ is an ε_n-net in K_n. Define K_{n+1} as

  $$K_{n+1} = \mathrm{conv}(K_n \cup \{g_i + e_{m_n+i} \colon 1 \leq i \leq l_n\}).$$

  Consider $m_{n+1} = m_n + l_n$ and choose $\{g_{l_n+1}, \ldots, g_{l_{n+1}}\} \subset K_{n+1}$ so that they form an ε_{n+1}-net in K_{n+1}.

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Finally, we define $K_0 = \overline{\cup_n K_n}$.

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Theorem (G. López-Pérez, E. Martínez Vañó and A.R.Z. (2025))

Given $\varepsilon > 0$ there exists an equivalent norm on $c_0 \oplus \mathbb{R}$ with the rBWOP and such that

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Then $X = (\oplus_{n=1}^{\infty} X_n)_1$ has the rBWOP and

$$\inf\{\operatorname{diam}(S) : S \text{ slice of } B_X\} = 1.$$

Connection with the convex point of continuity property

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Question (intermediate)

Does there exists a Banach space X with the rBWOP and being strongly regular?

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