

# Positively norming sets in Banach lattices

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1. POSITIVELY NORMING SETS
2. BANACH LATTICES AND VECTOR MEASURES
3. POSITIVELY NORMING SETS AND VECTOR MEASURES

## 1. POSITIVELY NORMING SETS

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A set  $B \subseteq B_{X^*}$  is said to be **norming** if there is  $c > 0$  such that

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- For  $X = C(K)$  and a dense set  $D \subseteq K$  of the compact Hausdorff topological space  $K$ , the set of all evaluation functionals  $\{\delta_t : t \in D\}$ .

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► Given a Banach lattice  $E$ , we write

$$E_+ := \{x \in E : x \geq 0\} \quad \text{and} \quad B_{E_+} := \{x^* \in B_{E^*} : x^* \geq 0\}.$$

Definition (Sánchez-Pérez and Tradacete, 2014)

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## Theorem (Sánchez-Pérez and Tradacete, 2014)

Let  $E$  be a Banach lattice. TFAE:

- 1 There is a **relatively norm compact** positively norming set  $H \subseteq B_{E^*}$ .
- 2 There is an **order bounded** positively norming set  $H \subseteq B_{E^*}$ .
- 3  $E$  is lattice isomorphic to  $L_1(\mu)$  for some non-negative measure  $\mu$ .

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### Theorem (Curbera, 1992)

Let  $E$  be an order continuous Banach lattice having a weak order unit.

Then there exist a Banach space  $X$ , a measurable space  $(\Omega, \Sigma)$  and a vector measure  $\nu : \Sigma \rightarrow X$  such that  $E$  is lattice isometric to  $L_1(\nu)$ .

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The linear space  $L_1(\nu)$  of all (equivalence classes of)  $\nu$ -integrable functions is a **Banach lattice** when equipped with the  $\|\nu\|$ -a.e. order and the norm

$$\|f\|_{L_1(\nu)} := \sup_{x^* \in B_{X^*}} \int_{\Omega} |f| d|x^* \circ \nu|.$$

$\|\nu\|$  = semivariation of  $\nu$

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### General question

Which kind of properties of  $\nu$  could have consequences for  $L_1(\nu)$ ?

### 3. POSITIVELY NORMING SETS AND VECTOR MEASURES

## On vector measures with values in $\ell_\infty$

by

S. OKADA (Canberra), J. RODRÍGUEZ (Murcia and Albacete) and  
E. A. SÁNCHEZ-PÉREZ (Valencia)

**Abstract.** We study some aspects of countably additive vector measures with values in  $\ell_\infty$  and the Banach lattices of real-valued functions that are integrable with respect to such a vector measure. On the one hand, we prove that if  $W \subseteq \ell_\infty^*$  is a total set not containing sets equivalent to the canonical basis of  $\ell_1(\mathfrak{c})$ , then there is a non-countably-additive  $\ell_\infty$ -valued map  $\nu$  defined on a  $\sigma$ -algebra such that the composition  $x^* \circ \nu$  is countably additive for every  $x^* \in W$ . On the other hand, we show that a Banach lattice  $E$  is separable whenever it admits a countable, positively norming set and both  $E$  and  $E^*$  are order continuous. As a consequence, if  $\nu$  is a countably additive vector measure defined on a  $\sigma$ -algebra and taking values in a separable Banach space, then the space  $L_1(\nu)$  is separable whenever  $L_1(\nu)^*$  is order continuous.

From now on:

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Theorem 1 (Okada, R., Sánchez-Pérez)

$\left. \begin{array}{l} \nu \text{ has separable range} \\ L_1(\nu)^* \text{ is order continuous} \end{array} \right\} \implies L_1(\nu) \text{ is separable.}$

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$\left. \begin{array}{l} \nu \text{ has separable range} \\ L_1(\nu)^* \text{ is order continuous} \end{array} \right\} \implies L_1(\nu) \text{ is separable.}$

Corollary

$\left. \begin{array}{l} \nu \text{ has separable range} \\ L_1(\nu) \text{ is reflexive} \end{array} \right\} \implies L_1(\nu) \text{ is separable.}$

Theorem 1 (Okada, R., Sánchez-Pérez)

$v$  has separable range &  $L_1(v)^*$  is order continuous  $\implies L_1(v)$  is separable.

Proposition

① For each  $x^* \in B_{X^*}$  we have a functional  $\varphi_{x^*} \in B_{L_1(v)_+}^*$  defined by

$$\varphi_{x^*}(f) := \int_{\Omega} f d|x^* \circ v| \quad \text{for all } f \in L_1(v).$$

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### Theorem 1 (Okada, R., Sánchez-Pérez)

$v$  has **separable** range &  $L_1(v)^*$  is **order continuous**  $\implies L_1(v)$  is **separable**.

### Proposition

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- ③ If  $v$  has **separable** range,  
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Theorem 1 is then a consequence of the following:

### Theorem 2 (Okada, R., Sánchez-Pérez)

Let  $E$  be a Banach lattice such that:

- both  $E$  and  $E^*$  are **order continuous**,
- there is a **countable** positively norming subset of  $B_{E_+^*}$ .

Then  $E$  is **separable**.