

A positive and a negative recent cases of the weak Banach-Saks property

Mauro Sanchiz

Workshop on Geometric and Functional Analysis

20 march 2026

Supported by NAWA under the Ulam postdoctoral program 2024/1/00064



- 1 Introduction
- 2 Results on the weak Banach-Saks property
- 3 $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks
- 4 The Hölder spaces are not weak Banach-Saks

- 1 Introduction
Background
- 2 Results on the weak Banach-Saks property
- 3 $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks
- 4 The Hölder spaces are not weak Banach-Saks

The motivation

- A Banach space X has a compact unit ball $\Leftrightarrow X$ is finite dimensional.
- X has a weakly compact unit ball $\Leftrightarrow X$ is reflexive ($X^{**} = X$).
- X^* has always a weakly* compact unit ball.

The definitions

Definition

A sequence (x_n) in X is Cesàro convergent or mean convergence if the sequence of the arithmetic mean of the n -th first terms is norm convergent, i.e. there exists an $x \in X$ such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n \frac{x_i}{n} - x \right\|_X = 0.$$

The definitions

Definition

A sequence (x_n) in X is Cesàro convergent or mean convergence if the sequence of the arithmetic mean of the n -th first terms is norm convergent, i.e. there exists an $x \in X$ such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n \frac{x_i}{n} - x \right\|_X = 0.$$

The standard sequence (e_n) is Cesàro convergent to 0 in ℓ_p for $1 < p \leq \infty$.
 $(1, 0, 0, \dots)$, $(\frac{1}{2}, \frac{1}{2}, 0, 0, \dots)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots)$, ... converges to 0 in norm.

The definitions

Definition

A sequence (x_n) in X is Cesàro convergent or mean convergence if the sequence of the arithmetic mean of the n -th first terms is norm convergent, i.e. there exists an $x \in X$ such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n \frac{x_i}{n} - x \right\|_X = 0.$$

The standard sequence (e_n) is Cesàro convergent to 0 in ℓ_p for $1 < p \leq \infty$. $(1, 0, 0, \dots)$, $(\frac{1}{2}, \frac{1}{2}, 0, 0, \dots)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots)$, ... converges to 0 in norm.

Definition

A Banach space X is Banach-Saks (respectively weakly Banach-Saks), or a subset $A \subset X$ has the Banach-Saks property, if every bounded sequence (x_n) in X (resp. weakly convergent sequence) has a subsequence (x_{n_k}) which is Cesàro convergent.

- 1 Introduction
Background
- 2 Results on the weak Banach-Saks property
- 3 $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks
- 4 The Hölder spaces are not weak Banach-Saks

The origin of it

Theorem (Banach and Saks, 1930)

The spaces L_p and ℓ_p are Banach-Saks for $1 < p < \infty$.

Theorem (Schreier, 1930)

The space of continuous functions $C[0, 1]$ is not weakly Banach-Saks.

The origin of it

Theorem (Banach and Saks, 1930)

The spaces L_p and ℓ_p are Banach-Saks for $1 < p < \infty$.

Theorem (Schreier, 1930)

The space of continuous functions $C[0, 1]$ is not weakly Banach-Saks.

Theorem (Szlenk, 1965)

L_1 is weakly Banach-Saks.

Theorem (Baernstein, 1972)

There exists a reflexive space X that is not weakly Banach-Saks.

Theorem (Farnum, 1974)

For a compact metric space M , $C(M)$ is weakly Banach-Saks $\Leftrightarrow M^{(\omega)} = \bigcap_{n \in \mathbb{N}} M^{(n)} = \emptyset$ ($M^{(n)}$ being the n -th derivated set).

1 Introduction

2 Results on the weak Banach-Saks property

More recent results on the weak Banach-Saks property

3 $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

4 The Hölder spaces are not weak Banach-Saks

Basic properties

Proposition

The weak Banach-Saks property is hereditary.

Proposition

Since every separable Banach space can be embedded in ℓ_∞ and there exist some separable non-weak Banach-Saks spaces, ℓ_∞ is not weak Banach-Saks.

Proposition

Every Banach space X containing an ℓ_∞ copy or a $C[0, 1]$ copy is not weak Banach-Saks.

Results on the weak Banach-Saks property

Theorem (Kakutani, 1938)

Every uniformly convex Banach space is weakly Banach-Saks (and Banach-Saks).

Theorem (Nishiura and Waterman, 1963)

X is Banach-Saks $\Leftrightarrow X$ is weakly Banach-Saks and reflexive (Banach-Saks \Rightarrow reflexive).

Results on the weak Banach-Saks property

Theorem (Kakutani, 1938)

Every uniformly convex Banach space is weakly Banach-Saks (and Banach-Saks).

Theorem (Nishiura and Waterman, 1963)

X is Banach-Saks $\Leftrightarrow X$ is weakly Banach-Saks and reflexive (Banach-Saks \Rightarrow reflexive).

That is why the weak Banach-Saks property is the most studied.

① Introduction

② Results on the weak Banach-Saks property

More recent results on the weak Banach-Saks property

③ $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

④ The Hölder spaces are not weak Banach-Saks

Theorem (Mercourakis, 1995)

X is weak Banach-Saks \Leftrightarrow every weakly convergent sequence has a subsequence uniformly weakly convergent (i.e. for every ε there exists $n(\varepsilon)$ with $\text{card}(\{n \in \mathbb{N} : x^*(x_n - x)| \geq \varepsilon\}) \leq n(\varepsilon)$).

Theorem (Rosenthal, 1976?)

X is weak Banach-Saks if no weakly convergent sequence (x_n) generates an ℓ_1 -spreading model, meaning the existence of $\delta > 0$ such that

$$\left\| \sum_{n \in s} a_n (x_n - x) \right\| \geq \delta \sum_{n \in s} |a_n|$$

for every $s \subset \mathbb{N}$ with $\text{card}(s) \leq \min(s)$ and every sequence of scalars (a_n) .

Theorem (Kamińska and Lee, 2013)

For a Musielak-Orlicz function Φ , ℓ_Φ is weak Banach-Saks \Leftrightarrow it is separable.

Theorem (Kamińska and Lee, 2014)

Any separable Orlicz-Lorentz space $\Lambda_{\varphi,\omega}(0,a)$, $a \leq \infty$, is weak Banach-Saks.

Theorem (Lopez-Abad, Ruiz and Tradacete, 2014)

If X is weak Banach-Saks, then the convex hull of a Banach subset $A \subset X$ is also Banach-Saks (the hypothesis is necessary).

Theorem (Kamińska and Lee, 2013)

For a Musielak-Orlicz function Φ , ℓ_Φ is weak Banach-Saks \Leftrightarrow it is separable.

Theorem (Kamińska and Lee, 2014)

Any separable Orlicz-Lorentz space $\Lambda_{\varphi,\omega}(0, a)$, $a \leq \infty$, is weak Banach-Saks.

Theorem (Lopez-Abad, Ruiz and Tradacete, 2014)

If X is weak Banach-Saks, then the convex hull of a Banach subset $A \subset X$ is also Banach-Saks (the hypothesis is necessary).

There has also been some quantification on the Banach-Saks property in:

- Bendová, Kalenda, Spurný. Quantification of the Banach-Saks property. *Journal of Functional Analysis* 268, 1733–1754 (2015).
- Silber. Quantification of Banach-Saks properties of higher orders. *Journal of Mathematical Analysis and Applications* 517, 126590 (2023).

① Introduction

② Results on the weak Banach-Saks property

③ $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

How to prove that a Banach space is weak Banach-Saks?

④ The Hölder spaces are not weak Banach-Saks

Variable Lebesgue spaces

Definition

Given $p(\cdot) : \Omega \rightarrow [1, \infty)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is the Banach function space consisting of all $f \in L_0(\Omega, \mu)$ with

$$\rho_{p(\cdot)}\left(\frac{f}{r}\right) := \int_{\Omega} \left|\frac{f(t)}{r}\right|^{p(t)} d\mu(t) < \infty, \text{ for some } r > 0,$$

endowed with the Luxemburg norm

$$\|f\|_{p(\cdot)} := \inf \left\{ r > 0 : \rho\left(\frac{f}{r}\right) \leq 1 \right\}.$$

$$p^+ := \operatorname{ess\,sup}_{t \in \Omega} p(t).$$

$$p^- := \operatorname{ess\,inf}_{t \in \Omega} p(t).$$

① Introduction

② Results on the weak Banach-Saks property

③ $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

How to prove that a Banach space is weak Banach-Saks?

④ The Hölder spaces are not weak Banach-Saks

How to prove that a Banach space is weak Banach-Saks?

To prove the weak Banach-Saks property of a Banach space X try to use any property satisfied by weakly compact sets.

How to prove that a Banach space is weak Banach-Saks?

To prove the weak Banach-Saks property of a Banach space X try to use any property satisfied by weakly compact sets.

Theorem (Hernández, Ruiz, MS, 2021, 2024)

Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $p(t) \neq 1$ a.e.- μ . A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact \Leftrightarrow

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega} |\lambda f(t)|^{p(t)} d\mu(t) = 0.$$

How to prove that a Banach space is weak Banach-Saks?

To prove the weak Banach-Saks property of a Banach space X try to use any property satisfied by weakly compact sets.

Theorem (Hernández, Ruiz, MS, 2021, 2024)

Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $p(t) \neq 1$ a.e.- μ . A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact \Leftrightarrow

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega} |\lambda f(t)|^{p(t)} d\mu(t) = 0.$$

Weirdly enough, if the space X is reflexive, any boundedness condition is a weak compactness condition.

How to prove that a Banach space is weak Banach-Saks?

To prove the weak Banach-Saks property of a Banach space X try to use any property satisfied by weakly compact sets.

Theorem (Hernández, Ruiz, MS, 2021, 2024)

Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $p(t) \neq 1$ a.e.- μ . A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact \Leftrightarrow

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega} |\lambda f(t)|^{p(t)} d\mu(t) = 0.$$

Weirdly enough, if the space X is reflexive, any boundedness condition is a weak compactness condition.

Theorem (Hernández, Ruiz, MS, 2021, 2024)

$L^{p(\cdot)}(\Omega)$ is weak Banach-Saks $\Leftrightarrow p^+ < \infty$.

An idea of the proof

Lemma

$L^{p(\cdot)}(\Omega)$ has the subsequence splitting property $\Leftrightarrow p^+ < \infty$ (it is a p^+ -concave lattice).

Lemma (Flores and Ruiz, 2006)

A Banach lattice X with the subsequence splitting property is weakly Banach-Saks \Leftrightarrow every pairwise disjointly weakly convergent sequence in X has a Cesàro convergent subsequence.

An idea of the proof

Lemma

$L^{p(\cdot)}(\Omega)$ has the subsequence splitting property $\Leftrightarrow p^+ < \infty$ (it is a p^+ -concave lattice).

Lemma (Flores and Ruiz, 2006)

A Banach lattice X with the subsequence splitting property is weakly Banach-Saks \Leftrightarrow every pairwise disjointly weakly convergent sequence in X has a Cesàro convergent subsequence.

Proof: (Step $(*)$ is done using that the sequence is disjoint)

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho_{p(\cdot)} \left(\frac{f_1 + \dots + f_n}{n} \right) &\stackrel{(*)}{=} \lim_{n \rightarrow \infty} \sum_{k=1}^n \rho_{p(\cdot)} \left(\frac{f_k}{n} \right) \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n \sup_{m \in \mathbb{N}} \rho_{p(\cdot)} \left(\frac{f_m}{n} \right) \\ &= \lim_{n \rightarrow \infty} \sup_{m \in \mathbb{N}} \left(n \cdot \rho_{p(\cdot)} \left(\frac{f_m}{n} \right) \right) = \lim_{\lambda \rightarrow 0} \sup_{f \in (f_n)} \frac{1}{\lambda} \rho_{p(\cdot)}(\lambda f_n) = 0. \end{aligned}$$

① Introduction

② Results on the weak Banach-Saks property

③ $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

④ The Hölder spaces are not weak Banach-Saks

How to proof that a Banach space is not weak Banach-Saks?

How to proof that ℓ_∞ is not weak Banach-Saks?

Hölder spaces

Definition

Let $0 < \alpha \leq 1$. The Hölder space $C^\alpha(\Omega)$ is the Banach space

$$C^\alpha(\Omega) := \left\{ f \in C(\Omega) : \rho_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty \right\},$$

endowed with the norm

$$\|f\|_{C^\alpha(\Omega)} = \|f\|_\infty + \rho_\alpha(f).$$

Hölder spaces

Definition

Let $0 < \alpha \leq 1$. The Hölder space $C^\alpha(\Omega)$ is the Banach space

$$C^\alpha(\Omega) := \left\{ f \in C(\Omega) : \rho_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty \right\},$$

endowed with the norm

$$\|f\|_{C^\alpha(\Omega)} = \|f\|_\infty + \rho_\alpha(f).$$

For $\alpha = 1$ we have the space of Lipschitz functions $Lip(\Omega)$.

① Introduction

② Results on the weak Banach-Saks property

③ $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

④ The Hölder spaces are not weak Banach-Saks

How to proof that a Banach space is not weak Banach-Saks?

How to proof that ℓ_∞ is not weak Banach-Saks?

How to proof that a Banach space is not weak Banach-Saks?

Proposition

ℓ_∞ is not weak Banach-Saks.

Proposition

The weak Banach-Saks property is hereditary.

Proposition

It X has an ℓ_∞ -copy, then it is not weakly Banach-Saks (or a $C(M)$ -copy with $M^{(\omega)} \neq \emptyset$)

How to proof that a Banach space is not weak Banach-Saks?

Proposition

ℓ_∞ is not weak Banach-Saks.

Proposition

The weak Banach-Saks property is hereditary.

Proposition

It X has an ℓ_∞ -copy, then it is not weakly Banach-saks (or a $C(M)$ -copy with $M^{(\omega)} \neq \emptyset$)

Theorem (Górka and MS, 2026)

If $\text{card}(M) = \infty$, then ℓ_∞ embeds into $C^\alpha(M)$

Corollary (Górka and MS, 2026)

The space $C^\alpha(M)$ is weak Banach-Saks $\Leftrightarrow \text{card}(M) < \infty$.

Sketch of the proof

Lemma (Cúth, Doucha and Wojtaszczyk, 2016)

For an infinite complete metric space (M, d) , there exists a constant $0 < K \leq 1$ and a sequence of pair of points (x_n, y_n) such that:

- $x_n \neq y_n$ for all $n \in \mathbb{N}$,
 - $x_m \notin B(y_n, K d(x_n, y_n))$ for all $n, m \in \mathbb{N}$, and
 - $B(y_n, K d(x_n, y_n)) \cap B(y_m, K d(x_m, y_m)) = \emptyset$ for all $n \neq m$.
-

Sketch of the proof

Lemma (Cúth, Doucha and Wojtaszczyk, 2016)

For an infinite complete metric space (M, d) , there exists a constant $0 < K \leq 1$ and a sequence of pair of points (x_n, y_n) such that:

- $x_n \neq y_n$ for all $n \in \mathbb{N}$,
- $x_m \notin B(y_n, K d(x_n, y_n))$ for all $n, m \in \mathbb{N}$, and
- $B(y_n, K d(x_n, y_n)) \cap B(y_m, K d(x_m, y_m)) = \emptyset$ for all $n \neq m$.

Define the sequence (f_n) by

$$f_n(x) := \max \left\{ \min \left\{ 1, d^\alpha(x, y_n) - \frac{d^\alpha(x, y_n)}{K^\alpha} \right\}, 0 \right\}, \quad x \in M$$

Sketch of the proof

Lemma (Cúth, Doucha and Wojtaszczyk, 2016)

For an infinite complete metric space (M, d) , there exists a constant $0 < K \leq 1$ and a sequence of pair of points (x_n, y_n) such that:

- $x_n \neq y_n$ for all $n \in \mathbb{N}$,
- $x_m \notin B(y_n, K d(x_n, y_n))$ for all $n, m \in \mathbb{N}$, and
- $B(y_n, K d(x_n, y_n)) \cap B(y_m, K d(x_m, y_m)) = \emptyset$ for all $n \neq m$.

Define the sequence (f_n) by

$$f_n(x) := \max \left\{ \min \left\{ 1, d^\alpha(x, y_n) - \frac{d^\alpha(x, y_n)}{K^\alpha} \right\}, 0 \right\}, \quad x \in M$$

Show that (f_n) is semi-normalized in $C^\alpha(M)$.

Sketch of the proof

Lemma (Cúth, Doucha and Wojtaszczyk, 2016)

For an infinite complete metric space (M, d) , there exists a constant $0 < K \leq 1$ and a sequence of pair of points (x_n, y_n) such that:

- $x_n \neq y_n$ for all $n \in \mathbb{N}$,
- $x_m \notin B(y_n, K d(x_n, y_n))$ for all $n, m \in \mathbb{N}$, and
- $B(y_n, K d(x_n, y_n)) \cap B(y_m, K d(x_m, y_m)) = \emptyset$ for all $n \neq m$.

Define the sequence (f_n) by

$$f_n(x) := \max \left\{ \min \left\{ 1, d^\alpha(x, y_n) - \frac{d^\alpha(x, y_n)}{K^\alpha} \right\}, 0 \right\}, \quad x \in M$$

Show that (f_n) is semi-normalized in $C^\alpha(M)$.

Use the lemma to show that $\text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset$.

Sketch of the proof

Lemma (Cúth, Doucha and Wojtaszczyk, 2016)

For an infinite complete metric space (M, d) , there exists a constant $0 < K \leq 1$ and a sequence of pair of points (x_n, y_n) such that:

- $x_n \neq y_n$ for all $n \in \mathbb{N}$,
- $x_m \notin B(y_n, K d(x_n, y_n))$ for all $n, m \in \mathbb{N}$, and
- $B(y_n, K d(x_n, y_n)) \cap B(y_m, K d(x_m, y_m)) = \emptyset$ for all $n \neq m$.

Define the sequence (f_n) by

$$f_n(x) := \max \left\{ \min \left\{ 1, d^\alpha(x, y_n) - \frac{d^\alpha(x, y_n)}{K^\alpha} \right\}, 0 \right\}, \quad x \in M$$

Show that (f_n) is semi-normalized in $C^\alpha(M)$.

Use the lemma to show that $\text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset$.

Show that (f_n) in $C^\alpha(M)$ is equivalent to (e_n) in ℓ_∞ .

① Introduction

② Results on the weak Banach-Saks property

③ $L^{p(\cdot)}(\Omega)$ are weak Banach-Saks

④ The Hölder spaces are not weak Banach-Saks

How to proof that a Banach space is not weak Banach-Saks?

How to proof that ℓ_∞ is not weak Banach-Saks?

How to proof that ℓ_∞ is not weak Banach-Saks?

Or that $C[0, 1]$ is not weak Banach-Saks

Proof: Every separable Banach Space embeds into ℓ_∞ . In particular $C[0, 1]$, which is not weak Banach-Saks while the property is hereditary. \square

How to proof that ℓ_∞ is not weak Banach-Saks?

Or that $C[0, 1]$ is not weak Banach-Saks

Proof: Every separable Banach Space embeds into ℓ_∞ . In particular $C[0, 1]$, which is not weak Banach-Saks while the property is hereditary. \square

Idea of a direct proof of ℓ_∞ (and $C[0, 1]$ and $C(M)$) not being weak Banach-Saks: We take the family $\mathcal{A} \subset 2^{\mathbb{N}} \setminus \{\emptyset\}$ of maximal Schreier sets

$$\mathcal{A} = \{A \subset \mathbb{N} : |A| = \min(A)\}$$

Take $T : \mathbb{N} \rightarrow \mathcal{A}$ an enumeration of \mathcal{A} , and define the sequence (u_k) in ℓ_∞ as follows

$$u_k(i) := \begin{cases} 1 & \text{if } k \in T(i) \\ 0 & \text{otherwise.} \end{cases}$$

It is shown that:

- (u_n) is weakly-null.
- (u_n) has no Cesàro convergent subsequence.

How to proof that ℓ_∞ is not weak Banach-Saks?

Fix any (k_j) strictly increasing and $N \in \mathbb{N}$. Observe that $k_{N+1} > N$. Let

$$A_N := \{k_{N+1}, \dots, k_{N+N}, k_{N+N+1}, \dots, k_{N+k_{N+1}}\} \in \mathcal{A}.$$

There exists i_0 with $T(i_0) = A$. So $u_{k_j}(i_0) = 1$ for $j \in \{N+1, \dots, N+N\}$.

$$\left\| \frac{1}{2N} \sum_{j=1}^{2N} u_{k_j} \right\|_{\ell_\infty} \geq \frac{1}{2N} \sum_{j=1}^{2N} u_{k_j}(i_0) \geq \frac{1}{2N} \sum_{j=N+1}^{2N} u_{k_j}(i_0) = \frac{1}{2},$$

so the subsequence (u_{k_j}) is not Cèsaro null.

How to proof that ℓ_∞ is not weak Banach-Saks?

Fix any (k_j) strictly increasing and $N \in \mathbb{N}$. Observe that $k_{N+1} > N$. Let

$$A_N := \{k_{N+1}, \dots, k_{N+N}, k_{N+N+1}, \dots, k_{N+k_{N+1}}\} \in \mathcal{A}.$$

There exists i_0 with $T(i_0) = A$. So $u_{k_j}(i_0) = 1$ for $j \in \{N+1, \dots, N+N\}$.

$$\left\| \frac{1}{2N} \sum_{j=1}^{2N} u_{k_j} \right\|_{\ell_\infty} \geq \frac{1}{2N} \sum_{j=1}^{2N} u_{k_j}(i_0) \geq \frac{1}{2N} \sum_{j=N+1}^{2N} u_{k_j}(i_0) = \frac{1}{2},$$

so the subsequence (u_{k_j}) is not Cèsaro null.

Lemma (cf. Toland, 2020)

Let (u_k) in ℓ_∞ with $u_k(i) \xrightarrow{k \rightarrow \infty} 0$ for each $i \in \mathbb{N}$. If for every $\alpha > 0$ and for all strictly increasing natural sequences (k_j) , (i_n) and, there exists $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, n\}$ such that $|u_{k_j}(i_n)| \leq \alpha$, then $u_k \rightarrow 0$ weakly in ℓ_∞ .

How to proof that ℓ_∞ is not weak Banach-Saks?

Fix any (k_j) strictly increasing and $N \in \mathbb{N}$. Observe that $k_{N+1} > N$. Let

$$A_N := \{k_{N+1}, \dots, k_{N+N}, k_{N+N+1}, \dots, k_{N+k_{N+1}}\} \in \mathcal{A}.$$

There exists i_0 with $T(i_0) = A$. So $u_{k_j}(i_0) = 1$ for $j \in \{N+1, \dots, N+N\}$.

$$\left\| \frac{1}{2N} \sum_{j=1}^{2N} u_{k_j} \right\|_{\ell_\infty} \geq \frac{1}{2N} \sum_{j=1}^{2N} u_{k_j}(i_0) \geq \frac{1}{2N} \sum_{j=N+1}^{2N} u_{k_j}(i_0) = \frac{1}{2},$$

so the subsequence (u_{k_j}) is not Cèsaro null.

Lemma (cf. Toland, 2020)

Let (u_k) in ℓ_∞ with $u_k(i) \xrightarrow{k \rightarrow \infty} 0$ for each $i \in \mathbb{N}$. If for every $\alpha > 0$ and for all strictly increasing natural sequences (k_j) , (i_n) and, there exists $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, n\}$ such that $|u_{k_j}(i_n)| \leq \alpha$, then $u_k \rightarrow 0$ weakly in ℓ_∞ .

Fix $\alpha > 0$, (k_j) , (i_n) . Take $n > k_1$. If $k_1 \notin T(i_n)$, then $u_{k_1}(i_n) = 0 < \alpha$, so suppose that $k_1 \in T(i_n)$. This forces $|T(i_n)| = \min(T(i_n)) \leq k_1$. Now, since $k_1 < n$ and $k_1 < k_2 < \dots < k_n$, there must exist at least some $j \in \{1, \dots, n\}$ such that $k_j \notin T(i_n)$. Hence, $|u_{k_j}(i_n)| = 0 < \alpha$. This concludes that $u_k \xrightarrow{w} 0$.

Bibliography

- P. Górká & M. Sanchiz, The weak Banach-Saks property for Hölder spaces, arXiv:2603.02977 (2026)
 - Hernández, F., Ruiz, C. & Sanchiz, M. Weak compactness in variable exponent spaces. *J. Func. Anal.* **281**, 109087 (2021)
 - Hernández, F., Ruiz, C. & Sanchiz, M. Weak compactness and representation in variable exponent Lebesgue spaces on infinite measure spaces. *RACSAM*. **116** (2022), 152
 - F.L. Hernández, C. Ruiz & M. Sanchiz, Remarks on weak compactness criteria in variable exponent Lebesgue spaces, *Quaestiones Mathematicae* **47** (2024) 195–211.
-
- S. Banach & S. Saks, Sur la convergence forte dans les champs L_p . *Studia Mathematica*. **2** (1930) 51-57.
 - J. Schreier, Ein Gegenbeispiel zur Theorie der schwachen Konvergenz, *Studia Mathematica* **2** (1930) 58–62.
 - Dodds, P.G., Semenov, E.M., Sukochev, F.A., The Banach–Saks property in rearrangement invariant spaces. *Studia Mathematica* **162** (2004) 263–294.
 - Lopez-Abad, J., C. Ruiz, & P. Tradacete, The convex hull of a Banach–Saks set. *Journal of Functional Analysis* **266** (4) (2014) 2251-2280.

Thanks!