Comparison geometry and isoperimetric inequalities

Toni Ikonen
University of Fribourg
Switzerland

Workshop on Geometric and Functional Analysis
Madrid, Spain, March 18-20

18.03.2026

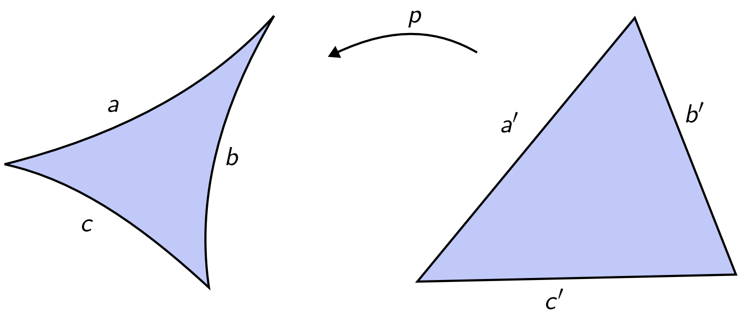
Introduction

Toni Ikonen and Stefan Wenger.
Ultralimits of Sobolev maps and stability of Dehn functions (preprint, 2026).

The CAT(0)-condition

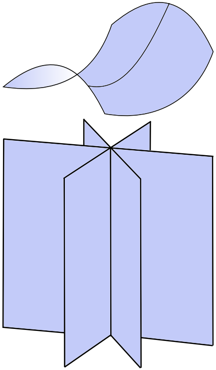
CAT(0)-spaces/Hadamard spaces

A (complete) geodesic space X is $\mathrm{CAT}(0)$ if for every geodesic triangle $T \subset X$ and a comparison triangle $T' \subset \mathbb{R}^2$, there exists a 1-Lipschitz map $T' \to T$ that restricts to an isometry on each side of the triangle.



Examples

- complete and simply connected Riemannian manifolds with non-positive sectional curvature
- stable under gluing along a totally geodesic set
- stable under (ultra)limits
- Euclidean buildings (a particular class of simplicial complexes arising in geometric group theory)
- non-proper examples: Hilbert spaces, some (ultra)limits of CAT(0) spaces
- $\mathrm{CAT}(0)$ short for the last names of Cartan, Aleksandrov and Toponogov.



Euclidean isoperimetric inequality

Euclidean isoperimetric inequality for curves

A geodesic metric space X satisfies the Euclidean isoperimetric inequality if for every Lipschitz $\gamma\colon \mathbb{S}^1 \to X$, there exists a Sobolev map $u \in W^{1,2}(\mathbb{D}, X)$ with boundary trace/values $\mathrm{tr}(u) = \gamma$ and area $\mathrm{Area}(u) \leq (1/(4\pi))\ell^2(\gamma)$.

- $\mathrm{Area}(u)$ is the Hausdorff area (with multiplicity) and $\ell(\gamma)$ is the length of γ.
- metric-valued Sobolev theory $W^{1,2}(\mathbb{D}, X)$ by Reshetnyak, Korevaar–Schoen and Heinonen–Koskela–Shanmugalingam–Tyson (1990s and early 2000s).

Euclidean isoperimetric inequality

Euclidean isoperimetric inequality for curves

A geodesic metric space X satisfies the Euclidean isoperimetric inequality if for every Lipschitz $\gamma\colon \mathbb{S}^1 \to X$, there exists a Sobolev map $u \in W^{1,2}(\mathbb{D}, X)$ with boundary trace/values $\mathrm{tr}(u) = \gamma$ and area $\mathrm{Area}(u) \leq (1/(4\pi))\ell^2(\gamma)$.

- $\mathrm{Area}(u)$ is the Hausdorff area (with multiplicity) and $\ell(\gamma)$ is the length of γ.
- metric-valued Sobolev theory $W^{1,2}(\mathbb{D}, X)$ by Reshetnyak, Korevaar–Schoen and Heinonen–Koskela–Shanmugalingam–Tyson (1990s and early 2000s).
- Constant $1/(4\pi)$ is sharp in Euclidean spaces $\mathbb{R}^n$ and other Hilbert spaces.

Euclidean isoperimetric inequality

Euclidean isoperimetric inequality for curves

A geodesic metric space X satisfies the Euclidean isoperimetric inequality if for every Lipschitz $\gamma\colon \mathbb{S}^1 \to X$, there exists a Sobolev map $u \in W^{1,2}(\mathbb{D}, X)$ with boundary trace/values $\mathrm{tr}(u) = \gamma$ and area $\mathrm{Area}(u) \leq (1/(4\pi))\ell^2(\gamma)$.

- $\mathrm{Area}(u)$ is the Hausdorff area (with multiplicity) and $\ell(\gamma)$ is the length of γ.
- metric-valued Sobolev theory $W^{1,2}(\mathbb{D}, X)$ by Reshetnyak, Korevaar–Schoen and Heinonen–Koskela–Shanmugalingam–Tyson (1990s and early 2000s).
- Constant $1/(4\pi)$ is sharp in Euclidean spaces $\mathbb{R}^n$ and other Hilbert spaces.
- Reshetnyak's majorization principle (1960s): every $\mathrm{CAT}(0)$ space satisfies the property above for a Lipschitz u.

Euclidean isoperimetric inequality

Euclidean isoperimetric inequality for curves

A geodesic metric space X satisfies the Euclidean isoperimetric inequality if for every Lipschitz $\gamma\colon \mathbb{S}^1 \to X$, there exists a Sobolev map $u \in W^{1,2}(\mathbb{D}, X)$ with boundary trace/values $\mathrm{tr}(u) = \gamma$ and area $\mathrm{Area}(u) \leq (1/(4\pi))\ell^2(\gamma)$.

- $\mathrm{Area}(u)$ is the Hausdorff area (with multiplicity) and $\ell(\gamma)$ is the length of γ.
- metric-valued Sobolev theory $W^{1,2}(\mathbb{D}, X)$ by Reshetnyak, Korevaar–Schoen and Heinonen–Koskela–Shanmugalingam–Tyson (1990s and early 2000s).
- Constant $1/(4\pi)$ is sharp in Euclidean spaces $\mathbb{R}^n$ and other Hilbert spaces.
- Reshetnyak's majorization principle (1960s): every $\mathrm{CAT}(0)$ space satisfies the property above for a Lipschitz u.
- Lytchak–Wenger (2019): if X is also proper, i.e. closed and bounded sets are compact, the Euclidean isoperimetric inequality implies that X is CAT(0).

Euclidean isoperimetric inequality

Euclidean isoperimetric inequality for curves

A geodesic metric space X satisfies the Euclidean isoperimetric inequality if for every Lipschitz $\gamma\colon \mathbb{S}^1 \to X$, there exists a Sobolev map $u \in W^{1,2}(\mathbb{D}, X)$ with boundary trace/values $\mathrm{tr}(u) = \gamma$ and area $\mathrm{Area}(u) \leq (1/(4\pi))\ell^2(\gamma)$.

- $\mathrm{Area}(u)$ is the Hausdorff area (with multiplicity) and $\ell(\gamma)$ is the length of γ.
- metric-valued Sobolev theory $W^{1,2}(\mathbb{D}, X)$ by Reshetnyak, Korevaar–Schoen and Heinonen–Koskela–Shanmugalingam–Tyson (1990s and early 2000s).
- Constant $1/(4\pi)$ is sharp in Euclidean spaces $\mathbb{R}^n$ and other Hilbert spaces.
- Reshetnyak's majorization principle (1960s): every $\mathrm{CAT}(0)$ space satisfies the property above for a Lipschitz u.
- Lytchak–Wenger (2019): if X is also proper, i.e. closed and bounded sets are compact, the Euclidean isoperimetric inequality implies that X is $\mathrm{CAT}(0)$.
- Question: why properness?

The proper setting

The Plateau problem in spaces with Euclidean isoperimetric inequalities

Consider a finite length Jordan curve $\Gamma \subset X$. Does Γ admit a filling $\mathbb{D} \to X$ with minimal area in the class $\Lambda(\Gamma, X)$? That is, does there exist $u \in \Lambda(\Gamma, X)$ such that $\mathrm{Area}(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}(v)$?

- Here $\Lambda(\Gamma, X)$ denotes the collection of Sobolev maps $u \in W^{1,2}(\mathbb{D}, X)$ for which the boundary data/trace $\mathrm{tr}(u)\colon \mathbb{S}^1 \to \Gamma$ is a uniform limit of homeomorphisms.

The proper setting

The Plateau problem in spaces with Euclidean isoperimetric inequalities

Consider a finite length Jordan curve $\Gamma \subset X$. Does Γ admit a filling $\mathbb{D} \to X$ with minimal area in the class $\Lambda(\Gamma, X)$? That is, does there exist $u \in \Lambda(\Gamma, X)$ such that $\mathrm{Area}(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}(v)$?

- Here $\Lambda(\Gamma, X)$ denotes the collection of Sobolev maps $u \in W^{1,2}(\mathbb{D}, X)$ for which the boundary data/trace $\mathrm{tr}(u)\colon \mathbb{S}^1 \to \Gamma$ is a uniform limit of homeomorphisms.
- Existence is known if X is proper (Lytchak–Wenger (2010s)). The theory uses compactness results due to Korevaar–Schoen (1990s). The argument is as follows.

Solving the Plateau problem

The standard argument (Douglas–Radó (Euclidean, 30s), Morrey (Riemannian, 40s), Lytchak–Wenger (metric, 2010s))

- Consider a minimizing sequence $(u_i)_{i\geq 1}$ in $\Lambda(\Gamma, X)$.

Solving the Plateau problem

The standard argument (Douglas–Radó (Euclidean, 30s), Morrey (Riemannian, 40s), Lytchak–Wenger (metric, 2010s))

- Consider a minimizing sequence $(u_i)_{i\geq 1}$ in $\Lambda(\Gamma, X)$.
- Morrey's ε-conformality: there exists a diffeomorphism (conformal near the boundary) $\phi_i \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ for which $E_+^2(u_i \circ \phi_i) \leq \mathrm{Area}(u_i) + \varepsilon_i$.
- Here $\mathrm{Area}(u_i) = \mathrm{Area}(u_i \circ \phi_i)$ by invariance of area so $(u_i \circ \phi_i)$ is area- and energy-minimizing sequence in $\Lambda(\Gamma, X)$!

Solving the Plateau problem

The standard argument (Douglas–Radó (Euclidean, 30s), Morrey (Riemannian, 40s), Lytchak–Wenger (metric, 2010s))

- Consider a minimizing sequence $(u_i)_{i\geq 1}$ in $\Lambda(\Gamma, X)$.
- Morrey's ε-conformality: there exists a diffeomorphism (conformal near the boundary) $\phi_i \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ for which $E_+^2(u_i \circ \phi_i) \leq \mathrm{Area}(u_i) + \varepsilon_i$.
- Here $\mathrm{Area}(u_i) = \mathrm{Area}(u_i \circ \phi_i)$ by invariance of area so $(u_i \circ \phi_i)$ is area- and energy-minimizing sequence in $\Lambda(\Gamma, X)$!
- Up to precomposing with a Möbius transformation, the boundary data/traces of $u_i \circ \phi_i$ form an equicontinuous family (Lebesgue–Courant lemma).

Solving the Plateau problem

The standard argument (Douglas–Radó (Euclidean, 30s), Morrey (Riemannian, 40s), Lytchak–Wenger (metric, 2010s))

- Consider a minimizing sequence $(u_i)_{i\geq 1}$ in $\Lambda(\Gamma, X)$.
- Morrey's ε-conformality: there exists a diffeomorphism (conformal near the boundary) $\phi_i\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ for which $E_+^2(u_i \circ \phi_i) \leq \mathrm{Area}(u_i) + \varepsilon_i$.
- Here $\mathrm{Area}(u_i) = \mathrm{Area}(u_i \circ \phi_i)$ by invariance of area so $(u_i \circ \phi_i)$ is area- and energy-minimizing sequence in $\Lambda(\Gamma, X)$!
- Up to precomposing with a Möbius transformation, the boundary data/traces of $u_i \circ \phi_i$ form an equicontinuous family (Lebesgue–Courant lemma).
- In the proper case, up to passing to a subsequence, there exists a limit $u = \lim_i u_i \circ \phi_i$ in $\Lambda(\Gamma, X)$ with minimal area. (Metric-valued Rellich–Kondrachov compactness theorem, pioneered by Korevaar–Schoen 1990s)

Solving the Plateau problem

The standard argument (Douglas–Radó (Euclidean, 30s), Morrey (Riemannian, 40s), Lytchak–Wenger (metric, 2010s))

- Consider a minimizing sequence $(u_i)_{i \geq 1}$ in $\Lambda(\Gamma, X)$.
- Morrey's ε-conformality: there exists a diffeomorphism (conformal near the boundary) $\phi_i \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ for which $E_+^2(u_i \circ \phi_i) \leq \mathrm{Area}(u_i) + \varepsilon_i$.
- Here $\mathrm{Area}(u_i) = \mathrm{Area}(u_i \circ \phi_i)$ by invariance of area so $(u_i \circ \phi_i)$ is area- and energy-minimizing sequence in $\Lambda(\Gamma, X)$!
- Up to precomposing with a Möbius transformation, the boundary data/traces of $u_i \circ \phi_i$ form an equicontinuous family (Lebesgue–Courant lemma).
- In the proper case, up to passing to a subsequence, there exists a limit $u = \lim_i u_i \circ \phi_i$ in $\Lambda(\Gamma, X)$ with minimal area. (Metric-valued Rellich–Kondrachov compactness theorem, pioneered by Korevaar–Schoen 1990s)
- Analyze the limiting map u.

The proper case

The conclusion of the standard argument

Overall, the limiting map u of (a subsequence of) $(u_i \circ \phi_i)$ is in $\Lambda(\Gamma, X)$ and satisfies

$$\mathrm{Area}(u) = E_+^2(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}(v)$$

The proper case

The conclusion of the standard argument

Overall, the limiting map u of (a subsequence of) $(u_i \circ \phi_i)$ is in $\Lambda(\Gamma, X)$ and satisfies

$$\mathrm{Area}(u) = E_+^2(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}(v)$$

- The limiting map is a 'conformal/angle-preserving' area-(and energy-)minimizer.

The proper case

The conclusion of the standard argument

Overall, the limiting map u of (a subsequence of) $(u_i \circ \phi_i)$ is in $\Lambda(\Gamma, X)$ and satisfies

$$\mathrm{Area}(u) = E_+^2(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}(v)$$

- The limiting map is a 'conformal/angle-preserving' area-(and energy-)minimizer.
- Fine analysis by Lytchak–Wenger (2019): given a (Jordan) geodesic triangle $\Gamma \subset X$ in a complete geodesic space X satisfying the Euclidean isoperimetric inequality, the existence of a conformal area-minimizer in $\Lambda(\Gamma, X)$ implies that Γ satisfies the comparison triangle condition!

The proper case

The conclusion of the standard argument

Overall, the limiting map u of (a subsequence of) $(u_i \circ \phi_i)$ is in $\Lambda(\Gamma, X)$ and satisfies

$$\mathrm{Area}(u) = E_+^2(u) = \inf_{v \in \Lambda(\Gamma, X)} \mathrm{Area}(v)$$

- The limiting map is a 'conformal/angle-preserving' area-(and energy-)minimizer.
- Fine analysis by Lytchak–Wenger (2019): given a (Jordan) geodesic triangle $\Gamma \subset X$ in a complete geodesic space X satisfying the Euclidean isoperimetric inequality, the existence of a conformal area-minimizer in $\Lambda(\Gamma, X)$ implies that Γ satisfies the comparison triangle condition!
- In the non-proper case, the compactness theorem can fail! Is it possible to reduce to the Lytchak–Wenger result?

The problem of non-properness for the Plateau problem

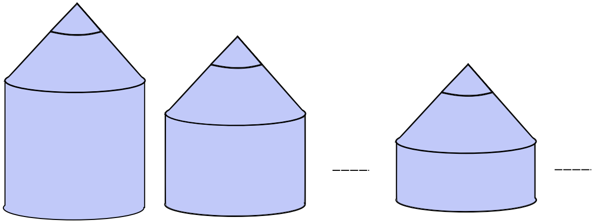


Figure: Plateau problem on a space X formed by gluing the bottom circles

The problem of non-properness for the Plateau problem

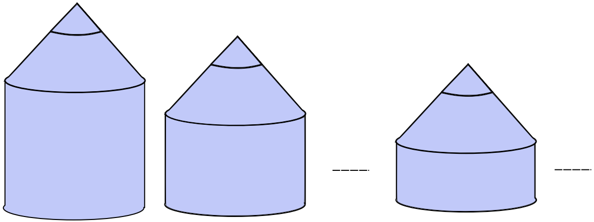


Figure: Plateau problem on a space X formed by gluing the bottom circles

- Non-compactness: need to enlarge the space to find a minimizer!

The problem of non-properness for the Plateau problem

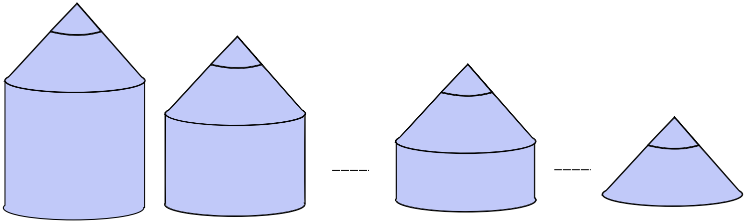


Figure: Plateau problem on a space X formed by gluing the bottom circles

- Non-compactness: need to enlarge the space to find a minimizer!
- Fact: there is a general method of obtaining a natural enlargement of X, called the ultracompletion X_ω, using a non-principal ultrafilter ω.

The problem of non-properness for the Plateau problem

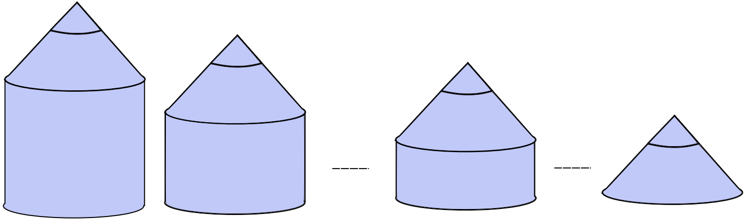


Figure: Plateau problem on a space X formed by gluing the bottom circles

- Non-compactness: need to enlarge the space to find a minimizer!
- Fact: there is a general method of obtaining a natural enlargement of X, called the ultracompletion X_ω, using a non-principal ultrafilter ω.
- Fact: there is also a general method of taking an ultralimit X_ω of a sequence of (pointed) metric/geodesic spaces $(X_i)_{i\geq 1}$.

Non-principal ultrafilters and ultralimits

Comments on non-principal ultrafilters ω on $\mathbb{N}$:

- ω is a $\{0, 1\}$-valued finitely additive probability measure on $\mathbb{N}$ that gives zero measure to finite sets; existence follows from Zorn's lemma.

Non-principal ultrafilters and ultralimits

Comments on non-principal ultrafilters ω on $\mathbb{N}$:

- ω is a $\{0, 1\}$-valued finitely additive probability measure on $\mathbb{N}$ that gives zero measure to finite sets; existence follows from Zorn's lemma.
- essential tool in the structure theory of asymptotic cones of groups/spaces (geometry in the large à la Gromov; pioneered in geometric analysis by van den Dries and Wilkie in 1984 and by Gromov 1993)

Non-principal ultrafilters and ultralimits

Comments on non-principal ultrafilters ω on $\mathbb{N}$:

- ω is a $\{0, 1\}$-valued finitely additive probability measure on $\mathbb{N}$ that gives zero measure to finite sets; existence follows from Zorn's lemma.
- essential tool in the structure theory of asymptotic cones of groups/spaces (geometry in the large à la Gromov; pioneered in geometric analysis by van den Dries and Wilkie in 1984 and by Gromov 1993)
- reason for the utility: ω defines a *generalized limit*, called the *ultralimit* $\lim_\omega \lambda_i$, for any bounded sequence of real numbers (λ_i). The ultralimit is unique and corresponds to a subsequential limit of the sequence.

Non-principal ultrafilters and ultralimits

Comments on non-principal ultrafilters ω on $\mathbb{N}$:

- ω is a $\{0,1\}$-valued finitely additive probability measure on $\mathbb{N}$ that gives zero measure to finite sets; existence follows from Zorn's lemma.
- essential tool in the structure theory of asymptotic cones of groups/spaces (geometry in the large à la Gromov; pioneered in geometric analysis by van den Dries and Wilkie in 1984 and by Gromov 1993)
- reason for the utility: ω defines a *generalized limit*, called the *ultralimit* $\lim_\omega \lambda_i$, for any bounded sequence of real numbers (λ_i). The ultralimit is unique and corresponds to a subsequential limit of the sequence.
- Next: applications to ultralimits of spaces and mappings.

Ultralimits of spaces

We consider now a sequence of pointed spaces $X_i = (X_i, d_i, p_i)$ where $p_i \in X_i$ is arbitrary ('the origin').

Ultralimits of spaces

We consider now a sequence of pointed spaces $X_i = (X_i, d_i, p_i)$ where $p_i \in X_i$ is arbitrary ('the origin'). A sequence (x_i) is bounded if $x_i \in X_i$ and

$$\sup_{i \in \mathbb{N}} d_i(x_i, p_i) < \infty.$$

Ultralimits of spaces

We consider now a sequence of pointed spaces $X_i = (X_i, d_i, p_i)$ where $p_i \in X_i$ is arbitrary ('the origin'). A sequence (x_i) is bounded if $x_i \in X_i$ and

$$\sup_{i \in \mathbb{N}} d_i(x_i, p_i) < \infty.$$

We say that bounded sequences (x_i) and (y_i) are equivalent if

$$\lim_\omega d_i(x_i, y_i) = 0.$$

The equivalence class is denoted by $\lim_\omega x_i$ and their collection is denoted by X_ω.

Ultralimits of spaces

We consider now a sequence of pointed spaces $X_i = (X_i, d_i, p_i)$ where $p_i \in X_i$ is arbitrary ('the origin'). A sequence (x_i) is bounded if $x_i \in X_i$ and

$$\sup_{i \in \mathbb{N}} d_i(x_i, p_i) < \infty.$$

We say that bounded sequences (x_i) and (y_i) are equivalent if

$$\lim_\omega d_i(x_i, y_i) = 0.$$

The equivalence class is denoted by $\lim_\omega x_i$ and their collection is denoted by X_ω. Here $p_\omega = \lim_\omega p_i \in X_\omega$ and X_ω is a metric space when equipped with the distance

$$d_\omega(\lim_\omega x_i, \lim_\omega y_i) = \lim_\omega d_i(x_i, y_i).$$

Ultralimits of spaces

We consider now a sequence of pointed spaces $X_i = (X_i, d_i, p_i)$ where $p_i \in X_i$ is arbitrary ('the origin'). A sequence (x_i) is bounded if $x_i \in X_i$ and

$$\sup_{i \in \mathbb{N}} d_i(x_i, p_i) < \infty.$$

We say that bounded sequences (x_i) and (y_i) are equivalent if

$$\lim_\omega d_i(x_i, y_i) = 0.$$

The equivalence class is denoted by $\lim_\omega x_i$ and their collection is denoted by X_ω. Here $p_\omega = \lim_\omega p_i \in X_\omega$ and X_ω is a metric space when equipped with the distance

$$d_\omega(\lim_\omega x_i, \lim_\omega y_i) = \lim_\omega d_i(x_i, y_i).$$

The triple $(X_\omega, d_\omega, p_\omega)$ is always complete. Also being geodesic, a Banach space, or $\mathrm{CAT}(0)$ are stable under ultralimits!

Utility of ultralimits

Basic facts about the ultralimit X_ω of (X_i). The ultralimit is especially important in the following contexts.

- a sequence of $\mathrm{CAT}(0)$ spaces can fail to be compact in the pointed Gromov–Hausdorff topology, or related topologies (e.g. groups with exponential growth, infinite-dimensional spaces);

Utility of ultralimits

Basic facts about the ultralimit X_ω of (X_i). The ultralimit is especially important in the following contexts.

- a sequence of $\mathrm{CAT}(0)$ spaces can fail to be compact in the pointed Gromov–Hausdorff topology, or related topologies (e.g. groups with exponential growth, infinite-dimensional spaces);
- a sequence of volume-diameter bounded sequences of closed Riemannian manifolds have similar problems.

Utility of ultralimits

Basic facts about the ultralimit X_ω of (X_i). The ultralimit is especially important in the following contexts.

- a sequence of $\mathrm{CAT}(0)$ spaces can fail to be compact in the pointed Gromov–Hausdorff topology, or related topologies (e.g. groups with exponential growth, infinite-dimensional spaces);
- a sequence of volume-diameter bounded sequences of closed Riemannian manifolds have similar problems.

We are motivated by the following Question: are Euclidean isoperimetric inequalities stable? For this, we need to take (ultra)limits of maps!

Ultralimits of maps

Given sequences of pointed metric spaces X_i and Y_i, a sequence of Lipschitz maps $u_i\colon Y_i \to X_i$ is called Lipschitz bounded if

$$\sup_{i\in\mathbb{N}} \left(d_i(p_i, u_i(q_i)) + \mathrm{LIP}(u_i)\right) < \infty.$$

Ultralimits of maps

Given sequences of pointed metric spaces X_i and Y_i, a sequence of Lipschitz maps $u_i\colon Y_i \to X_i$ is called Lipschitz bounded if

$$\sup_{i\in\mathbb{N}} \left(d_i(p_i, u_i(q_i)) + \mathrm{LIP}(u_i)\right) < \infty.$$

Fact: there exists a unique pointwise ultralimit $u_\omega\colon Y_\omega \to X_\omega$ with Lipschitz constant $\mathrm{LIP}(u) \leq \lim_\omega \mathrm{LIP}(u_i)$: $u_\omega(\lim_\omega x_i) = \lim_\omega u_i(x_i)$ ('ultralimit Arzelà–Ascoli').

Ultralimits of maps

Given sequences of pointed metric spaces X_i and Y_i, a sequence of Lipschitz maps $u_i \colon Y_i \to X_i$ is called Lipschitz bounded if

$$\sup_{i\in\mathbb{N}} \left(d_i(p_i, u_i(q_i)) + \mathrm{LIP}(u_i)\right) < \infty.$$

Fact: there exists a unique pointwise ultralimit $u_\omega \colon Y_\omega \to X_\omega$ with Lipschitz constant $\mathrm{LIP}(u) \leq \lim_\omega \mathrm{LIP}(u_i)$: $u_\omega(\lim_\omega x_i) = \lim_\omega u_i(x_i)$ ('ultralimit Arzelà–Ascoli').
Similarly, let $p > 1$. A sequence of Sobolev maps $u_i \in W^{1,p}(\Omega, X_i)$, from a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, is called p-bounded if

$$\sup_{i\in\mathbb{N}} \left(\|d_i(p_i, u_i)\|^p_{L^p(\Omega)} + E^p_+(u_i) \right) < \infty.$$

Question: is there a Sobolev analog of u_ω ('ultralimit weak compactness')?

A limiting result

Theorem (I. and Wenger, 2026)

There is a unique assignment of a map $u_\omega \in W^{1,p}(\Omega, X_\omega)$ to each p-bounded sequence of Sobolev maps $(u_i \colon \Omega \to X_i)$ such that the following properties hold:

A limiting result

Theorem (I. and Wenger, 2026)

There is a unique assignment of a map $u_\omega \in W^{1,p}(\Omega, X_\omega)$ to each p-bounded sequence of Sobolev maps $(u_i : \Omega \to X_i)$ such that the following properties hold:

(i) If the sequence (u_i) is Lipschitz bounded, then u_ω agrees with the pointwise ultralimit of (u_i).

(ii) If $(\varphi_i : X_i \to Y_i)$ is a Lipschitz bounded sequence of maps, then the sequence of maps $v_i = \varphi_i \circ u_i$ satisfies $v_\omega = \varphi_\omega \circ u_\omega$ almost everywhere.

(iii) We have

$$\|d_\omega(u_\omega, v_\omega)\|_{L^p(\Omega)} = \lim_\omega \|d_i(u_i, v_i)\|_{L^p(\Omega)}$$

for all p-bounded sequences $(u_i : \Omega \to X_i)$ and $(v_i : \Omega \to X_i)$.

(iv) Compatability with traces and lower semicontinuity of volumes and energies.

- Properties (i)-(iii) above uniquely characterize the ultralimit.

Ingredients of the proof

- Consider a p-bounded sequence $(u_i\colon \Omega \to X_i)$. Embed X_i to the space of bounded functions $\ell^\infty(X_i)$ sending p_i to the origin; here $\ell^\infty(X_i)$ has strong componentwise Lipschitz extension properties (injective Banach space).

Ingredients of the proof

- Consider a p-bounded sequence $(u_i \colon \Omega \to X_i)$. Embed X_i to the space of bounded functions $\ell^\infty(X_i)$ sending p_i to the origin; here $\ell^\infty(X_i)$ has strong componentwise Lipschitz extension properties (injective Banach space).
- Quantitative Lusin–Lipschitz approximation: for each $k \in \mathbb{N}$, there exists Lipschitz bounded sequence $(u_i^k \colon \Omega \to \ell^\infty(X_i))$ so that

$$\|d_i(u_i^k, u_i)\|_{L^p(\Omega)}^p + E_+^p(u_i^k - u_i) \leq c2^{-kp}$$

for a c depending only on Ω, p, and the bound on the Sobolev data.

Ingredients of the proof

- Consider a p-bounded sequence $(u_i\colon \Omega \to X_i)$. Embed X_i to the space of bounded functions $\ell^\infty(X_i)$ sending p_i to the origin; here $\ell^\infty(X_i)$ has strong componentwise Lipschitz extension properties (injective Banach space).
- Quantitative Lusin–Lipschitz approximation: for each $k \in \mathbb{N}$, there exists Lipschitz bounded sequence $(u_i^k\colon \Omega \to \ell^\infty(X_i))$ so that

$$\|d_i(u_i^k, u_i)\|_{L^p(\Omega)}^p + E_+^p(u_i^k - u_i) \leq c2^{-kp}$$

  for a c depending only on Ω, p, and the bound on the Sobolev data.
- A key property: we may require that there are $E_i^1 \subset E_i^2 \subset \cdots \subset \Omega$ such that

$$|\Omega \setminus E_i^k| \leq |\Omega| 2^{-kp} \quad \text{and} \quad u_i|_{E_i^k} = u_i^k|_{E_i^k}.$$

  In fact, u_i^k is defined as a Lipschitz extension of $u_i|_{E_i^k}$.

Problems in the construction

A candidate for the ultralimit: the ultralimits $u^k_\omega\colon \Omega \to (\ell^\infty(X_i))_\omega$ are Lipschitz, converging to some u_ω in $L^p(\Omega, (\ell^\infty(X_i))_\omega)$ as $k \to \infty$.

Problems in the construction

A candidate for the ultralimit: the ultralimits $u^k_\omega \colon \Omega \to (\ell^\infty(X_i))_\omega$ are Lipschitz, converging to some u_ω in $L^p(\Omega, (\ell^\infty(X_i))_\omega)$ as $k \to \infty$.

- Problem A: $(X_i)_\omega$ is typically a tiny subspace of $(\ell^\infty(X_i))_\omega$. Why does u_ω take values in $(X_i)_\omega$?
- Problem B: is the resulting map in $W^{1,p}(\Omega, (\ell^\infty(X_i))_\omega)$ (every Sobolev exponent $q < p$ is much easier)? What about semicontinuity of energies and volumes/areas and traces?

Problems in the construction

A candidate for the ultralimit: the ultralimits $u_\omega^k \colon \Omega \to (\ell^\infty(X_i))_\omega$ are Lipschitz, converging to some u_ω in $L^p(\Omega, (\ell^\infty(X_i))_\omega)$ as $k \to \infty$.

- Problem A: $(X_i)_\omega$ is typically a tiny subspace of $(\ell^\infty(X_i))_\omega$. Why does u_ω take values in $(X_i)_\omega$?
- Problem B: is the resulting map in $W^{1,p}(\Omega, (\ell^\infty(X_i))_\omega)$ (every Sobolev exponent $q < p$ is much easier)? What about semicontinuity of energies and volumes/areas and traces?

Regarding Problem A, we show that for each $k \in \mathbb{N}$ there exists a compact set $E_\omega^k \subset \overline{\Omega}$ that is an ultralimit of (E_i^k) that are nested and

$$|\Omega \setminus E_\omega^k| \leq |\Omega| 2^{-kp}, \quad u_\omega|_{\Omega \cap E_\omega^k} = u_\omega^k|_{\Omega \cap E_\omega^k}, \quad \text{and} \quad u_\omega(\Omega \cap E_\omega^k) \subset (X_i)_\omega.$$

Problems in the construction

A candidate for the ultralimit: the ultralimits $u_\omega^k \colon \Omega \to (\ell^\infty(X_i))_\omega$ are Lipschitz, converging to some u_ω in $L^p(\Omega, (\ell^\infty(X_i))_\omega)$ as $k \to \infty$.

- Problem A: $(X_i)_\omega$ is typically a tiny subspace of $(\ell^\infty(X_i))_\omega$. Why does u_ω take values in $(X_i)_\omega$?
- Problem B: is the resulting map in $W^{1,p}(\Omega, (\ell^\infty(X_i))_\omega)$ (every Sobolev exponent $q < p$ is much easier)? What about semicontinuity of energies and volumes/areas and traces?

Regarding Problem A, we show that for each $k \in \mathbb{N}$ there exists a compact set $E_\omega^k \subset \overline{\Omega}$ that is an ultralimit of (E_i^k) that are nested and

$$|\Omega \setminus E_\omega^k| \leq |\Omega| 2^{-kp}, \quad u_\omega|_{\Omega \cap E_\omega^k} = u_\omega^k|_{\Omega \cap E_\omega^k}, \quad \text{and} \quad u_\omega(\Omega \cap E_\omega^k) \subset (X_i)_\omega.$$

Problem B is solved by extending a Gromov's embedding theorem (1981) to a stronger version, tailored to ultralimits (technical).

Stability of isoperimetric profiles

Theorem: lower semicontunuity of area (I. and Wenger, 2026)

Let $(X_i)_{i\geq 1}$ be a sequence of (pointed) geodesic spaces and $X_\omega = \lim_\omega X_i$ an ultralimit, and let $\gamma_i\colon \mathbb{S}^1 \to X_i$ be Lipschitz bounded, with ultralimit $\gamma_\omega = \lim_\omega \gamma_i$. Then there exists $u \in W^{1,2}(\mathbb{D}, X_\omega)$ with

$$\mathrm{tr}(u) = \gamma_\omega \quad \text{and} \quad \mathrm{Area}(u) \leq \lim_\omega \inf_{\substack{\mathrm{tr}(v)=\gamma_i \\ v\in W^{1,2}(\mathbb{D},X_i)}} \mathrm{Area}(v).$$

Stability of isoperimetric profiles

Theorem: lower semicontunuity of area (I. and Wenger, 2026)

Let $(X_i)_{i\geq 1}$ be a sequence of (pointed) geodesic spaces and $X_\omega = \lim_\omega X_i$ an ultralimit, and let $\gamma_i\colon \mathbb{S}^1 \to X_i$ be Lipschitz bounded, with ultralimit $\gamma_\omega = \lim_\omega \gamma_i$. Then there exists $u \in W^{1,2}(\mathbb{D}, X_\omega)$ with

$$\operatorname{tr}(u) = \gamma_\omega \quad \text{and} \quad \operatorname{Area}(u) \leq \lim_\omega \inf_{\substack{\operatorname{tr}(v)=\gamma_i \\ v\in W^{1,2}(\mathbb{D},X_i)}} \operatorname{Area}(v).$$

Corollary: stability of isoperimetric inequalities (I. and Wenger, 2026)

Let $(X_i)_{i\geq 1}$ be a sequence of (pointed) geodesic spaces and $X_\omega = \lim_\omega X_i$ an ultralimit. If each $(X_i)_{i\geq 1}$ satisfies a Euclidean isoperimetric inequality, then so does X_ω.

- Answers several questions by Wenger (2019), Lytchak–Wenger–Young (2020), Stadler–Wenger (2025).

Ideas of the proof

Let $L > \sup_{i \in \mathbb{N}} \mathrm{LIP}(\gamma_i)$.

- A mapping cylinder of γ_i: glue to X_i the cylinder $Z = [0, 3L] \times \mathbb{S}^1(L) \subset \mathbb{R}^3$ and identify $\gamma_i(s)$ with $(0, Ls)$ for $s \in \mathbb{S}^1(L)$.

Ideas of the proof

Let $L > \sup_{i\in\mathbb{N}} \mathrm{LIP}(\gamma_i)$.

- A mapping cylinder of γ_i: glue to X_i the cylinder $Z = [0, 3L] \times \mathbb{S}^1(L) \subset \mathbb{R}^3$ and identify $\gamma_i(s)$ with $(0, Ls)$ for $s \in \mathbb{S}^1(L)$.
- Almost minimizers: we consider $u_i \in W^{1,2}(\mathbb{D}, X_i)$ that satisfy

$$\mathrm{tr}(u_i) = \gamma_i \quad \text{and} \quad \mathrm{Area}(u_i) \leq 2^{-i} + \inf_{\substack{\mathrm{tr}(v)=\gamma_i \\ v\in W^{1,2}(\mathbb{D},X_i)}} \mathrm{Area}(v).$$

Ideas of the proof

Let $L > \sup_{i\in\mathbb{N}} \mathrm{LIP}(\gamma_i)$.

- A mapping cylinder of γ_i: glue to X_i the cylinder $Z = [0, 3L] \times \mathbb{S}^1(L) \subset \mathbb{R}^3$ and identify $\gamma_i(s)$ with $(0, Ls)$ for $s \in \mathbb{S}^1(L)$.
- Almost minimizers: we consider $u_i \in W^{1,2}(\mathbb{D}, X_i)$ that satisfy

$$\mathrm{tr}(u_i) = \gamma_i \quad \text{and} \quad \mathrm{Area}(u_i) \leq 2^{-i} + \inf_{\substack{\mathrm{tr}(v)=\gamma_i \\ v\in W^{1,2}(\mathbb{D},X_i)}} \mathrm{Area}(v).$$

- Extending to the cylinder: we define

$$v_i(z) = \begin{cases} u_i(2z), & \text{for } |z| < 1/2 \\ \left(3L(2|z|-1), L\dfrac{z}{|z|}\right), & \text{for } 1/2 \leq |z| < 1. \end{cases}$$

Ideas of the proof

Let $L > \sup_{i\in\mathbb{N}} \mathrm{LIP}(\gamma_i)$.

- A mapping cylinder of γ_i: glue to X_i the cylinder $Z = [0, 3L] \times \mathbb{S}^1(L) \subset \mathbb{R}^3$ and identify $\gamma_i(s)$ with $(0, Ls)$ for $s \in \mathbb{S}^1(L)$.
- Almost minimizers: we consider $u_i \in W^{1,2}(\mathbb{D}, X_i)$ that satisfy

$$\mathrm{tr}(u_i) = \gamma_i \quad \text{and} \quad \mathrm{Area}(u_i) \leq 2^{-i} + \inf_{\substack{\mathrm{tr}(v)=\gamma_i \\ v\in W^{1,2}(\mathbb{D},X_i)}} \mathrm{Area}(v).$$

- Extending to the cylinder: we define

$$v_i(z) = \begin{cases} u_i(2z), & \text{for } |z| < 1/2 \\ \left(3L(2|z|-1), L\dfrac{z}{|z|}\right), & \text{for } 1/2 \leq |z| < 1. \end{cases}$$

- From area to energy bounds: for suitable diffeomorphisms $\phi_i \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, the sequence $w_i = v_i \circ \phi_i$ is 2-bounded and equicontinuous at the boundary.

Ideas of the proof

We proceed as follows.

- The ultralimit: we consider the ultralimit w_ω and note that $(Y_i)_\omega$ is sometric to the mapping cylinder of γ_ω.

Ideas of the proof

We proceed as follows.

- The ultralimit: we consider the ultralimit w_ω and note that $(Y_i)_\omega$ is sometric to the mapping cylinder of γ_ω.
- Problem with traces: the trace of w_ω may fail to be $z \mapsto (3L, Lz)$ in $z \in \mathbb{S}^1$.

Ideas of the proof

We proceed as follows.

- The ultralimit: we consider the ultralimit w_ω and note that $(Y_i)_\omega$ is sometric to the mapping cylinder of γ_ω.
- Problem with traces: the trace of w_ω may fail to be $z \mapsto (3L, Lz)$ in $z \in \mathbb{S}^1$.
- The point of the mapping cylinder: we can homotope the ultralimit w_ω to a map $h \in W^{1,2}(\mathbb{D}, (Y_i)_\omega)$, only changing the map w_ω near the boundary $\mathbb{S}^1$, while only increasing area in the cylinder $(Y_i)_\omega \setminus (X_i)_\omega$, so that $\mathrm{tr}(h)(z) = (3L, Lz)$

Ideas of the proof

We proceed as follows.

- The ultralimit: we consider the ultralimit w_ω and note that $(Y_i)_\omega$ is sometric to the mapping cylinder of γ_ω.
- Problem with traces: the trace of w_ω may fail to be $z \mapsto (3L, Lz)$ in $z \in \mathbb{S}^1$.
- The point of the mapping cylinder: we can homotope the ultralimit w_ω to a map $h \in W^{1,2}(\mathbb{D}, (Y_i)_\omega)$, only changing the map w_ω near the boundary $\mathbb{S}^1$, while only increasing area in the cylinder $(Y_i)_\omega \setminus (X_i)_\omega$, so that $\mathrm{tr}(h)(z) = (3L, Lz)$
- The final map: the post-composition of h and the projection $(Y_i)_\omega \to (X_i)_\omega$ defines $u \in W^{1,2}(\mathbb{D}, (X_i)_\omega)$ with

$$\mathrm{tr}(u) = \gamma_\omega \quad \text{and} \quad \mathrm{Area}(u) \leq \lim_\omega \inf_{\substack{\mathrm{tr}(v)=\gamma_i \\ v \in W^{1,2}(\mathbb{D}, X_i)}} \mathrm{Area}(v).$$

An applications of the methods

Theorem (Stadler–Wenger, 2025)

If a geodesic and complete metric space X satisfies the Euclidean isoperimetric inequality, then X is CAT(0).

An applications of the methods

Theorem (Stadler–Wenger, 2025)

If a geodesic and complete metric space X satisfies the Euclidean isoperimetric inequality, then X is CAT(0).

- Extends the result by Lytchak–Wenger (2019) to the general case: the CAT(0) condition is equivalent to the Euclidean isoperimetric inequality (geometric vs analytic conditions).

An applications of the methods

Theorem (Stadler–Wenger, 2025)

If a geodesic and complete metric space X satisfies the Euclidean isoperimetric inequality, then X is CAT(0).

- Extends the result by Lytchak–Wenger (2019) to the general case: the CAT(0) condition is equivalent to the Euclidean isoperimetric inequality (geometric vs analytic conditions).
- Idea of proof: the standard minimization argument leads to a conformal area-minimizer in X_ω. The Stadler–Wenger argument involved delicate analysis on the continuity of the minimizer and proving that the minimizer satisfies an intrinsic Euclidean isoperimetric inequality.

An applications of the methods

Theorem (Stadler–Wenger, 2025)

If a geodesic and complete metric space X satisfies the Euclidean isoperimetric inequality, then X is CAT(0).

- Extends the result by Lytchak–Wenger (2019) to the general case: the CAT(0) condition is equivalent to the Euclidean isoperimetric inequality (geometric vs analytic conditions).
- Idea of proof: the standard minimization argument leads to a conformal area-minimizer in X_ω. The Stadler–Wenger argument involved delicate analysis on the continuity of the minimizer and proving that the minimizer satisfies an intrinsic Euclidean isoperimetric inequality.
- Our stability result leads to a significantly simplified proof.

An applications of the methods

Theorem (Stadler–Wenger, 2025)

If a geodesic and complete metric space X satisfies the Euclidean isoperimetric inequality, then X is CAT(0).

- Extends the result by Lytchak–Wenger (2019) to the general case: the CAT(0) condition is equivalent to the Euclidean isoperimetric inequality (geometric vs analytic conditions).
- Idea of proof: the standard minimization argument leads to a conformal area-minimizer in X_ω. The Stadler–Wenger argument involved delicate analysis on the continuity of the minimizer and proving that the minimizer satisfies an intrinsic Euclidean isoperimetric inequality.
- Our stability result leads to a significantly simplified proof.
- These methods are robust enough to characterize locally $\mathrm{CAT}(\kappa)$-spaces and yields stability of more general isoperimetric profiles — previous stability only known under very strong assumptions.

Thank You!