

Orbits of isometry groups in ω -categorical Banach spaces



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de Ciencias

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in joint work with V. Ferenczi and J. López-Abad
Part of my PhD Thesis

WORKSHOP ON GEOMETRIC AND FUNCTIONAL ANALYSIS
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Warning!

All Banach spaces will be assumed to be separable.

Ultrahomogeneity

Definition

A Banach space X is **ultrahomogeneous** (**uH**) if for every isometric embedding $\gamma : E \rightarrow X$ of a finite-dimensional subspace $E \subseteq X$ there exists $T \in \text{Isom}(X)$ (surjective isometry) such that

$$T|_E = \gamma$$

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There is just one orbit.

Definition

A Banach space X is **approximately ultrahomogeneous (AuH)** if for every $\varepsilon > 0$ and for every isometric embedding $\gamma : E \rightarrow X$ of a finite-dimensional subspace $E \subseteq X$ there exists $T \in \text{Isom}(X)$ (surjective isometry) such that

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Every orbit is dense.

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All known separable examples are:

- ▶ The **Hilbert space** ℓ_2 (exact ultrahomogeneity)
- ▶ The **Gurarij space** \mathbb{G} (the unique separable space of almost universal disposition for finite-dimensional spaces)
- ▶ $L_p[0, 1]$ for $1 \leq p < \infty$, $p \neq 4, 6, 8, \dots$ [Lusky, 1978]

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But these spaces satisfy a stronger property...

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Fraïssé spaces

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Open questions

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Theorem (Ferenczi–Rincón–Villamizar, 2025)

Fraïssé \iff AuH + ω -categorical

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X is ω -categorical if and only if it is a unique ultraroot, that is, for every separable Y and every ultrafilter \mathcal{U} ,

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If a group G acts by isometries on a metric space S , we denote by $S // G$ the set of orbit closures. This is a metric space when endowed with the distance between the orbits,

$$d(\overline{G \cdot x}, \overline{G \cdot y}) := \inf_{g \in G} d(gx, y).$$

Let G act by isometries on S^n diagonally, i.e. $g \cdot (s_1, \dots, s_n) := (gs_1, \dots, gs_n)$.

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has the Heine–Borel property (equivalently, if $B_X^n // \text{Isom}(X)$ is compact).

ω -categorical Banach spaces

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Space of “positions” of tuples (x_1, \dots, x_n) .

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Non-examples

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- ▶ Sequence spaces c_0 and ℓ_p , $1 \leq p < \infty$.

Isometries \leftrightarrow Rearranging coordinates and changing signs.

The sequence

$$x_k := (k^{-1/p}, \dots, k^{-1/p}, 0, \dots) \in S_{\ell_p}, \quad k \geq 1,$$

does not have a subsequence whose orbits converge $\implies B_{\ell_p} // \text{Isom}(\ell_p)$ is not compact. Similarly with c_0 .

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- ▶ Sequence space

Isometries \leftrightarrow \mathbb{R}

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Same shape,
but different positions.

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- ▶ $C[0, 1]$
Isometries \leftrightarrow Composing with a homeomorphism of $[0, 1]$ and multiplying by ± 1 .
But every homeomorphism of $[0, 1]$ is strictly monotone, so they cannot “kill” oscillations $\implies B_{C[0,1]} // \text{Isom}(C[0, 1])$ is not compact.

Continuous functions and Banach–Stone isometries

Let K be a compact Hausdorff space.

Theorem (Banach–Stone)

Every linear surjective isometry $T : C(K) \rightarrow C(K)$ is of the form

$$T(f)(t) = v(t) \cdot f(\gamma(t))$$

where $v \in C(K; \{\pm 1\})$ and $\gamma \in \text{Homeo}(K)$. Thus

$$\text{Isom}(C(K)) \cong \text{Homeo}(K) \times C(K; \{\pm 1\}).$$

$$\begin{array}{ccc} K & \xrightarrow{f} & \mathbb{R} \\ \gamma \uparrow & & \downarrow \pm 1 \\ K & \xrightarrow{Tf} & \mathbb{R} \end{array}$$

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Definition

For $G \leq \text{Isom}(X)$, the **G -Banach–Stone group** of the space $C(K; X)$ is the closed subgroup of $\text{Isom}(C(K; X))$ given by

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Closed in the *strong operator topology* (SOT).

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Let $\pi : X \rightarrow X // G$ be the projection. Every Banach–Stone isometry preserves the image under π , that is,

$$\pi(f(K)) = \pi(Tf(K)) \quad \text{for every } f \in C(K; X) \text{ and } T \in \text{BS}_G.$$

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Definition (The image map)

Let $\iota : C(K; X) \rightarrow \mathcal{K}(X // G)$ be given by

$$\iota(f) := \pi(f(K)).$$

$K = 2^{\mathbb{N}}$ is the right choice

Theorem (V. Ferenczi, J. López–Abad, V. O–P)

For $K = 2^{\mathbb{N}}$, the factored map

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Key properties of the compact space $2^{\mathbb{N}}$:

- ▶ **Universality**: for every compact metric K there is $f : 2^{\mathbb{N}} \rightarrow K$ continuous and onto.
- ▶ **Ultrahomogeneity**: if $(U_i)_{i=1}^n$ and $(V_i)_{i=1}^n$ are two finite partitions of $2^{\mathbb{N}}$ into clopen sets, there exists $\gamma \in \text{Homeo}(2^{\mathbb{N}})$ such that $\gamma(U_i) = V_i$ for all i .

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Consequence of the ultrahomogeneity of the *countable atomless Boolean algebra* $\text{Clop}(2^{\mathbb{N}})$ and Stone duality.

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Corollary

If X is ω -categorical then so is $C(2^{\mathbb{N}}; X)$, and

$$C(2^{\mathbb{N}}; X)^n // \text{BS}_{\text{Isom}(X)} \equiv \mathcal{K}(X^n // \text{Isom}(X)).$$

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The space of positions $C(2^{\mathbb{N}}; X)^n // \text{Isom}(C(2^{\mathbb{N}}; X))$ is a quotient of this.

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Bochner spaces and Banach–Lamperti isometries

We say that $\gamma : [0, 1] \rightarrow [0, 1]$ is an **automorphism** of $[0, 1]$ (as a measure space) if

- ▶ It is measurable and has a measurable inverse (a.e.).
- ▶ Both γ and γ^{-1} map null sets to null sets, i.e. $\lambda \ll \gamma_*\lambda \ll \lambda$.

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Theorem (Banach–Lamperti Theorem)

Every linear surjective isometry $T : L_p \rightarrow L_p$, $p \neq 2$, is of the form

$$T(f)(t) = v(t) \cdot h_\gamma(t) \cdot f(\gamma(t))$$

where $v : [0, 1] \rightarrow \{\pm 1\}$, $\gamma \in \text{Aut}^*([0, 1])$ and $h_\gamma := \frac{d\lambda}{d\gamma_*\lambda} \circ \gamma : [0, 1] \rightarrow [0, \infty)$. Thus

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Definition

For $G \leq \text{Isom}(X)$, the **G–Banach–Lamperti group** of the space $L_p([0, 1]; X)$ is the closed subgroup of $\text{Isom}(L_p([0, 1]; X))$ given by

$$\text{BL}_G := \text{Aut}^*([0, 1]) \times L_0([0, 1]; G).$$

A more complicated invariant

How to identify $L_p([0, 1]; X) // \text{BL}_G$?

What property do elements of BL_G preserve?

$$\begin{array}{ccccc} [0, 1] & \xrightarrow{f} & X & \xrightarrow{\cdot h_\gamma} & X \\ \gamma \uparrow & & & & \downarrow v(t) \in G \\ [0, 1] & \xrightarrow{Tf} & & & X \end{array}$$

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Definition (The **normalized distribution map**)

Let $\tau : L_p([0, 1]; X) \rightarrow \mathcal{P}(X // G)$ (space of probability measures, with weak topology) be given by

$$\tau(F) := \mathcal{L}(\pi \circ \hat{T}F) \quad (\text{for } F \text{ of full support}).$$

Why this is the right choice

Let $\mathcal{P}_S(X // G)$ be the family of **spherical measures** on $X // G$, that is, those supported on some sphere $\alpha \cdot S_X // G$, $\alpha \geq 0$.

Theorem (V. Ferenczi, J. López–Abad, V. O–P)

The map τ factors through $L_p([0, 1]; X) // \text{BL}_G$ and the resulting

$$\tilde{\tau} : L_p([0, 1]; X) // \text{BL}_G \rightarrow \mathcal{P}_S(X // G)$$

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$$W_p(\mu, \nu) := \inf \left\{ d_{L_p}(f, g) \mid f, g : [0, 1] \rightarrow X // G, \mathcal{L}(f) = \mu, \mathcal{L}(g) = \nu \right\}.$$

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Key properties of the measure space $([0, 1], \mathcal{B}([0, 1]), \lambda)$:

- **Universality**: for every standard Borel measure space $(\Omega, \mathcal{A}, \mu)$ there is $f : [0, 1] \rightarrow \Omega$ measurable and with $\mathcal{L}(f) = \mu$.

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- ▶ **Ultrahomogeneity**: if $(U_i)_{i \in \mathbb{N}}$ and $(V_i)_{i \in \mathbb{N}}$ are two countable partitions of $[0, 1]$ into measurable sets, there exists $\gamma \in \text{Aut}^*([0, 1])$ **measure-preserving** such that

$$\lambda(\gamma(U_i) \triangle V_i) = |\lambda(U_i) - \lambda(V_i)| \quad \text{for all } i \in \mathbb{N}.$$

Corollary

If X is ω -categorical then so is $L_p([0, 1]; X)$, and

$$L_p([0, 1]; X)^n // \text{BL}_{\text{Isom}(X)} \cong \mathcal{P}_S(X^n // \text{Isom}(X)).$$

The space of positions $L_p([0, 1]; X)^n // \text{Isom}(L_p([0, 1]; X))$ is a quotient of this.

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All L_p spaces, $1 \leq p < \infty$, are ω -categorical. But for $p = 4, 6, 8, \dots$ they are **not AuH**. Why?

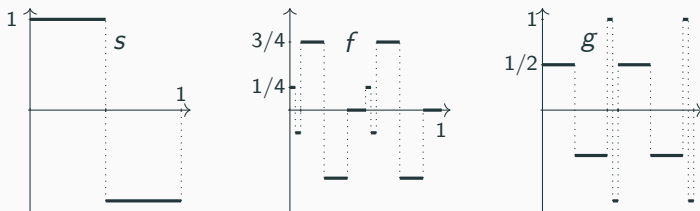
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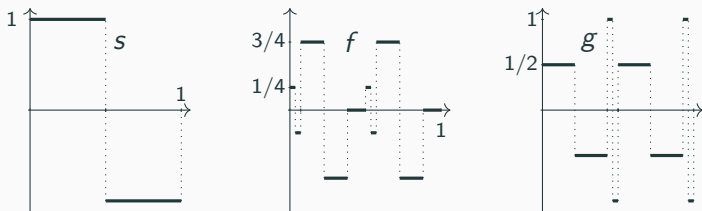
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The pairs (s, f) and (s, g) are normalized and isometrically equivalent. But their distributions are clearly different, so by the previous corollary they are not in the same closed orbit under $\text{Isom}(L_p)$.

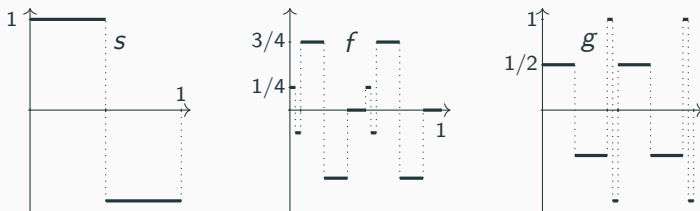
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Thank you for your attention!
