Level sets of prevalent Weierstrass functions

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A function $f\colon [0,1] \to \mathbb{R}$ is *α-Hölder* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leqslant C|x - y|^{\alpha}$$

for all $x, y \in [0,1]$.

A subset A of a Banach space X is *shy* if there exists a Borel measure μ and a Borel set $B \subset X$ containing A such that $0 < \mu(K) < \infty$ for some compact set $K \subset X$ and $\mu(x + B) = 0$ for all $x \in X$.
A set A is *prevalent* if its complement is shy.

A is said to be *d-prevalent* if there is a d-dimensional subspace $S \subset X$, called the *probe space*, such that for any $x \in X$, we have $x + s \in B$ for Lebesgue-almost every $s \in S$, where the Lebesgue measure $\mathcal{L}^d$ is defined on S and B is a Borel set contained in A.

d-prevalent $\Rightarrow$ prevalent: taking μ as the Lebesgue measure on S and K as the closed unit ball in S, we have

$$\mu(x + (X \setminus B)) = \mu(\{s \in S : s - x \notin B\}) = 0$$

for all $x \in X$.

For an integer $b \geqslant 2$, $0 < \alpha \leqslant 1$, and a non-constant $\mathbb{Z}$-periodic Lipschitz function g on $\mathbb{S}^1$, where $\mathbb{S}^1$ is identified with $\mathbb{R}/\mathbb{Z}$,
the *Weierstrass function* $W_g^{\alpha,b}\colon \mathbb{S}^1 \to \mathbb{R}$ is defined by

$$W_g^{\alpha,b}(x) = \sum_{k=0}^{\infty} b^{-\alpha k} g(b^k x).$$

For fixed $b \geqslant 2$ and $0 < \alpha \leqslant 1$, the family of such Weierstrass functions is denoted by $\mathcal{W}^{\alpha,b}$.
These functions are known to be α-Hölder implying that

$$\overline{\dim}_{\mathrm{M}}(\mathrm{graph}(W_g^{\alpha,b})) \leqslant 2 - \alpha,$$

where $\overline{\dim}_{\mathrm{M}}(A)$ denotes the upper Minkowski dimension of a bounded set A.

The functions in $\mathcal{W}^{\alpha,b}$ are in one-to-one correspondence with the Banach space $\mathrm{Lip}(\mathbb{S}^1)$ of Lipschitz functions on $\mathbb{S}^1$, equipped with the norm

$$\|g\|_{\mathrm{Lip}} = \sup_{x\in\mathbb{S}^1} |g(x)| + \sup_{(x,y)\in\mathbb{S}^1\times\mathbb{S}^1} \frac{|g(x)-g(y)|}{|x-y|}.$$

Thus, we identify $\mathcal{W}^{\alpha,b}$ with $\mathrm{Lip}(\mathbb{S}^1)$, and prevalence in $\mathcal{W}^{\alpha,b}$ refers to prevalence under this identification.

The family $\mathcal{W}^{\alpha,b}$ includes well-studied examples such as the classical Weierstass function, where $g(x) = \cos(2\pi x)$, and the Takagi function, where $g(x) = \mathrm{dist}(x, \mathbb{Z})$.

By Marstrand's slicing theorem and $\overline{\dim}_{\mathrm{M}}(\mathrm{graph}(W_g^{\alpha,b})) \leqslant 2-\alpha$ we have

$$\dim_{\mathrm{H}}((W_g^{\alpha,b})^{-1}(\{y\})) = \dim_{\mathrm{H}}(\mathrm{graph}(W_g^{\alpha,b}) \cap \mathrm{proj}_2^{-1}(\{y\}))$$

$$\leqslant \dim_{\mathrm{H}}(\mathrm{graph}(W_g^{\alpha,b})) - 1 \leqslant 1-\alpha$$

for $\mathcal{L}^1$-almost every $y \in \mathbb{R}$, where $\mathrm{proj}_2 \colon \mathbb{R}^2 \to \mathbb{R}$, $\mathrm{proj}_2(x,y) = y$, is the orthogonal projection onto the y-axis.

The level sets of prevalent α-Hölder functions were recently investigated by Anttila, Bárány and Käenmäki:

T.: *A prevalent α-Hölder function f on the unit interval satisfies*

- $\overline{\dim}_{\mathrm{M}}(f^{-1}(\{y\})) \leqslant 1-\alpha$ *for all* $y \in \mathbb{R}$ *provided that* $0 < \alpha < \frac{1}{2}$,
- $\mathcal{L}^1(\{y \in f([0,1]) : \dim_{\mathrm{H}}(f^{-1}(\{y\})) = 1-\alpha\}) > 0$ *provided that* $0 < \alpha < 1$.

It was conjectured that prevalent α-Weierstrass functions exhibit similar behavior.
Indeed, we prove:

T.: *For any integer $b \geqslant 2$, a prevalent function $g \in \mathrm{Lip}(\mathbb{S}^1)$ satisfies*
$\overline{\dim}_{\mathrm{M}}((W_g^{\alpha,b})^{-1}(\{y\})) \leqslant 1-\alpha$ *for all* $y \in \mathbb{R}$ *provided that* $0 < \alpha < \frac{1}{2}$.

and

T.: *For any integer $b \geqslant 2$, a prevalent function $g \in \mathrm{Lip}(\mathbb{S}^1)$ satisfies*
$\mathcal{L}^1(\{y \in W_g^{\alpha,b}(\mathbb{S}^1) : \dim_{\mathrm{H}}((W_g^{\alpha,b})^{-1}(\{y\})) = 1-\alpha\}) > 0$
provided that $0 < \alpha < 1$.

To prove our main results we analyze the occupation measures of Weierstrass functions.

Occupation measures are commonly studied in the context of stochastic processes.

For the one-dimensional Brownian motion B, the occupation measure, defined as $\mu(A) = \mathcal{L}^1(B^{-1}(A))$, is almost surely absolutely continuous with respect to the Lebesgue measure, satisfying the *local time (LT)* condition.

Geman and Horowitz (1980) noted that:
"...results can be successfully applied to random functions and fields, it is difficult to apply them to particular nonrandom functions. For example, an interesting open problem is *to determine which functions representable as Fourier series* (for instance) *are (LT)...*"

Bertoin (1988), (1990) analyzed the occupation measures and Hausdorff dimensions of level sets for certain self-affine functions.

Depending on parameter values, these functions either satisfy the (LT) condition or have a singular occupation measure.

Z.B. (2008) has shown that the occupation measure of the Takagi function is singular, and Lebesgue almost every level set is finite.

For the Weierstrass-Cellerier function, $W(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x)$ it was established that the occupation measure of $W(x) + cx$ is singular for almost every $c \in \mathbb{R}$.

In Z.B. (2010) it was proved that there are no exceptional c values, including the case $c = 0$, confirming that the occupation measure of W is singular.

The Weierstrass-Cellerier function and the Takagi function belong to the class $\mathcal{W}^{1,2}$, whereas our new result addresses prevalent functions in the spaces $\mathcal{W}^{\alpha,b}$, with $0 < \alpha < 1$, showing that these functions have absolutely continuous occupation measures.

Weierstrass embedding

D.: We say that $\Phi\colon [0,1] \to \mathbb{R}^d$ is an *α-bi-Hölder map* if there are constants $c_1, c_2 > 0$ such that $c_1 \leqslant \dfrac{\|\Phi(x) - \Phi(y)\|}{|x-y|^\alpha} \leqslant c_2$ for all $x, y \in [0,1]$ with $x \neq y$.
We choose $\|\cdot\|$ to be the ℓ^∞ norm.

Let $\mathcal{G} = \{g_0, \ldots, g_{d-1}\}$ be a finite collection of Lipschitz functions on $\mathbb{S}^1$.
The *Weierstrass embedding* $\Phi_{\mathcal{G}}^{\alpha,b}\colon \mathbb{S}^1 \to \mathbb{R}^d$ associated to the collection $\mathcal{G}$, integer $b \geqslant 2$, and $0 < \alpha < 1$ is defined by

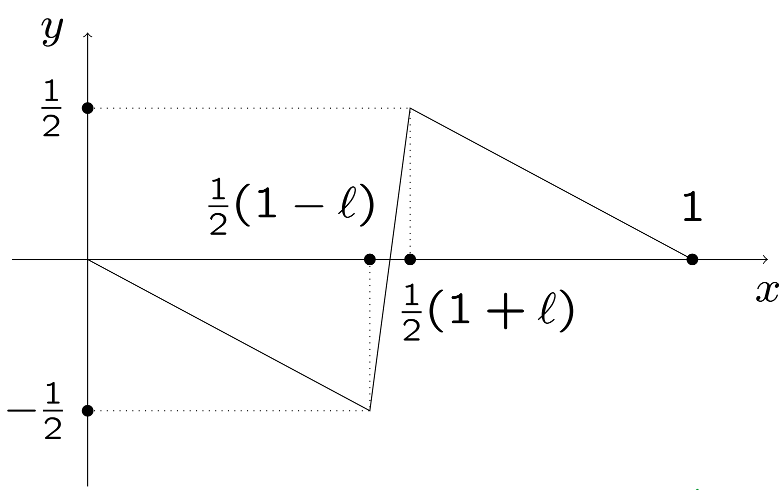
$$\Phi_{\mathcal{G}}^{\alpha,b}(x) = (W_{g_0}^{\alpha,b}(x), W_{g_1}^{\alpha,b}(x), \ldots, W_{g_{d-1}}^{\alpha,b}(x)).$$

T.: *For every integer $b \geqslant 2$ and $0 < \alpha < 1$ there exist $d \in \mathbb{N}$ and a finite collection $\mathcal{G} = \{g_0, \ldots, g_{d-1}\}$ of Lipschitz functions on $\mathbb{S}^1$ such that the Weierstrass embedding $\Phi_{\mathcal{G}}^{\alpha,b}\colon \mathbb{S}^1 \to \mathbb{R}^d$ is α-bi-Hölder.*

Prop.: *Let $b \geqslant 2$ and $d \geqslant 1$ be integers, and let $0 < \alpha < 1$. If $d < \frac{1}{\alpha}$ and $\mathcal{G} = \{g_0, \ldots, g_{d-1}\}$ is a finite collection of Lipschitz functions on $\mathbb{S}^1$, then the Weierstrass embedding $\Phi_{\mathcal{G}}^{\alpha,b}\colon \mathbb{S}^1 \to \mathbb{R}^d$ is not α-bi-Hölder.*

Recall: **T**.: *For every integer $b \geqslant 2$ and $0 < \alpha < 1$ there exist $d \in \mathbb{N}$ and a finite collection $\mathcal{G} = \{g_0, \dots, g_{d-1}\}$ of Lipschitz functions on $\mathbb{S}^1$ such that the Weierstrass embedding $\Phi_{\mathcal{G}}^{\alpha,b} \colon \mathbb{S}^1 \to \mathbb{R}^d$ is α-bi-Hölder.*

Ideas of the proof:



The function h_ℓ with $\ell = \frac{1}{15}$.

Let $0 < \ell \leqslant \frac{1}{2}$ and define a function $\overline{h}_\ell \colon \mathbb{R} \to \mathbb{R}$ by setting $\overline{h}_\ell(x) = h_\ell(x - [x])$ for all $x \in \mathbb{R}$, where $[x]$ is the integer part of x and $h_\ell \colon [0,1) \to \mathbb{R}$ satisfies

$$h_\ell(x) = \begin{cases} -\frac{1}{1-\ell}x, & \text{if } 0 \leqslant x < \frac{1}{2}(1-\ell), \\ \frac{1}{\ell}x - \frac{1}{2\ell}, & \text{if } \frac{1}{2}(1-\ell) \leqslant x < \frac{1}{2}(1+\ell), \\ -\frac{1}{1-\ell}x + \frac{1}{1-\ell}, & \text{if } \frac{1}{2}(1+\ell) \leqslant x < 1 \end{cases}$$

for all $x \in [0,1)$.

Choose suitable $\ell_0 \in \mathbb{N}$.

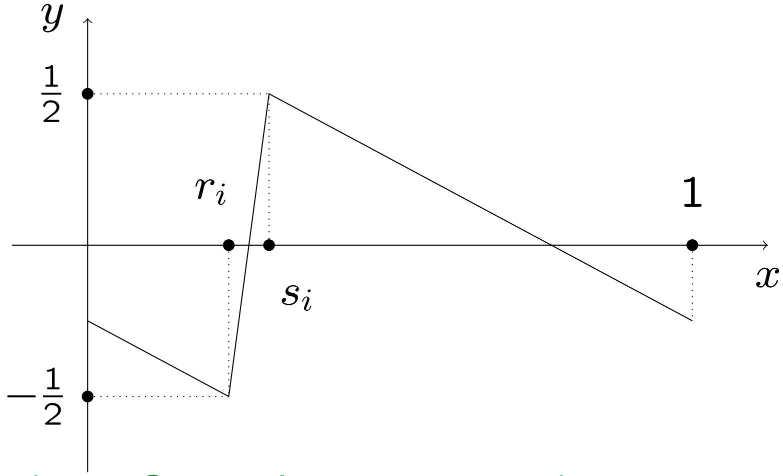
Write $r_i = ib^{-\ell_0-3}$, $s_i = r_i + b^{-\ell_0}$, and $\ell = s_i - r_i = b^{-\ell_0} < \frac{1}{2}$, and define a function $g_i\colon [0,1) \to \mathbb{R}$ by setting

$$g_i(x) = \overline{h}_\ell|_{[0,1)}(x - \tfrac{1}{2}(r_i + s_i - 1))$$

for all $x \in [0,1)$ and $i \in \{0, \ldots, b^{\ell_0+3} - 1\}$.

Let $d_0 = b^{\ell_0+3}$ and $\mathcal{G}_0 = \{g_i\}_{i=0}^{d_0-1}$.

We need to add some more functions g_i $i = d_0, ..., d-1$ by using this lemma with $D = b^{-\ell_0-1}$:



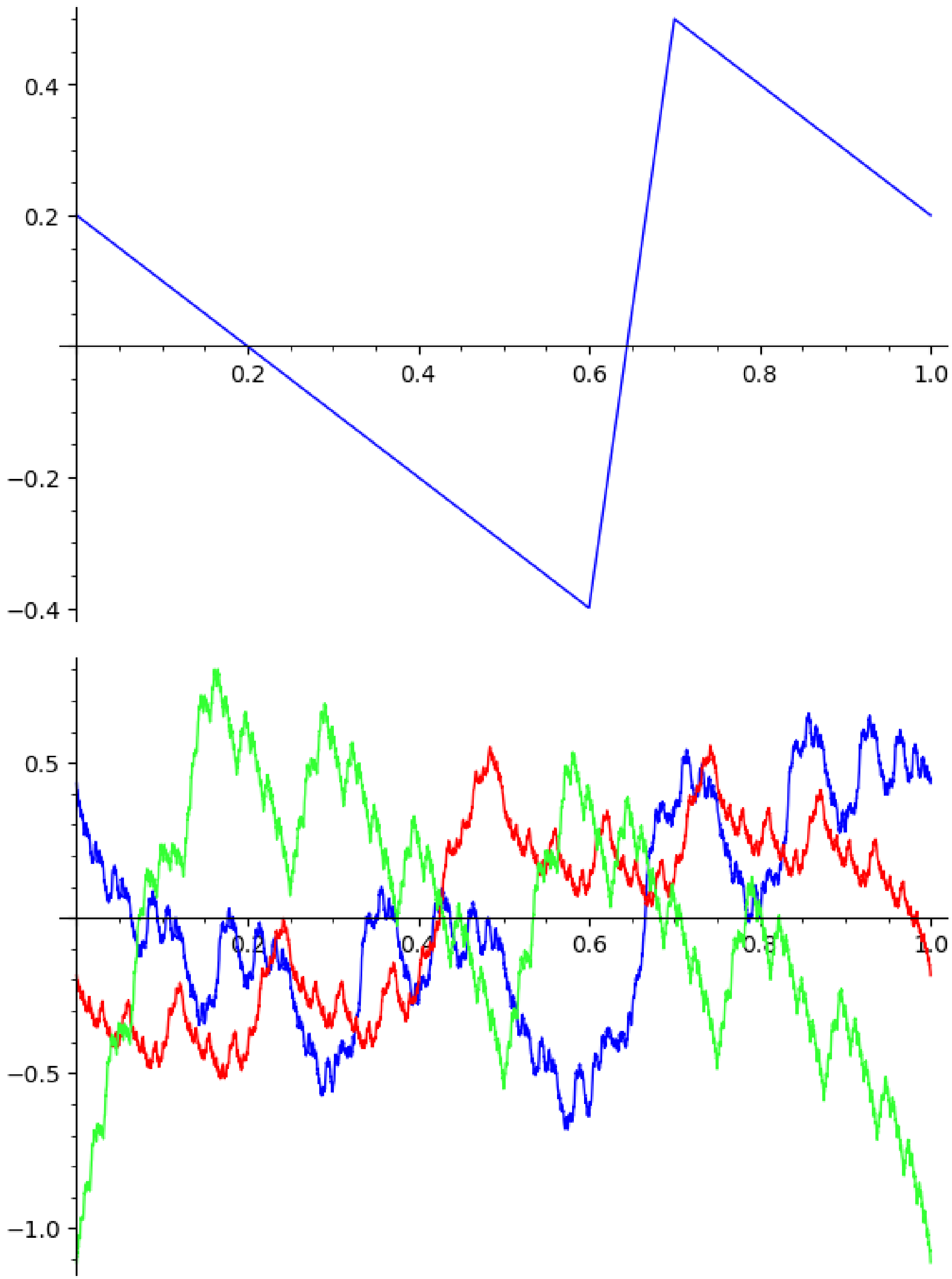
The function g_i, where we have chosen $r_i = \frac{7}{30}$ and $s_i = \frac{9}{30}$ for illustrative purposes.

L.: *Suppose $b \geqslant 2$ is an integer, $0 < \alpha < 1$ and $D > 0$ is given. Then there exist finitely many Lipschitz, and hence α-Hölder functions $g_{d_0}, ..., g_{d-1}$ such that for all $x, y \in \mathbb{S}^1$ with $|x-y| \geqslant D$ there exists $i \in \{d_0, ..., d-1\}$ such that $|W^{\alpha,b}_{g_i}(x) - W^{\alpha,b}_{g_i}(y)| \geqslant \frac{1-b^{-\alpha}}{2}$.*

it remains to show that the Weierstrass embedding $\Phi^{\alpha,b}_{\mathcal{G}}\colon \mathbb{S}^1 \to \mathbb{R}^d$,

$\Phi^{\alpha,b}_{\mathcal{G}}(x) = (W^{\alpha,b}_{g_0}(x), W^{\alpha,b}_{g_1}(x), \ldots, W^{\alpha,b}_{g_{d-1}}(x))$ is α-bi-Hölder...

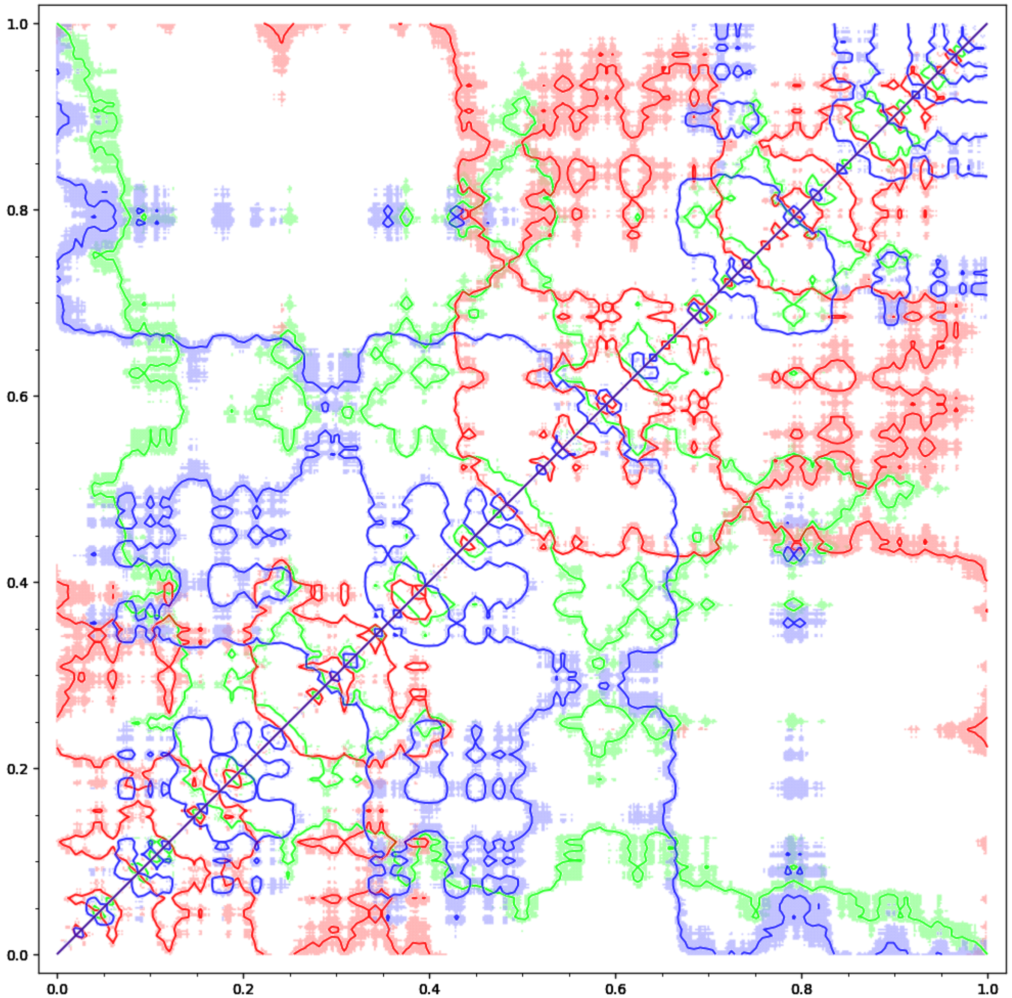
It suffices to verify that for each $x < y$ with $y - x < b^{-\ell_0-1} = D$ in the Lemma there is i such that ★ $|W^{\alpha,b}_{g_i}(y) - W^{\alpha,b}_{g_i}(x)| \geqslant c|x-y|^\alpha$ where the constant $c > 0$ depends only on b and α.



We selected the function g depicted in the upper part of the figure and chose $b = 2$ and $\alpha = 0.7$ to define $W_g^{\alpha,b}$, shown in blue in the lower part of the figure.

We generated three translated copies of g, denoted by g_i for $i \in \{1, 2, 3\}$, with translations by 0 (blue function), 0.214 (red function), and 0.534 (green function).

The latter two values were chosen randomly to avoid unintended symmetry or correlation in the resulting images.



The zero loci of $W^{\alpha,b}_{g_i}(y) - W^{\alpha,b}_{g_i}(x)$ for $i \in \{1,2,3\}$ are plotted in blue, red, and green, respectively.

Regions violating

$$\bigstar \quad |W^{\alpha,b}_{g_i}(y) - W^{\alpha,b}_{g_i}(x)| \geqslant c|x-y|^{\alpha}.$$

To satisfy $\bigstar$, there must exist a constant c such that no point off the diagonal is simultaneously shaded by all three colors.

In the figure, we used $c = 0.2$.

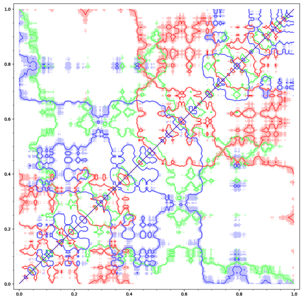
As $c \to 0$, the lightly shaded regions contract toward the zero loci, which can become highly complex near the diagonal.

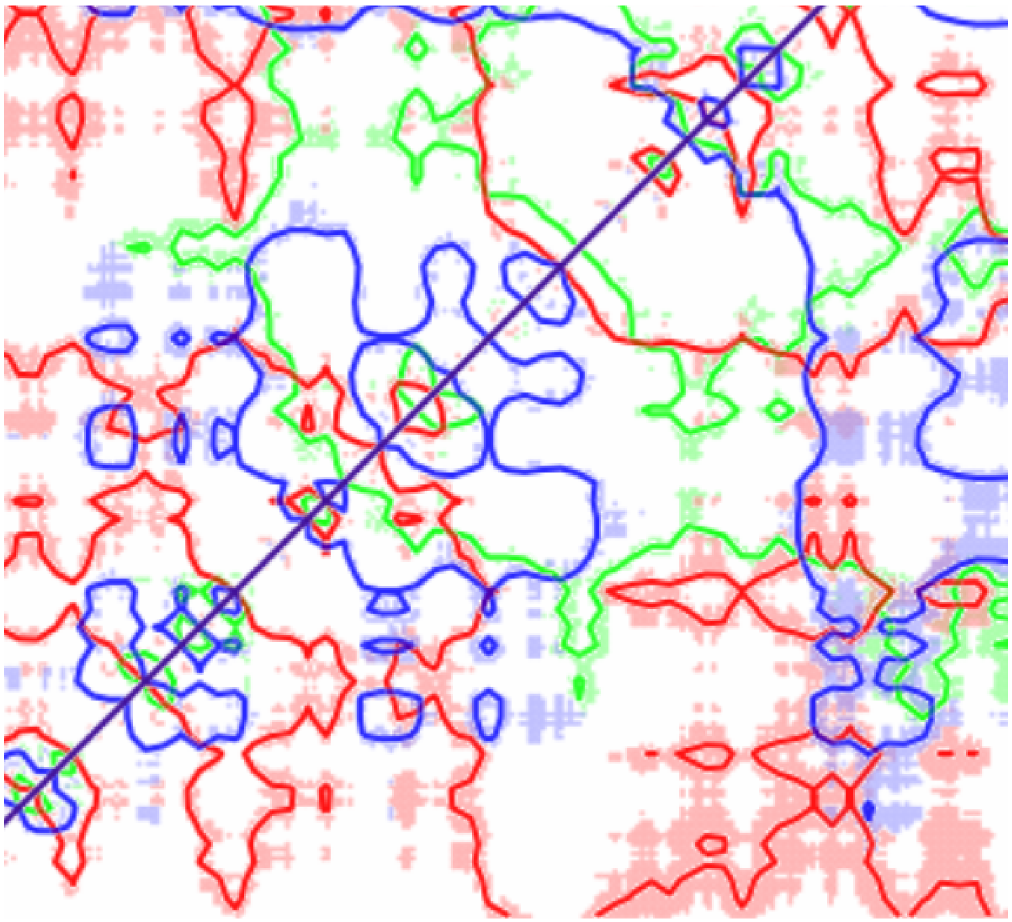
If $W^{\alpha,b}_{g_i}(y) - W^{\alpha,b}_{g_i}(x) = 0$ at a pixel corresponding to (x,y) off the diagonal for some i, then $\bigstar$ cannot be satisfied for that i with any c. This represents a significant obstacle to achieving the Weierstrass embedding with these three functions.

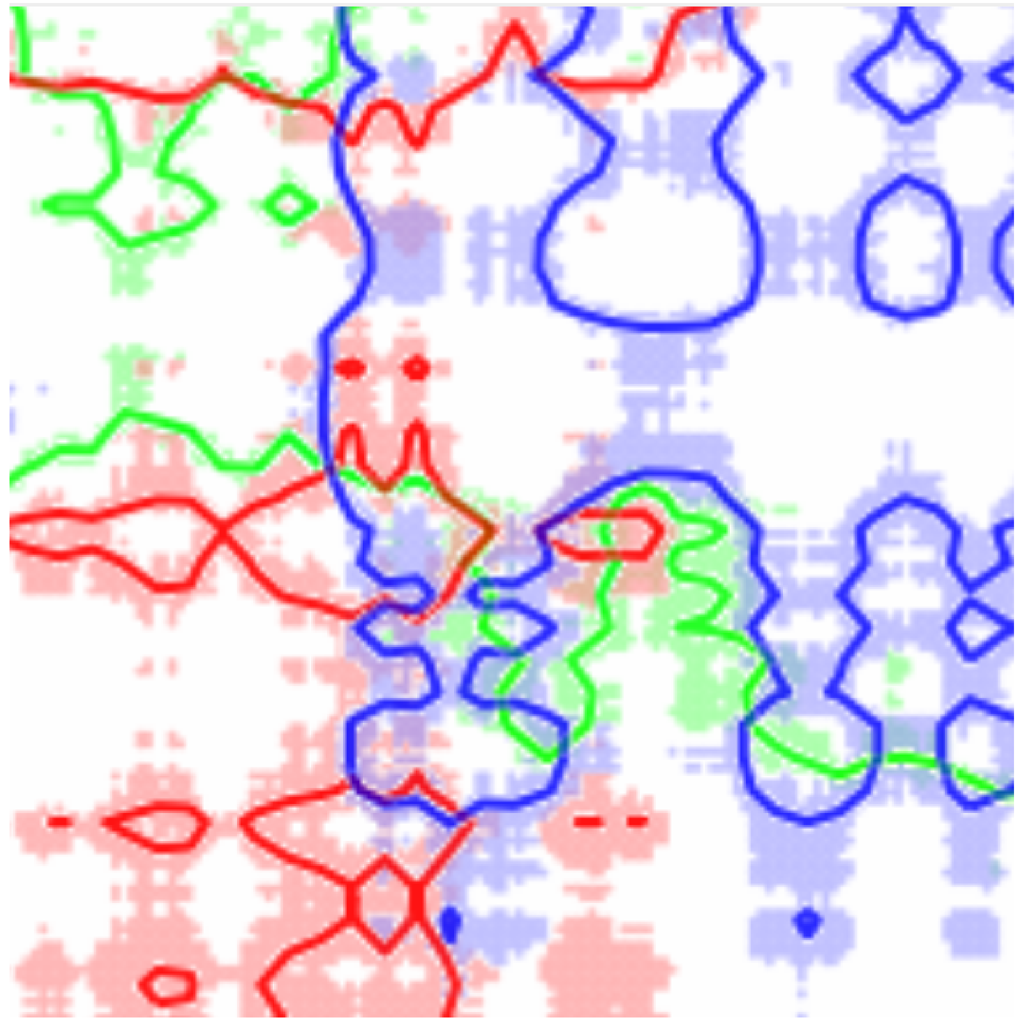
To prove $\bigstar$ the idea is to find a dominant summand in

$$\sum_{k=0}^{\infty} b^{-\alpha k} g_i(b^k y) - b^{-\alpha k} g_i(b^k x) = W^{\alpha,b}_{g_i}(y) - W^{\alpha,b}_{g_i}(x)$$

and then to show that the other summands do not contribute too much.

Some  details:





Fourier analysis and occupation measures

Let $b \geqslant 2$ be an integer and $0 < \alpha < 1$.
Let $\Phi = \Phi_{\mathcal{G}}^{\alpha,b}$ be the α-bi-Hölder Weierstrass embedding associated with $\mathcal{G} = \{g_0, \ldots, g_{d-1}\}$, as given earlier.
For any $W \in \mathcal{W}^{\alpha,b}$ and $\mathbf{t} \in \mathbb{R}^d$, the function $W_{\mathbf{t}} \colon \mathbb{S}^1 \to \mathbb{R}$, defined by $W_{\mathbf{t}}(x) = W(x) + \langle \mathbf{t}, \Phi(x) \rangle$, belongs to $\mathcal{W}^{\alpha,b}$,
where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$.
Put $\mu_{\mathbf{t}} = (\mathrm{Id}, W_{\mathbf{t}})_* \mathcal{L}^1$ $\mu_{\mathbf{t}}$ is the lift of the Lebesgue measure from the unit interval onto the graph of $W_{\mathbf{t}}$.
$\mathrm{proj}_2 \colon \mathbb{R}^2 \to \mathbb{R}$, $\mathrm{proj}_2(x, y) = y$, is the projection onto the y-axis, the *occupation measure* $\lambda_{\mathbf{t}}$ associated with $W_{\mathbf{t}}$ is $\lambda_{\mathbf{t}} = (\mathrm{proj}_2)_* \mu_{\mathbf{t}}$.
That is, the occupation measure is the pushforward of the Lebesgue measure, $\lambda_{\mathbf{t}} = (W_{\mathbf{t}})_* \mathcal{L}^1$.

The Fourier transform of the occupation measure $\lambda_{\mathbf{t}}$ is defined as
$\widehat{\lambda}_{\mathbf{t}}(\xi) = \int_{\mathbb{R}} e^{i\xi x} \, \mathrm{d}\lambda_{\mathbf{t}}(x)$ for all $\xi \in \mathbb{R}$.
The Fourier transform of an integrable function $\psi \colon \mathbb{R}^d \to \mathbb{R}$ is given by
$\widehat{\psi}(\zeta) = \int_{\mathbb{R}^d} \psi(\mathbf{t}) e^{i\langle \mathbf{t}, \zeta \rangle} \, \mathrm{d}\mathbf{t}$ for all $\zeta \in \mathbb{R}^d$.

Let $C_0^\infty(\mathbb{R}^d)$ denote the space of smooth functions with compact support on $\mathbb{R}^d$.

By the Riemann-Lebesgue lemma and integration by parts, for any $\psi \in C_0^\infty(\mathbb{R}^d)$ and $s > 0$, there exists a constant $C = C(\psi, s) > 0$ such that $|\widehat{\psi}(\zeta)| \leqslant C(1 + \|\zeta\|)^{-s}$ for all $\zeta \in \mathbb{R}^d$.

T.: *Let $b \geqslant 2$ be an integer, $0 < \alpha < 1$, and $W \in \mathcal{W}^{\alpha,b}$. Then for $\mathcal{L}^d$-almost every $\mathbf{t} \in \mathbb{R}^d$ the occupation measure $\lambda_{\mathbf{t}}$ associated with $W_{\mathbf{t}}$ is absolutely continuous with density in $L^2(\mathbb{R})$.*

Proof.: Since $0 < \alpha < 1$, choose $s > 1$ such that $\alpha s < 1$. Let $\varrho > 0$ and let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a smooth function satisfying $0 \leqslant \psi \leqslant 1$ and $\psi(\mathbf{t}) = 1$ for all $\mathbf{t} \in B(\mathbf{0}, \varrho)$.

By Fubini's theorem and the estimate coming from the Riemann–Lebesgue lemma, there exists a constant $C = C(\psi, s) > 0$ such that

$$
\begin{aligned}
\left| \int_{B(\mathbf{0},\varrho)} \int |\widehat{\lambda}_{\mathbf{t}}(\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}\mathbf{t} \right| &\leqslant \left| \iiint \int_0^1 \int_0^1 \psi(\mathbf{t}) e^{i\xi(W(x)-W(y)+\langle \mathbf{t}, \Phi(x)-\Phi(y)\rangle)} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \,\mathrm{d}\mathbf{t} \right| \\
&= \left| \int \int_0^1 \int_0^1 e^{i\xi(W(x)-W(y))} \left(\int \psi(\mathbf{t}) e^{i\langle \mathbf{t}, \xi(\Phi(x)-\Phi(y))\rangle} \,\mathrm{d}\mathbf{t} \right) \mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \right| \\
&= \left| \int \int_0^1 \int_0^1 e^{i\xi(W(x)-W(y))} \widehat{\psi}(\xi(\Phi(x)-\Phi(y))) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \right| \\
&\leqslant \int \int_0^1 \int_0^1 |\widehat{\psi}(\xi(\Phi(x)-\Phi(y)))| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \\
&\leqslant \int \int_0^1 \int_0^1 \frac{C}{(1+|\xi|\|\Phi(x)-\Phi(y)\|)^s} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi.
\end{aligned}
$$

Assuming that the constant $c_1 \leqslant 1$ in the α-bi-Hölder condition, and using that Φ is α-bi-Hölder with $|x-y| \leqslant 1$, we obtain

$$\begin{aligned}
&\left| \int_{B(\mathbf{0},\varrho)} \int |\widehat{\lambda}_{\mathbf{t}}(\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}\mathbf{t} \right| \\
&\qquad \leqslant \int \int_0^1 \int_0^1 \frac{C}{(1+|\xi| \|\Phi(x)-\Phi(y)\|)^s} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \\
&\qquad \leqslant \int \int_0^1 \int_0^1 \frac{C}{(1+c_1|\xi| |x-y|^\alpha)^s} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \\
&\qquad \leqslant \int \int_0^1 \int_0^1 \frac{C}{\left(\frac{1}{|x-y|^\alpha} + c_1|\xi|\right)^s |x-y|^{\alpha s}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\xi \\
&\qquad \leqslant \int \frac{C}{c_1^s(1+|\xi|)^s} \,\mathrm{d}\xi \int_0^1 \int_0^1 \frac{1}{|x-y|^{\alpha s}} \,\mathrm{d}x \,\mathrm{d}y < \infty
\end{aligned}$$

since $\alpha s < 1$. Thus, for $\mathcal{L}^d$-almost every $\mathbf{t}$, the Fourier transform $\widehat{\lambda}_{\mathbf{t}}$ is in $L^2(\mathbb{R})$.

By well-known theorems, this implies that the Radon-Nikodym derivative $\mathrm{d}\lambda_{\mathbf{t}}/\mathrm{d}y$ belongs to $L^2(\mathbb{R})$.

Since the derivative is compactly supported, the Cauchy–Schwarz inequality ensures that $\mathrm{d}\lambda_{\mathbf{t}}/\mathrm{d}y \in L^1(\mathbb{R})$, confirming that $\lambda_{\mathbf{t}}$ is absolutely continuous. $\square$

Recall:

T.: *Let $b \geqslant 2$ be an integer, $0 < \alpha < 1$, and $W \in \mathcal{W}^{\alpha,b}$. Then for $\mathcal{L}^d$-almost every $\mathbf{t} \in \mathbb{R}^d$ the occupation measure $\lambda_{\mathbf{t}}$ associated with $W_{\mathbf{t}}$ is absolutely continuous with density in $L^2(\mathbb{R})$.*

It follows directly that α-Weierstrass functions satisfying the local time (LT) condition are d-prevalent in $\mathcal{W}^{\alpha,b}$.

The proof of this theorem builds on [Proposition 3.2] from a paper of Anttila, Bárány and Käenmäki (2025), which applies only for $0 < \alpha < \frac{1}{2}$ and yields a stronger result not explicitly stated but derived within its proof: namely, the occupation measure is not only absolutely continuous but also has a bounded and continuous density.

By combining these techniques with our methods in the present work, we establish the following theorem:

T.: *Let $b \geqslant 2$ be an integer, $0 < \alpha < \frac{1}{2}$, and $W \in \mathcal{W}^{\alpha,b}$. Then for $\mathcal{L}^d$-almost every $\mathbf{t} \in \mathbb{R}^d$ the occupation measure $\lambda_{\mathbf{t}}$ associated with $W_{\mathbf{t}}$ is absolutely continuous with bounded and continuous density.*

$\Rightarrow$ $\mathcal{L}^d$-almost every $\mathrm{t} \in \mathbb{R}^d$, there exists a function $h_\mathrm{t} \in L^2(\mathbb{R})$ such that $\lambda_\mathrm{t}(A) = \int_A h_\mathrm{t}(y)\,\mathrm{d}y$ for all Borel sets $A \subset \mathbb{R}$.

Applying Rohlin's disintegration theorem to the projection $\mathrm{proj}_2 \colon \mathbb{R}^2 \to \mathbb{R}$, defined by $\mathrm{proj}_2(x, y) = y$, and the measure μ_t, we obtain, for λ_t-almost every y, a Borel measure $(\mu_\mathbf{t})_y$ supported on $\mathrm{graph}(W_\mathrm{t}) \cap \mathrm{proj}_2^{-1}(\{y\})$.

We have $\mu_\mathrm{t}(A) = \iint \mathbb{1}_A(a)\,\mathrm{d}(\mu_\mathrm{t})_y(a)\,\mathrm{d}\lambda_\mathrm{t}(y)$ for all Borel sets $A \subset \mathbb{R}^2$, where the parameter a is in $\mathbb{R}^2$.

For each y, define a Borel measure by $(\nu_\mathrm{t})_y = h_\mathrm{t}(y)(\mu_\mathrm{t})_y$, and note that $\mu_\mathrm{t}(A) = \iint \mathbb{1}_A(a)\,\mathrm{d}(\nu_\mathrm{t})_y(a)\,\mathrm{d}y$ for all Borel sets $A \subset \mathbb{R}^2$.

Hence for μ_t-integrable functions f, we have

$$(\bigstar\bigstar) \qquad \int f(a)\,\mathrm{d}\mu_\mathrm{t}(a) = \iint f(a)\,\mathrm{d}(\nu_\mathrm{t})_y(a)\,\mathrm{d}y.$$

Recall that for $\mu_{\mathbf{t}}$-integrable functions f, we have

$$(\bigstar\bigstar) \quad \int f(a)\,\mathrm{d}\mu_{\mathbf{t}}(a) = \iint f(a)\,\mathrm{d}(\nu_{\mathbf{t}})_y(a)\,\mathrm{d}y.$$

Furthermore, for all $g \in L^1(\mathbb{R})$, we have for $\lambda_{\mathbf{t}}$ almost every y

$$\int g(a)\,\mathrm{d}(\mu_{\mathbf{t}})_y(a) = \lim_{r\downarrow 0} \frac{\int_{B^\top(y,r)} g(a)\,\mathrm{d}\mu_{\mathbf{t}}(a)}{\lambda_{\mathbf{t}}(B(y,r))} = \lim_{r\downarrow 0} \frac{1}{2r} \frac{\int_{B^\top(y,r)} g(a)\,\mathrm{d}\mu_{\mathbf{t}}(a)}{h_{\mathbf{t}}(y)},$$

where $B^\top(y,r) = \mathrm{proj}_2^{-1}(B(y,r))$.

Since the measures $(\mu_{\mathbf{t}})_y$ are supported on $\mathrm{graph}(W_{\mathbf{t}}) \cap \mathrm{proj}_2^{-1}(\{y\})$ for given $y_0 \in \mathbb{R}$ and $r > 0$ we can deduce from $(\bigstar\bigstar)$ that

$$\int_{B^\top(y_0,r)} f(a)\,\mathrm{d}\mu_{\mathbf{t}}(a) = \int \mathbb{1}_{B(y_0,r)}(\mathrm{proj}_2(a)) f(a)\,\mathrm{d}\mu_{\mathbf{t}}(a)$$

$$= \int_{y_0-r}^{y_0+r} \int f(a)\,\mathrm{d}(\nu_{\mathbf{t}})_y(a)\,\mathrm{d}y.$$

Prop.: *Let $b \geqslant 2$ be an integer, $0 < \alpha < 1$, and $W \in \mathcal{W}^{\alpha,b}$. Then for $\mathcal{L}^d$-almost every $\mathbf{t} \in \mathbb{R}^d$,*

$$\dim_{\mathrm{H}}(W_{\mathbf{t}}^{-1}(\{y\})) \geqslant 1 - \alpha$$

for $\lambda_{\mathbf{t}}$-almost all $y \in \mathbb{R}$.

Open questions

- The primary open question is to strengthen

T.: *For any integer* $b \geqslant 2$*, a prevalent function* $g \in \mathrm{Lip}(\mathbb{S}^1)$ *satisfies*
$\mathcal{L}^1(\{y \in W_g^{\alpha,b}(\mathbb{S}^1) : \dim_{\mathrm{H}}((W_g^{\alpha,b})^{-1}(\{y\})) = 1-\alpha\}) > 0$
provided that $0 < \alpha < 1$*.*

by proving that for prevalent functions in $\mathcal{W}^{\alpha,b}$, with $b \geqslant 2$ and $0 < \alpha < 1$, the Hausdorff dimension of the level set $(W_g^{\alpha,b})^{-1}(\{y\})$ equals $1-\alpha$ for Lebesgue almost all $y \in W_g^{\alpha,b}(\mathbb{S}^1)$.

Additionally, could the Hausdorff dimension $\dim_{\mathrm{H}}((W_g^{\alpha,b})^{-1}(\{y\}))$ equal $1-\alpha$ for all $y \in \mathrm{int}(W_g^{\alpha,b}(\mathbb{S}^1))$?

To establish the Lebesgue almost all claim, it suffices to show that for prevalent functions $W_g^{\alpha,b} \in \mathcal{W}^{\alpha,b}$, the occupation measure has a density function which is positive Lebesgue almost everywhere on $W_g^{\alpha,b}(\mathbb{S}^1)$.

• Further investigation into the properties of the density function of the occupation measure would be valuable. For instance, extending

T.: *Let $b \geqslant 2$ be an integer, $0 < \alpha < \frac{1}{2}$, and $W \in \mathcal{W}^{\alpha,b}$. Then for $\mathcal{L}^d$-almost every $\mathbf{t} \in \mathbb{R}^d$ the occupation measure $\lambda_{\mathbf{t}}$ associated with $W_{\mathbf{t}}$ is absolutely continuous with bounded and continuous density.*

to $\frac{1}{2} \leqslant \alpha < 1$ would be of interest.

Additionally, verifying for the density function stronger properties than continuity across certain ranges of α is a compelling challenge.

Heuristically, there appears to be a dichotomy: when α is small, the prevalent Weierstrass function exhibits "wilder" behavior with a smaller Hölder exponent, yet its occupation measure's density function may possess more favorable properties.

• Instead of focusing solely on the occupation measure, one can consider slices and projections in various directions.
For $\theta \in [0, 2\pi)$, let $\mathrm{pr}_\theta \colon \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection defined by $\mathrm{pr}_\theta(x, y) = x\cos(\theta) + y\sin(\theta)$ for all $(x, y) \in \mathbb{R}^2$. Note that $\mathrm{proj}_2 = \mathrm{pr}_{\frac{\pi}{2}}$.
Given the measure μ_t defined earlier, we consider the projections of μ_t for $\theta \in [0, 2\pi)$, defined as
$\lambda_t^\theta = (\mathrm{pr}_\theta)_* \mu_t$. These measures λ_t^θ are referred to as *θ-oblique occupation measures*. Note that the occupation measure is $\lambda_t = \lambda_t^{\frac{\pi}{2}}$.

By combining techniques from this paper and Anttila, Bárány and Käenmäki (2025), one can show that for prevalent functions in $\mathcal{W}^{\alpha,b}$, the θ-oblique occupation measure is absolutely continuous with respect to the Lebesgue measure for Lebesgue almost every θ.

A compelling open problem is to determine whether this property holds for all θ, i.e., whether the θ-oblique occupation measure is absolutely continuous for every θ.

Furthermore, one could explore whether questions analogous to the previous two questions admit positive answers for all θ in this context.
In particular, can it be shown that for prevalent functions in $\mathcal{W}^{\alpha,b}$, the non-empty, or non-extremal slices in all non-vertical directions simultaneously have Hausdorff dimension $1 - \alpha$?

• **T.**: *For every integer $b \geqslant 2$ and $0 < \alpha < 1$ there exist $d \in \mathbb{N}$ and a finite collection $\mathcal{G} = \{g_0, \ldots, g_{d-1}\}$ of Lipschitz functions on $\mathbb{S}^1$ such that the Weierstrass embedding $\Phi_{\mathcal{G}}^{\alpha,b}\colon \mathbb{S}^1 \to \mathbb{R}^d$ is α-bi-Hölder.*

This ensures the existence of α-bi-Hölder Weierstrass embeddings only when the dimension d is sufficiently large.

In

Prop.: *Let $b \geqslant 2$ and $d \geqslant 1$ be integers, and let $0 < \alpha < 1$. If $d < \frac{1}{\alpha}$ and $\mathcal{G} = \{g_0, \ldots, g_{d-1}\}$ is a finite collection of Lipschitz functions on $\mathbb{S}^1$, then the Weierstrass embedding $\Phi_{\mathcal{G}}^{\alpha,b}\colon \mathbb{S}^1 \to \mathbb{R}^d$ is not α-bi-Hölder.*

a lower bound on d is established as a function of α. Is this bound sharp? What is the minimal dimension d required for a given α?