A characterization of p-nuclear operators on spaces of continuous functions

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Table of contents

Introduction

Let $(X, \|\cdot\|_X)$ be a Banach space and let X^* denote the Banach dual of X. By $\mathcal{B}o$ we denote the σ-algebra of Borel sets in a compact Hausdorff space K.

The **variation** $|\mu|$ of the measure $\mu : \mathcal{B}o \to X$ on $A \in \mathcal{B}o$ is defined by:

$$|\mu|(A) := \sup \sum \|\mu(A_i)\|_X,$$

where the supremum is taken over all finite $\mathcal{B}o$-partitions (A_i) of A (see [DU, Definition 4, p. 2]).

[DU] Diestel, J. and J.J. Uhl, Vector Measures, Amer. Math. Soc., Math. Surveys 15, Providence, RI, 1977.

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By $M(K, X^*)$ we denote the Banach space of all countably additive regular measures $\mu : \mathcal{B}o \to X^*$ of bounded variation ($|\mu|(K) < \infty$), equipped with the variation norm $\|\mu\| := |\mu|(K)$.

Theorem

Every continuous linear functional on $C(K, X)$ can be identified with $\mu \in M(K, X^*)$ through the isomorphism

$$\Phi : M(K, X^*) \ni \mu \mapsto \Phi_\mu \in C(K, X)^*,$$

where $\Phi_\mu(f) = \int_K f(t)\, d\mu$ for $f \in C(K, X)$ and $\|\Phi_\mu\| = |\mu|(K)$.

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Let $(Y, \|\cdot\|_Y)$ be a Banach space. By $\mathcal{L}(X, Y)$ we denote the space of all bounded linear operators from X to Y.

Assume that $T : C(K, X) \to Y$ is a continuous linear operator. Then the biconjugate operator $T^{**} : C(K, X)^{**} \to Y^{**}$ of T is defined by

$$T^{**}(\Psi)(y^*) := \Psi(y^* \circ T) \text{ for } \Psi \in C(K, X)^{**}, y^* \in Y^*.$$

For $A \in \mathcal{B}o$ and $x \in X$, let

$$\Psi_{A,x}(\Phi_\mu) := \mu(A)(x) \text{ for } \mu \in M(K, X^*).$$

Then $\Psi_{A,x} \in C(K, X)^{**}$ and one can define a measure $\hat{m} : \mathcal{B}o \to \mathcal{L}(X, Y^{**})$ (called the **representing measure** of T) by

$$\hat{m}(A)(x) := T^{**}(\Psi_{A,x}) \text{ for } A \in \mathcal{B}o, x \in X.$$

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The **semivariation** $\|\hat{m}\|$ of $\hat{m} : \mathcal{B}o \to \mathcal{L}(X, Y^{**})$ on $A \in \mathcal{B}o$ is defined by

$$\|\hat{m}\|(A) := \sup \left\| \sum \hat{m}(A_i)(x_i) \right\|_{Y^{**}},$$

where the supremum is taken over all finite $\mathcal{B}o$-partitions (A_i) of A and x_i is from the unit ball B_X in X for each i (see [DU, Definition 4, p. 2]).

Riesz representation theorem for operators

Let $i_Y : Y \to Y^{**}$ denote the canonical embedding and j_Y be the left inverse of i_Y.

Dinculeanu–Singer Theorem [DU, p.182]

Assume that $T : C(K, X) \to Y$ is a continuous linear operator. Then there exists a unique representing measure $\hat{m} : \mathcal{B}o \to \mathcal{L}(X, Y^{**})$ such that:

(i) $\|\hat{m}\|(K) < \infty$;

(ii) for every $y^* \in Y^*$, the measure $\hat{m}_{y^*} : \mathcal{B}o \to X^*$ is regular, where $\hat{m}_{y^*}(A) := (\hat{m}(A)(x))(y^*)$ for $A \in \mathcal{B}o$, $x \in X$;

(iii) the mapping $Y^* \ni y^* \mapsto \hat{m}_{y^*} \in M(K, X^*)$ is $(\sigma(Y^*, Y), \sigma(M(K, X^*), C(K, X)))$-continuous;

(iv) for $f \in C(K, X)$, $\int_K f(t)\, d\hat{m} \in i_Y(Y)$ and $T(f) = j_Y\left(\int_K f(t)\, d\hat{m}\right)$;

(v) for each $y^* \in Y^*$, $y^*(T(f)) = \int_K f(t)\, d\hat{m}_{y^*}$ for $f \in C(K, X)$.

Conversely, assume that a measure $\hat{m} : \mathcal{B}o \to \mathcal{L}(X, Y^{**})$ satisfies the conditions (i), (ii) and (iii). Then there exists a unique continuous linear operator $T : C(K, X) \to Y$ such that (iv) and (v) hold.

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Riesz representation theorem for operators

Corollary [BrL, Corollary 4.4.1]

If a continuous linear operator $T : C(K, X) \to Y$ is weakly compact, then $\hat{m}(A)(x) \in i_Y(Y)$ for $A \in \mathcal{B}o$, $x \in X$.

Then one can define a measure $m : \mathcal{B}o \to \mathcal{L}(X, Y)$ by

$$m(A)(x) := j_Y(\hat{m}(A)(x)) \ \text{ for } \ A \in \mathcal{B}o, x \in X.$$

[BrL] Brooks, J. K. and P. Lewis, *Linear Operators and Vector Measures*, Trans. Amer. Math. Soc., 192 (1974), 139–162.

Different classes of operators

A linear operator $U : X \to Y$ is **nuclear** if there exist sequences (x_n^*) in X^* and (y_n) in Y such that

$$U(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \quad \text{for} \quad x \in X,$$

where $\sum_{n=1}^{\infty} \|x_n^*\|_{X^*} \|y_n\|_Y < \infty$.

The **nuclear norm** of a nuclear operator $U : X \to Y$ is defined by

$$\|U\|_{nuc} := \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\|_{X^*} \|y_n\|_Y \right\},$$

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Theorem [Po$_1$, Theorem 1], [SS, Proposition 1]

Suppose X^* has the Radon–Nikodym Property. Then $T : C(K, X) \to Y$ is nuclear if and only if

(i) m takes value in $\mathcal{N}(X, Y)$;

(ii) $m : \mathcal{B}o \to \mathcal{N}(X, Y)$ is $\|\cdot\|_{nuc}$-countably additive;

(iii) $|m|_{nuc}(K) < \infty$;

(iv) m has Bochner derivative with respect to $|m|_{nuc}$, i.e., there exists $H \in L^1(|m|_{nuc}, \mathcal{N}(X, Y))$ such that

$$m(A) = \int_A H(t)\, d|m|_{nuc} \quad \text{for} \quad A \in \mathcal{B}o.$$

[Po$_1$] Popa, D., *Nuclear operators on $C(T, X)$*, Stud. Cerc. Mat., 42 no. 1 (1990), 47–50.

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Let $1 < p < \infty$. A continuous linear operator $U : X \to Y$ is ***p*-absolutely summing** if $\sum_{n=1}^{\infty} \|U(x_n)\|_Y^p < \infty$ whenever (x_n) in X such that $\sum_{n=1}^{\infty} |x^*(x_n)|^p < \infty$ for $x^* \in X^*$.

Theorem [BiL, Theorem 4.1 and Theorem 2.5], [Po$_2$, Theorem 2]

If $T : C(K, X) \to Y$ is p-absolutely summing, then

(i) m takes value in $\mathcal{AS}_p(X, Y)$;

(ii) $|m|^p_{p\text{-}abs}(K) < \infty$.

Conversely, if $|m|_{p\text{-}abs}(K) < \infty$, then T is p-absolutely summing.

[BiL] Bilyeu, R. and P. Lewis, *Some mapping properties of representing measures*, Ann. Mat. Pur. Appl., 109 (1976), 273–287.

[Po$_2$] Popa, D., *(r, p)-absolutely summing operators on the space $C(T, X)$ and applications*, Abstr. Appl. Anal., 6(5) (2001), 309–315.

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Different classes of operators

Let $1 < p < \infty$ and $1/p + 1/p^* = 1$.

Definition [DJT, p. 112]

A linear operator $U : X \to Y$ is said to be p-**nuclear** if there exist sequences (x_n^*) in X^* and (y_n) in Y such that

$$(1) \qquad U(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \quad \text{for} \quad x \in X$$

with

$$\left(\sum_{n=1}^{\infty} \|x_n^*\|_{X^*}^p \right)^{\frac{1}{p}} < \infty \quad \text{and} \quad \sup_{\|y^*\|_{Y^*} \le 1} \left(\sum_{n=1}^{\infty} |y^* y_n|^{p^*} \right)^{\frac{1}{p^*}} < \infty.$$

[DJT] Diestel J., H. Jarchow and A. Tonge, Absolutely Summing Operators, Cambridge Stud. Adv. Math., vol. 43, Cambridge Univ. Press, 1995.

The p-**nuclear norm** of a p-nuclear operator $U : X \to Y$ is defined by

$$\|U\|_{p\text{-}nuc} := \inf\left\{\left(\sum_{n=1}^{\infty}\|x_n^*\|_{X^*}^p\right)^{\frac{1}{p}} \sup_{\|y^*\|_{Y^*}\leq 1}\left(\sum_{n=1}^{\infty}|y^* y_n|^{p^*}\right)^{\frac{1}{p^*}}\right\},$$

where the infimum is taken over all representations (1) of U.

The linear space $\mathcal{N}_p(X, Y)$ of all p-nuclear operators $U : X \to Y$, equipped with the p-nuclear norm $\|\cdot\|_{p\text{-}nuc}$ is a Banach space.

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Following [Po$_2$, Theorem 2]:

Definition

The p-**variation** $|m|^p_{p\text{-}nuc}$ of a measure $m : \mathcal{B}o \to \mathcal{N}_p(X, Y)$ on $A \in \mathcal{B}o$ is defined by

$$|m|^p_{p\text{-}nuc}(A) := \sup \left\{ \left(\sum_{i=1}^{k} \|m(A_i)\|^p_{p\text{-}nuc} \right)^{1/p} \right\},$$

where the supremum is taken over all finite $\mathcal{B}o$-partitions (A_i) of A.

p-nuclear operators and the p-variation

Theorem

Assume that $T : C(K, X) \to Y$ is a p-nuclear operator and $\hat{m}$ is its representing measure. Then the following statements hold:

(i) For each $A \in \mathcal{B}o$, $m(A) : X \to Y$ is p-nuclear operator.

(ii) $m : \mathcal{B}o \to \mathcal{N}_p(X, Y)$ is $\|\cdot\|_{p\text{-}nuc}$-countably additive.

(iii) $|m|^p_{p\text{-}nuc}(K) \leq \|T\|_{p\text{-}nuc}$.

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(i) For each $A \in \mathcal{B}o$, $m(A) : X \to Y$ is p-nuclear operator.

(ii) $m : \mathcal{B}o \to \mathcal{N}_p(X, Y)$ is $\|\cdot\|_{p\text{-}nuc}$-countably additive.

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p-nuclear operators and the p-variation

Following [Po$_2$, Lemma 3]:

Example

Let $x^* \in X^*$ and $y \in Y$ be such that $\|x^*\|_{X^*} = 1 = \|y\|_Y$. For fixed $t \in [0,1]$, denote $\nu := 2\delta_t - \mu$, where δ_t is the Dirac measure and μ is the Lebesgue measure. We define the one rank operator $T : C([0,1], X) \to Y$ by

$$T(f) := \left(\int_0^1 x^*(f(t))\, d\nu \right) y \quad \text{for} \quad f \in C([0,1], X).$$

It follows that T is p-nuclear for $1 < p < \infty$ and $\|T\|_{p\text{-}nuc} = 3$.
On the other hand, the representing measure m is given by the formula

$$m(A) = (x^* \otimes y)\nu(A) \quad \text{for} \quad A \in \mathcal{B}o,$$

where $(x^* \otimes y)(x) := x^*(x)y$ for $x \in X$.
Thus $|m|^p_{p\text{-}nuc}([0,1]) = (2^p + 1)^{\frac{1}{p}}$.

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Pettis integrable derivative

Let $(K, \mathcal{B}o, \lambda)$ be a finite measure space.

The weakly measurable function $h : K \to Y$ is λ-**Pettis integrable** if for each $A \in \mathcal{B}o$, there exists $y_A \in Y$ such that

$$y^*(y_A) = \int_A y^*(h(t))\, d\lambda \quad \text{for every} \quad y^* \in Y^*.$$

Then $(P) \int_A h(t)\, d\lambda := y_A$ is called **the Pettis integral** of h over $A \in \mathcal{B}o$ with respect to λ.

Pettis integrable derivative

Let $X \otimes Y$ be the tensor product of Banach spaces X and Y.

The **left Chevet–Saphar p-norm** for $u \in X \otimes Y$ is defined by

$$g_p(u) := \inf \left\{ \left(\sum_{i=1}^{n} \|x_i\|_X^p \right)^{\frac{1}{p}} \sup_{\|y^*\|_{Y^*} \leq 1} \left(\sum_{i=1}^{n} |y^* y_i|^{p^*} \right)^{\frac{1}{p^*}} : u = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all sequences $(x_i)_{i=1}^n$ in X and $(y_i)_{i=1}^n$ in Y such that u admits the above representation (see [R, p. 135]).

By $X \hat{\otimes}_{g_p} Y$ we denote the completion of the normed space $(X \otimes Y, g_p)$.

[R] Ryan, R., Introduction to Tensor Products of Banach Spaces, Springer Monographs in Math., 2002.

Pettis integrable derivative

Proposition [R, Proposition 6.10]

For $u \in X\hat{\otimes}_{g_p} Y$, there exist sequences (x_n) in X and (y_n) in Y with $(\sum_{n=1}^{\infty} \|x_n\|_X^p)^{\frac{1}{p}} < \infty$ and $\sup_{\|y^*\|_{Y^*} \leq 1} (\sum_{n=1}^{\infty} |y^* y_n|^{p^*})^{\frac{1}{p^*}} < \infty$ such that

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

and

$$g_p(u) = \inf \left\{ \left(\sum_{n=1}^{\infty} \|x_n\|_X^p \right)^{\frac{1}{p}} \sup_{\|y^*\|_{Y^*} \leq 1} \left(\sum_{n=1}^{\infty} |y^* y_n|^{p^*} \right)^{\frac{1}{p^*}} \right\},$$

where the infimum is taken over all such representations.

Pettis integrable derivative

Proposition

If a function $h \in L^1(\lambda)\hat{\otimes}_{g_p} Y$ is such that $h(t) \in Y$ for every $t \in K$, then h is λ-Pettis integrable.

Pettis integrable derivative

Theorem [DU, Theorem 4, p. 11, and Chap. VI, Section 2]

If $T : C(K) \to Y$ is weakly compact, then $\hat{m}(A) \in i_Y(Y)$ for every $A \in \mathcal{B}o$, and there is a measure $\lambda \in M^+(K)$, called the **control measure** for $m := j_Y \circ \hat{m} : \mathcal{B}o \to Y$, so that

$$\sup\{|y^* \circ m|(A) : y^* \in B_{Y^*}\} \to 0$$

as $\lambda(A) \to 0$.

Pettis integrable derivative

Theorem

Let $T : C(K) \to Y$ be a p-nuclear operator. Then there exist a measure $\lambda \in M^+(K)$ and a sequence (h_n) in $L^1(\lambda) \otimes Y$, $h_n := \sum_{i=1}^n v_i \otimes y_i$ for $n \in \mathbb{N}$, converging to some $h \in L^1(\lambda)\hat{\otimes}_{g_p} Y$ such that

$$m(A) = \sum_{n=1}^{\infty} \left(\int_A v_n(t)\, d\lambda \right) y_n \quad \text{for} \quad A \in \mathcal{B}o.$$

Moreover, if $h(t) \in Y$ for every $t \in K$, then

$$m(A) = (P) \int_A h(t)\, d\lambda \quad \text{for} \quad A \in \mathcal{B}o.$$

In this case, $g_p(h) \leq \|T\|_{p\text{-}nuc}$.

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In this case, $g_p(h) \leq \| T \|_{p\text{-}nuc}$.

Pettis integrable derivative

Theorem

Let $T : C(K) \to Y$ be a weakly compact operator and $\lambda \in M^+(K)$ be a control measure for m. If there exists $h \in L^1(\lambda)\hat{\otimes}_{g_p} Y$ such that

(i) $h(t) \in Y$ for every $t \in K$,

(ii) $m(A) = (P)\displaystyle\int_A h(t)\, d\lambda$ for $A \in \mathcal{B}o$,

then T is p-nuclear and $\|T\|_{p\text{-}nuc} \le g_p(h)$.

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Thank you for your attention!