Transference and restriction of Fourier multipliers on Orlicz spaces

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XX EARCO
Encuentros de Análisis Real y Complejo
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**Multipliers on** $L^p$

Recall that a bounded measurable function $m : \mathbb{R} \to \mathbb{C}$ is said to be a $p$-multiplier in $\mathcal{M}_p(\mathbb{R})$, if

$$T_m(f)(y) = \int_{\mathbb{R}} m(x) \hat{f}(x) e^{ixy} \, dx$$

defines a bounded operator from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$. 

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Recall that a bounded sequence $(m_n) \subset \mathbb{C}$ (respect. a bounded periodic function $m : \mathbb{T} \rightarrow \mathbb{C}$ ) is said to be a $p$-multiplier in $M_p(\mathbb{Z})$ (respect. $M_p(\mathbb{T})$).
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$$T_m(f)(t) = \sum_{n \in \mathbb{Z}} m_n \hat{f}(n)e^{int}$$

(respect. $\left( T_m((\alpha_n)) \right)_n = \left( \int_0^{2\pi} m(t)(\sum_k \alpha_k e^{ikt})e^{int} \frac{dt}{2\pi} \right)_n$)
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Transference and restriction

Let $m : \mathbb{R} \rightarrow \mathbb{C}$ be continuous and bounded. Assume that $m \in \mathcal{M}_p(\mathbb{R})$. 

K. DeLeeuw, (1965) YES

Tool (Use the Bohr group $\mathbb{D}$, that is $\mathbb{R}$ with the discrete topology. 

$\mathcal{M}_p(\mathbb{R}) = \mathcal{M}_p(\mathbb{D})$

Aim: Similar questions for multipliers between Orlicz spaces

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for $\gamma \in \hat{G}$ whenever $f \in L^1(G)$. 
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Orlicz spaces

Given a Young function \( \Phi : [0, \infty) \rightarrow [0, \infty) \), that is convex, \( \Phi(0) = 0 \) and \( \lim_{x \to \infty} \Phi(x) = \infty \), we write

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Then the Orlicz space $L^\Phi(G)$ consists of the set of all measurable functions $f : G \to \mathbb{C}$ such that $\rho_\Phi(f/\lambda) < \infty$ for some $\lambda > 0$. 

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(Orlicz norm) \( \|f\|_\Phi = \sup\{\int_G |f(x)g(x)| dm_G(x) : \rho_\Psi(g) \leq 1\} \) where \( \Psi \) is the complementary Young function, i.e.

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(Amemiya norm) \( \|\|f\|\|_{\Phi} = \inf_{k > 0} \frac{1}{k}(1 + \rho_{\Phi}(kf)) \).
**Δ₂-condition**

A Young function $\Phi$ is said to satisfy $\Delta_2$-condition (globally) if there exists a constant $K > 0$ such that

$$\Phi(2x) \leq K\Phi(x), \quad x \geq 0.$$  \hfill (2)

A Young function $\Phi$ is said to satisfy $\nabla_2$-condition (globally) if there exists a constant $\ell > 1$ such that

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Let $\Phi_1$ and $\Phi_2$ be Young functions, and let $m$ be a bounded measurable function defined on $G$. The function $m$ is said to be a $(\Phi_1, \Phi_2)$-multiplier on $G$ if there exists $C > 0$ such that

$$N_{\Phi_2}(T_m(f)) \leq CN_{\Phi_1}(f) \quad (5)$$

for all $f \in A(\hat{G})$. 
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We write \( M_{\Phi_1,\Phi_2}(G) \) for the space of \((\Phi_1, \Phi_2)\)-multipliers on \( G \).

Whenever \( A(\hat{G}) \) is dense in \( L^{\Phi_1}(\hat{G}) \) we have that \( T_m \) extends to a bounded operator from \( L^{\Phi_1}(\hat{G}) \) to \( L^{\Phi_2}(\hat{G}) \) for any \((\Phi_1, \Phi_2)\)-multiplier \( m \).

Moreover \( \| T_m \|_{L^{\Phi_1}\to L^{\Phi_2}} = \| m \|_{(\Phi_1,\Phi_2)} \).
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Given a bounded measurable function $m$ defined on $G$ and $f \in A(\hat{G})$ we write

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We write $\mathcal{M}_{\Phi_1, \Phi_2}(G)$ for the space of $(\Phi_1, \Phi_2)$-multipliers on $G$. Whenever $A(\hat{G})$ is dense in $L^{\Phi_1}(\hat{G})$ we have that $T_m$ extends to a bounded operator from $L^{\Phi_1}(\hat{G})$ to $L^{\Phi_2}(\hat{G})$ for any $(\Phi_1, \Phi_2)$-multiplier $m$. Moreover

$$\|T_m\|_{L^{\Phi_1} \rightarrow L^{\Phi_2}} = \|m\|_{(\Phi_1, \Phi_2)}.$$  

If $\Phi$ is a Young function satisfying $\Delta_2$ condition then $A(G)$ is dense in $L^{\Phi}(G)$.
Basic Examples

As usual we denote $\hat{\mu}(x) = \int_{\hat{G}} \gamma^{-1}(x) d\mu(\gamma)$ for the Fourier transform of a regular Borel measure $\mu$ defined in $\hat{G}$.
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Proposition

Let \( \Phi_1, \Phi_2 \) and \( \Phi_3 \) be Young functions.

(i) Assume that there exists \( C > 0 \) such that

\[
\Phi_2(x) \leq C \Phi_1(x), \quad x > 0.
\]  

If \( m(x) = \hat{\mu}(x) \) for some regular Borel measure \( \mu \) defined on \( \hat{G} \) then \( m \in \mathcal{M}_{\Phi_1, \Phi_2}(G) \). Moreover \( \| m \|_{(\Phi_1, \Phi_2)} \leq C \| \mu \|_1 \).

(ii) Assume that

\[
\Phi_1^{-1}(x) \Phi_2^{-1}(x) \leq x \Phi_3^{-1}(x), \quad x \geq 0.
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If \( m(x) = \hat{g}(x) \) for some \( g \in L^1(\hat{G}) \cap L^{\Phi_2}(\hat{G}) \) then \( m \in \mathcal{M}_{\Phi_1, \Phi_3}(G) \) and

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\| m \|_{(\Phi_1, \Phi_3)} \leq 2N_{\Phi_2}(g).
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More Examples

Proposition

Let $\Phi, \Phi_i$ for $i = 1, 2$ be Young functions and $m \in \mathcal{M}_{\Phi_1, \Phi_2}(G)$. 

(i) If $\varphi \in L^1(G)$ then $\varphi \ast m \in \mathcal{M}_{\Phi_1, \Phi_2}(G)$. Moreover

$$\| \varphi \ast m \|_{(\Phi_1, \Phi_2)} \leq \| \varphi \|_1 \| m \|_{(\Phi_1, \Phi_2)}$$

(ii) If $\psi \in L^1(\hat{G})$ then $\hat{\psi} m \in \mathcal{M}_{\Phi_1, \Phi_2}(G)$. Moreover

$$\| \hat{\psi} m \|_{(\Phi_1, \Phi_2)} \leq \| \psi \|_1 \| m \|_{(\Phi_1, \Phi_2)}.$$
A bounded measurable function $m$ defined in $\mathbb{R}$ is $(\Phi_1, \Phi_2)$-multiplier on $\mathbb{R}$ if there exists $C > 0$ such that

$$T_m(f)(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi ix\xi} d\xi$$

satisfies $N_{\Phi_2}(T_m(f)) \leq CN_{\Phi_1}(f)$ for any $f \in \mathcal{S}(\mathbb{R})$, which in case that $\Phi_1$ satisfies $\Delta_2$ is equivalent to the fact that $T_m$ extends to a bounded operator from $L^{\Phi_1}(\mathbb{R})$ into $L^{\Phi_2}(\mathbb{R})$. 

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There are a lot of results known about $(p, q)$-multipliers corresponding to $\Phi_1(x) = x^p$ and $\Phi_2(x) = x^q$ and denoted by $\mathcal{M}_{p,q}(\mathbb{R})$. 

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- \( \text{sign}(\xi) \in \mathcal{M}_{p, p}(\mathbb{R}) \) for \( 1 < p < \infty \).
G = ℝ

A bounded measurable function m defined in ℝ is (Φ₁, Φ₂)-multiplier on ℝ if there exists C > 0 such that

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- \( \text{sign}(\xi) \in \mathcal{M}_{p,p}(\mathbb{R}) \) for \( 1 < p < \infty \).
- \( |2\pi \xi|^{-\alpha} \in \mathcal{M}_{p,q}(\mathbb{R}) \) for \( 0 < \alpha < 1 \), \( 1/q = 1/p - \alpha \).
A bounded measurable function $m$ defined in $\mathbb{R}$ is $(\Phi_1, \Phi_2)$-multiplier on $\mathbb{R}$ if there exists $C > 0$ such that

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satisfies $N_{\Phi_2}(T_m(f)) \leq C N_{\Phi_1}(f)$ for any $f \in \mathcal{S}(\mathbb{R})$, which in case that $\Phi_1$ satisfies $\Delta_2$ is equivalent to the fact that $T_m$ extends to a bounded operator from $L^{\Phi_1}(\mathbb{R})$ into $L^{\Phi_2}(\mathbb{R})$.

There are a lot of results known about $(p, q)$-multipliers corresponding to $\Phi_1(x) = x^p$ and $\Phi_2(x) = x^q$ and denoted by $\mathcal{M}_{p,q}(\mathbb{R})$.

- $\text{sign}(\xi) \in \mathcal{M}_{p,p}(\mathbb{R})$ for $1 < p < \infty$.
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- $\mathcal{M}_{p,q}(\mathbb{R}) = \{ 0 \}$ for $p > q$. 
The dilation operator $D_\lambda$

Denote $D_\lambda(f)(x) = f(\lambda x)$ for $\lambda > 0$.

$$C_\Phi(\lambda) = \| D_\lambda \|_{L^\Phi(\mathbb{R}) \to L^\Phi(\mathbb{R})} = \sup \{ N_\Phi(D_\lambda(f)) : N_\Phi(f) \leq 1 \}$$

Of course $C_\Phi(\lambda)$ is non-increasing, submultiplicative and $C_\Phi(1) = 1$. 

(Boyd indices) $\alpha(\Phi) > 0$ implies $\Phi$ satisfies $\Delta^2$ and $\beta(\Phi) < 1$ implies $\Phi$ satisfies $\nabla^2$. 

Oscar Blasco Transference and restriction of Fourier multipliers on Orlicz spaces
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$\alpha(\Phi) > 0$ implies $\Phi$ satisfies $\Delta_2$ and $\beta(\Phi) < 1$ implies $\Phi$ satisfies $\nabla_2$. 
New results

**Theorem**

Let $\Phi_1, \Phi_2$ be Young functions satisfying $\Delta_2$. If $\mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{R}) \neq \{0\}$ then $\beta(\Phi_1) \geq \alpha(\Phi_2)$.

**Corollary**

Let $\Phi_{p,q}(t) = \max\{t^p, t^q\}$. If $\max\{p_2, q_2\} < \min\{p_1, q_1\}$ then $\mathcal{M}_{\Phi_{p_1,q_1}, \Phi_{p_2,q_2}}(\mathbb{R}) = \{0\}$. 
The Bohr group

It is well-known that $\hat{D}$ is the Bohr compactification of $D$. We use the notation $AP(\mathbb{R})$ for the set of all continuous almost periodic functions on $\mathbb{R}$, that is to say uniform limits of polynomials $\sum_{k=1}^{n} \alpha_k e^{2\pi i x_k t}$ where $x_k \in \mathbb{R}$ and $\alpha_k \in \mathbb{C}$. 
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Recall now the Besicovich-Orlicz spaces for almost periodic functions: If \( f \in AP(\mathbb{R}) \) and \( \Phi \) is a Young function we define

\[
\tilde{\rho}_\Phi(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \Phi(|f(x)|)dx = \lim_{T \to \infty} \int_{-1/2}^{1/2} \Phi(|D_T f(x)|)dx
\]

and

\[
\|f\|_{B\Phi} = \inf\{k > 0 : \tilde{\rho}_\Phi(f/k) \leq 1\}.
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\]

A basic fact to use for the Bohr group is that if \( \mu \) is any measure defined on \( \mathbb{R} \) having support on a finite number of points, then \( \hat{\mu} \in AP(\mathbb{R}) \) and

\[
\|\hat{\mu}\|_{B\Phi(\mathbb{R})} = \|\mu\|_{L\Phi(\hat{D})}.
\]

(9)
Multipliers for $G = D$

Let $\Phi_1, \Phi_2$ be Young functions. A bounded function $m \in M_{\Phi_1, \Phi_2}(D)$ if there exists a constant $C > 0$ such that

$$N_{\Phi_2} \left( \sum \alpha_t m(t) \chi_t \right) \leq C N_{\Phi_1} \left( \sum \alpha_t \chi_t \right)$$

(10)

for any $\alpha = \sum \alpha_t \chi_t$ (finite sum).
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Assume that $\Phi_2$ satisfies $\nabla_2$ and $m$ is a bounded function on $\mathbb{R}$. The following are equivalent:

(i) $m \in \mathcal{M}_{\Phi_1, \Phi_2}(D)$.

(ii) There exists a constant $K$ such that

$$|\sum_{t \in \mathbb{R}} m(t)\mu(t)\lambda(t)dx| \leq C\|\hat{\mu}\|_{B_{\Phi_1}} \|\hat{\lambda}\|_{B_{\Psi_2}}$$

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Main results 1

**Theorem**

Let $m$ be a bounded continuous function on $\mathbb{R}$ and let $\Phi_1, \Phi_2$ be Young functions such that $\Phi_2$ satisfies $\nabla_2$ and

$$\sup_{\lambda > 1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < +\infty. \quad (12)$$

If $m \in \mathcal{M}_{\Phi_1, \Phi_2}(D)$ then $m \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{R})$. 
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If $m \in M_{\Phi_1, \Phi_2}(D)$ then $m \in M_{\Phi_1, \Phi_2}(\mathbb{R})$.

Corollary

Let $m$ be a bounded continuous function on $\mathbb{R}$ such that $m \in M_{\Phi_1, \Phi_2}(D)$ and let $\Phi_1, \Phi_2$ be Young functions such that $\alpha(\Phi_1) > \beta(\Phi_2)$. Then $m \in M_{\Phi_1, \Phi_2}(\mathbb{R})$.  

Oscar Blasco

Transference and restriction of Fourier multipliers on Orlicz spaces
Main results 2

Theorem

Let $m$ be a bounded continuous function on $\mathbb{R}$ and let $\Phi_1, \Phi_2$ be Young functions satisfying that $\Phi_2$ has $\nabla_2$ condition and

$$
\sup_{0<\lambda<1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < +\infty. \quad (13)
$$

If $m \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{R})$ then $m \in \mathcal{M}_{\Phi_1, \Phi_2}(D)$. 

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If $m \in \mathcal{M}_{\Phi_1,\Phi_2}(\mathbb{R})$ then $m \in \mathcal{M}_{\Phi_1,\Phi_2}(D)$.

Corollary

Let $m$ be a bounded continuous in $\mathbb{R}$ and $\Phi$ be a Young function satisfying $\nabla_2$ and

$$\sup_{\lambda > 0} C_{\Phi}(\lambda) C_{\Phi}(\frac{1}{\lambda}) < \infty. \quad (14)$$

Then $m \in \mathcal{M}_{\Phi}(\mathbb{R})$ iff $m \in \mathcal{M}_{\Phi}(D)$. 

Oscar Blasco
Transference and restriction of Fourier multipliers on Orlicz spaces
A bounded sequence \( m = (m_n)_{n \in \mathbb{Z}} \) is \((\Phi_1, \Phi_2)\)-multiplier on \( \mathbb{Z} \) if there exists \( C > 0 \) such that

\[
T_m(P)(t) = \sum_{k \in \mathbb{Z}} m_k \alpha_k e^{2\pi i k t}
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satisfies \( N_{\Phi_2}(T_m(P)) \leq CN_{\Phi_1}(P) \) for any \( P(t) = \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k t} \in P(\mathbb{T}) \), or equivalently, in case that \( \Phi_1 \) satisfies \( \Delta_2 \), extends to a bounded operator from \( L^{\Phi_1}(\mathbb{T}) \) to \( L^{\Phi_2}(\mathbb{T}) \).
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If \( \Phi_2 \) satisfying \( \nabla_2 \) and let \( m = (m_n) \) be a bounded sequence on \( \mathbb{Z} \). The following are equivalent:

(i) \( m \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{Z}) \).

(ii) There exists a constant \( K \) such that

\[
| \sum_{n \in \mathbb{Z}} m_n \alpha_n \beta_n | \leq CN_{\Phi_1}(P) N_{\Psi_2}(Q) \tag{16}
\]

for any \( P(t) = \sum_{n \in \mathbb{Z}} \alpha_n e^{2\pi int} \) and \( Q(t) = \sum_{n \in \mathbb{Z}} \beta_n e^{2\pi int} \) in \( P(\mathbb{T}) \).
Main results 3

**Theorem**

*Let $m$ be a bounded continuous function on $\mathbb{R}$ and $\Phi_1, \Phi_2$ be Young functions with $\Phi_2$ satisfying $\nabla_2$. (i) Assume that*

$$\sup_{0 < \lambda < 1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < \infty. \quad (17)$$

*If $m \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{R})$ then $m_n = (m(n)) \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{Z})$. (ii) Assume that*

$$\sup_{\lambda > 1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < \infty. \quad (18)$$

*If $(D_{1/N} m(n)) \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{Z})$ for all $N \in \mathbb{N}$ with $
abla_{\Phi_1, \Phi_2}(\mathbb{Z})$ then $m \in \mathcal{M}_{\Phi_1, \Phi_2}(\mathbb{R})$.***


THANK YOU VERY MUCH FOR YOUR ATTENTION!