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On the principle of pseudo-linearized stability: Applications to some delayed nonlinear parabolic equations ☆

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1. Introduction

We study the stabilization, as $t \to \infty$, of the solutions of the nonlinear abstract functional differential equation

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) + Bu(t) \ni F(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \tag{1}$$

on a Banach space X, where

$$u_t(\theta) = u(t+\theta), \theta \in [-\tau, 0],$$

to the associated equilibria: $w \in D(A) \subset D(B) \subset X$ such that

$$Aw + Bw \ni F(\widehat{w}(.)),$$

where $\widehat{w} \in C := C([-\tau, 0] : X)$ is the function which takes constant values equal to w. Our main goal is to extend, to a broad class of nonlinear operators A, the usual linearized stability principle saying, roughly speaking, that for the special case of A linear (single valued) and B and F are differentiable, the asymptotic stability of the zero solution of the

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linearized equation,

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t}(t) + Av(t) + \mathrm{D}B(w)v(t) = \mathrm{D}F(\widehat{w})v_t(.) & \text{in } X, \\ v(s) = u_0(s) & s \in [-\tau, 0], \end{cases}$$

implies that $u(t : u_0) \to w$ as $t \to \infty$, at least if $u_0(.)$ is close enough to $\widehat{w}$. We point out that our results seem to be new even without the delayed and nonlocal term (i.e. for $F \equiv 0$).

Our main motivation comes from some previous works by the authors and collaborators [11,12,15] dealing with the stabilization of the uniform oscillations for the *complex Ginzburg–Landau equation*. This stabilization takes place by means of some global delayed feedback. If, for instance,we consider the case in which the domain is $\Omega = (0, L_1) \times (0, L_2)$ with periodic boundary conditions, and define the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}, \;\; j = 1, 2,$$

this problem can be stated as follows:

$$(\mathrm{P}_1) \quad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - (1 + \mathrm{i}\varepsilon)\Delta\mathbf{u} = (1 - \mathrm{i}\omega)\mathbf{u} & \\ \quad -(1 + \mathrm{i}\beta)|\mathbf{u}|^2\mathbf{u} + \mu \mathrm{e}^{\mathrm{i}\chi_0}\mathbf{F}(\mathbf{u}, t, \tau) & \Omega \times (0, +\infty), \\ \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+2}}, \left(-\left.\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|_{\Gamma_j} = \right) \left.\dfrac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_j} & \\ \quad = \left.\dfrac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_{j+2}} \left(= \left.\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|_{\Gamma_{j+2}} \right) & \partial\Omega \times (0, +\infty), \\ \mathbf{u}(x, s) = \mathbf{u}_0(x, s) & \Omega \times [-\tau, 0], \end{cases}$$

where $\mathbf{n}$ is the outpointing normal unit vector and

$$\mathbf{F}(\mathbf{u}, t, \tau) = [m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + m_3\mathbf{u}(t - \tau, x) + m_4\overline{\mathbf{u}}(t - \tau)]$$
$$\text{with } \overline{\mathbf{u}}(s) = (1/|\Omega|) \int_\Omega \mathbf{u}(s, x)\,\mathrm{d}x.$$

Here the parameters $\varepsilon, \beta, \omega, \mu, \chi_0, m_i$ and τ are real numbers, in contrast with the solution $\mathbf{u}(x, t) = u_1(x, t) + \mathrm{i}u_2(x, t)$.

This type of equations (called as of Stuart–Landau in absence of the diffusion term) arise in the study of the stability of reaction diffusion equations such as $(\partial\mathbf{x}/\partial t) - \mathbf{D}\Delta\mathbf{X} = \mathbf{f}(\mathbf{X} : \eta)$ where $\mathbf{X} : \Omega \times (0, +\infty) \to \mathbb{R}^n$ and η is a real scalar parameter when the deviation $\mathbf{v}$ from the uniform state solution $\mathbf{X}_\infty$ is developed asymptotically in terms of some multiple scales (see [17]). Coefficient ε measures the degree to which the diffusion matrix $\mathbf{D}$ deviates from a scalar. With the basis of a sound experimental work, many recent studies of a more descriptive nature, but of a great originality and interest, have been written. In those studies the delay term $\mathbf{F}(\mathbf{u}, t, \tau)$ has been taken corresponding to $m_4 = 1, m_i = 0$ for $i = 1, 2, 3$ and introduced as a control mechanism (see [6,18]).

If we focus our attention on the so-called *slowly varying complex amplitudes* defined by $\mathbf{u}(x,t)=\mathbf{v}(x,t)\mathrm{e}^{-\mathrm{i}\omega t}$, thus, $\mathbf{v}$ satisfies

$$
(\mathrm{P}_2)\quad
\begin{cases}
\dfrac{\partial \mathbf{v}}{\partial t}-(1+\mathrm{i}\varepsilon)\Delta\mathbf{v}=\mathbf{v}-(1+\mathrm{i}\beta)|\mathbf{v}|^2\mathbf{v} \\
\quad +\mu\mathrm{e}^{\mathrm{i}\chi_0}[m_1\mathbf{v}+m_2\overline{\mathbf{v}} \\
\quad +\mathrm{e}^{\mathrm{i}\omega\tau}(m_3\mathbf{v}(t-\tau,x)+m_4\overline{\mathbf{v}}(t-\tau))] & \text{in } \Omega\times(0,+\infty), \\
\mathbf{v}|_{\Gamma_j}=\mathbf{v}|_{\Gamma_{j+2}},\ \left(-\left.\dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\right|_{\Gamma_j}=\right)\left.\dfrac{\partial\mathbf{v}}{\partial x_j}\right|_{\Gamma_j} \\
\quad =\left.\dfrac{\partial\mathbf{v}}{\partial x_j}\right|_{\Gamma_{j+2}}\left(=\left.\dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\right|_{\Gamma_{j+2}}\right) & \text{on } \partial\Omega\times(0,+\infty), \\
\mathbf{v}(x,s)=\mathbf{u}_0(x,s)\mathrm{e}^{\mathrm{i}\omega s} & \text{on } \Omega\times[-\tau,0].
\end{cases}
$$

The existence and uniqueness of a solution of (P_1) can be proven once we assume, for instance, that $\mathbf{u}_0\in \mathrm{C}([-\tau,0]:\mathbf{L}^2(\Omega))$ (see [15]). In the mentioned references we were interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x,t)=\rho_0\mathrm{e}^{-\mathrm{i}\theta t}$. We can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t)=\mathbf{v}(x,t)\mathrm{e}^{\mathrm{i}\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_2). Thus $\mathbf{z}(x,t)$ satisfies

$$
(\mathrm{P}_3)\quad
\begin{cases}
\dfrac{\partial \mathbf{z}}{\partial t}-(1+\mathrm{i}\varepsilon)\Delta\mathbf{z}=(1+\mathrm{i}\theta)\mathbf{z}-(1+\mathrm{i}\beta)|\mathbf{z}|^2\mathbf{z} \\
\quad +\mu\mathrm{e}^{\mathrm{i}\chi_0}[m_1\mathbf{z}+m_2\overline{\mathbf{z}} \\
\quad +\mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau,x)+m_4\overline{\mathbf{z}}(t-\tau))] & \text{in } \Omega\times(0,+\infty), \\
\mathbf{z}|_{\Gamma_j}=\mathbf{z}|_{\Gamma_{j+2}},\ \left(-\left.\dfrac{\partial\mathbf{z}}{\partial\vec{n}}\right|_{\Gamma_j}=\right)\left.\dfrac{\partial\mathbf{z}}{\partial x_j}\right|_{\Gamma_j} \\
\quad =\left.\dfrac{\partial\mathbf{z}}{\partial x_j}\right|_{\Gamma_{j+2}}\left(=\left.\dfrac{\partial\mathbf{z}}{\partial\vec{n}}\right|_{\Gamma_{j+2}}\right) & \text{on } \partial\Omega\times(0,+\infty), \\
\mathbf{z}(x,s)=\mathbf{u}_0(x,s)\mathrm{e}^{\mathrm{i}(\omega-\theta)s} & \text{on } \Omega\times[-\tau,0].
\end{cases}
$$

Notice that now, $\mathbf{v}_{uosc}(x,t)=\rho_0\mathrm{e}^{-\mathrm{i}\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t)=\mathbf{v}_{uosc}(x,t)\mathrm{e}^{\mathrm{i}\theta t}=\mathbf{y}=\rho_0$ is an stationary solution of (P_3): i.e. $\mathbf{0}=(1+\mathrm{i}\theta)\mathbf{y}-(1+\mathrm{i}\beta)|\mathbf{y}|^2\mathbf{y}+\mu\mathrm{e}^{\mathrm{i}\chi_0}[m_1+m_2+\mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3+m_4)]\mathbf{y}$.

The motivation to keep A nonlinear after the process of linearization (reason why we used the term of *pseudo-linearization principle*) comes from the fact that if we use the representation for the unknown of the delayed nonlinear equation (P_3) as $\mathbf{z}(x,t)=\rho(x,t)\mathrm{e}^{\mathrm{i}\phi(x,t)}$ then we arrive to a coupled nonlinear system of delayed equations for ρ and ϕ which can be described in terms of the representation operator given by $\mathbf{P}:\mathbb{R}^2\to\mathbb{C}$, $\mathbf{P}(\rho,\phi)=\rho\mathrm{e}^{\mathrm{i}\phi}$. Indeed, notice that $\mathbf{P}$ is nonlinear and that if $\mathbf{q}=(\rho,\phi)$ then $\mathbf{z}(x,t)=\mathbf{P}(\mathbf{q}(x,t))$ and the (P_3) can be formulated as $\mathrm{d}\mathbf{P}(\mathbf{q}(\cdot,t))/\mathrm{d}t+A\mathbf{P}(\mathbf{q}(\cdot,t))+B\mathbf{P}(\mathbf{q}(\cdot,t))=F(\mathbf{P}(\mathbf{q}(\cdot))_t)$. By using that the matrix $\mathbf{C}(\mathbf{q}(\cdot,t))=\operatorname{grad}\mathbf{P}(\mathbf{q}(\cdot,t))$ is not singular, we can arrive to the simpler formulation

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}(\cdot,t)+\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}[A\mathbf{P}(\mathbf{q}(\cdot,t))+B\mathbf{P}(q(\cdot,t))]=\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}F(\mathbf{P}(\mathbf{q}(\cdot))_t). \tag{2}$$

Notice that, although this delayed system can be also (formally) linearized (this is the procedure followed in [6,18]) the above diffusion operator $\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,t))$ becomes now quasilinear on $\mathbf{q}$ and thus the mathematical justification is much more delicate.

Other examples, given in Section 3, justify also the philosophy of keeping A nonlinear after linearizing the rest of the terms of the equation. For instance, this is the case when A is multivalued, or nondifferentiable or a degenerate quasilinear operator. We point out that some relevant examples of nonlinear functional equations arise in the most different contexts (see, for instance, [14] for one example in Climatology, [13] for a family of examples dealing with the wealth of nations and the general exposition made in [16]).

Coming back to the abstract formulation, the structural assumptions we shall assume in this paper are the following

(H1): $A \in \mathscr{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with

$$\mathscr{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathscr{P}(X) \text{ such that } A + \omega I \text{ is a } m\text{-accretive operator}\},$$

(see [9] for the case of $X = H$ a Hilbert space and the works by Benilan, Crandall, Pazy and others for the case of a general Banach space: see the monographs [8,21]),

(H2): The operators semigroup $T(t) : \overline{D_x(A)}^X \to X$, $t \geqslant 0$, generated by A, is compact (see [21]),

(H3): $B \in \mathscr{A}(0 : X)$, B is single valued, Fréchet differentiable, and B is dominated by A; i.e.

$$D_X(A) \subset D_X(B) \quad \text{and} \quad |Bu| \leqslant k|A^0u| + \sigma(|u|) \tag{3}$$

for any $u \in D_X(A)$ and for some $k < 1$ and some continuous function $\sigma : \mathbb{R} \to \mathbb{R}$, where, here and in what follows, $|.|$ denotes the norm in the space X (in contrast with the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $|A^0u| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$,

(H4): $F : C \to X$ satisfies a local Lipschitz condition, i.e., for any $R > 0$ there exists $L(R) > 0$ such that

$$|F(\phi) - F(\psi)| \leqslant L(R)\|\phi - \psi\| \quad \text{for any } \phi,\ \psi \in C \text{ and } \|\phi\|,\ \|\psi\| \leqslant R. \tag{4}$$

(H5): There exists $\delta^F > 0$ such that $F : B^X_{\delta^F}(\widehat{w}) \to X$ is Fréchet differentiable with the Fréchet derivative $\mathrm{D}F(\widehat{w})$ given by $\mathrm{D}(F(\widehat{w}))\phi = \int_{-\tau}^0 \mathrm{d}\eta(\theta)\phi(\theta)$, $\phi \in C$, for $\eta : [-\tau, 0] \to B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where $B^X_{\delta^F}(\widehat{w}) = \{\phi \in C; \|\phi - \widehat{x}\| < \delta^F\}$,

We further assume the main condition of our arguments:

(H6): The operator $y \to Ay + By - \mathrm{D}F(\widehat{w})(\mathrm{e}^{\omega\,\cdot} y)$ belongs to $\mathscr{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\operatorname{Re}\omega = \gamma < 0$ where $\mathrm{e}^{\omega\cdot}v \in C$ is defined by

$$(\mathrm{e}^{\omega\cdot}v)(s) = \mathrm{e}^{\omega s}\widehat{v}(s), \quad \text{with } \widehat{v}(s) = v \quad \text{for any } s \in [-\tau, 0] \text{ for } v \in X. \tag{5}$$

In order to treat the case in which B is differentiable we introduce the conditions

(H7): There exists a Banach space Y and there exists $\delta^B > 0$ such that B is Fréchet differentiable as function from $B_{\delta^B}(w) = \{z \in D(B);\ |w - z| < \delta^B\}$ into Y, with the Fréchet derivative $\mathrm{D}B(w)$ locally Lipschitz continuous,

(H8): The operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})(\mathrm{e}^{\omega^* \cdot} y)$ belongs to $\mathscr{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\operatorname{Re}\omega^* = \gamma^* < 0$.

2. The abstract results

Theorem 1. *Assume* (H1)–(H6). *Then there exists* $\alpha > 0$, $\varepsilon > 0$ *and* $M \geqslant 1$ *such that if* $u_0 \in B_\varepsilon^X(\widehat{w})$, $u_0(s) \in D_X(B)$ *for any* $s \in [-\tau, 0]$ *then the solution* $u(\cdot : u_0)$ *of* (1) *exists on* $[-\tau, +\infty)$ *and*

$$|u(t : u_0) - w| \leqslant M\mathrm{e}^{-\alpha t}\|u_0 - \widehat{w}\| \quad \text{for any } t > 0. \tag{6}$$

Moreover, if we also assume (H7), *that* (H1)–(H5) *holds on the space Y and* (H8) *then there exists* $\alpha^* > 0$, $\varepsilon^* \in (0, \varepsilon]$ *and* $M^* \geqslant 1$ *such that if* $u_0 \in B_{\varepsilon^*}^{X\cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ *for any* $s \in [-\tau, 0]$ *then*

$$\begin{aligned}&|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \\ &\quad \leqslant M^*\mathrm{e}^{-\alpha^* t}(\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y) \quad \text{for any } t > 0.\end{aligned} \tag{7}$$

Proof. From assumptions (H4) and (H5)

$$F(\phi) = F(\widehat{w}) + \mathrm{D}F(\widehat{w})(\phi - \widehat{w}) + G^F(\widehat{w}, \phi) \quad \text{for any } \phi \in B_{\delta^F}^X(\widehat{w}).$$

Moreover since $\mathrm{D}F(\widehat{w})$ is locally Lipschitz continuous, there exists a continuous increasing functions b_X^F such that

$$|G^F(\widehat{w}, \phi)| \leqslant b_X^F(\|\phi - \widehat{w}\|)\|\phi - \widehat{w}\| \quad \text{for any } \phi \in B_{\delta^F}^X(\widehat{w}). \tag{8}$$

Then

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) - \frac{\mathrm{d}w}{\mathrm{d}t} + Au(t) - Aw + Bu(t) - Bw - \mathrm{D}F(\widehat{w})(u_t - \widehat{w}) \ni -G^F(\widehat{w}, u_t). \tag{9}$$

We now use assumption (H6). We claim that we can find a constant $K \geqslant 1$ and such that

$$\|u_t - \widehat{w}\| \leqslant K\mathrm{e}^{\gamma t}\|u_0 - \widehat{w}\| + \int_0^t K\mathrm{e}^{\gamma(t-s)}|G^F(\widehat{w}, u_s)|\,\mathrm{d}s. \tag{10}$$

Indeed, as $u(t)$ and w are "integral solutions" in the sense of Benilan (see, e.g. [8]), then, by (H6), if we multiply (9) by $u(t) - w$ (by using the usual semiinner braket [,]: see, for instance [8] or [21, Section 1.4]) we get that

$$|u(t) - w| \leqslant K\mathrm{e}^{\gamma(t-t_0)}|u(t_0) - w| + \int_{t_0}^t K\mathrm{e}^{\gamma(t-s)}|G^F(\widehat{w}, u_s)|\,\mathrm{d}s \tag{11}$$

for any $t \geqslant t_0 \geqslant 0$ (see, for instance, [8] or [21, Theorem 1.7.5]). Then,

$$|u(t) - w| \leqslant K e^{\gamma t} \|u_0 - \widehat{w}\| + \int_0^t K e^{\gamma(t-s)} |G^F(\widehat{w}, u_s)| \, ds \tag{12}$$

for any $t \geqslant 0$. Finally, since (12) holds trivially for $t \in [-\tau, 0]$ we get (10) by taking the maximum, in (11), on intervals of the form $[t-\tau, t]$ for any $t \geqslant 0$.

Now, let $R \in (0, \delta^F)$ be chosen so that

$$b_X^F(R) < (-\gamma)/(4K). \tag{13}$$

Define $\varepsilon = \min\{R/(2K), \delta_X^F\}$. Let us show that if $u_0 \in B_\varepsilon^X(\widehat{w})$ then the associated solution u of (1) exists and $\|u_t - \widehat{w}\| < R$ for all $t \geqslant 0$. Thanks to assumption (H2) we can apply some maximal continuation results (see, for instance, Chapter 3 of [21], or Chapter 2 of [22] when A is linear), it suffices to show that there exists no $t_1 > 0$ so that $\|u_{t_1}\| = R$ and $\|u_t\| < R$ for $t \in [0, t_1)$. By contradiction, if there exists such a t_1, then on $[0, t_1]$ we have

$$\begin{aligned}\|u_t - \widehat{w}\| &\leqslant K e^{\gamma t} \|u_0 - \widehat{w}\| + \int_0^t K e^{\gamma(t-s)} |G^F(\widehat{w}, u_s)| \, ds \\ &\leqslant K e^{\gamma t} \|u_0 - \widehat{w}\| + 2K b_X^F(R) \int_0^t e^{\gamma(t-s)} \|u_s - \widehat{w}\| \, ds.\end{aligned}$$

In particular, at $t = t_1$ we have

$$\|u_{t_1} - \widehat{w}\| \leqslant K\varepsilon + \frac{2K b_X^F(R)}{(-\gamma)} R \leqslant R,$$

a contradiction to the choice of t_1.

Finally, to end the proof, let $u_0 \in B_\varepsilon^X(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ and let u the associated solution of (1). Since we have shown that $\|u_t - \widehat{w}\| \leqslant R$ for all $t \geqslant 0$ we get that

$$\|u_t - \widehat{w}\| \leqslant K e^{\gamma t} \|u_0 - \widehat{w}\| + K b_X^F(R) \int_0^t e^{\gamma(t-s)} \|u_s - \widehat{w}\| \, ds \tag{14}$$

holds for all $t \geqslant 0$. Thus, by using the Gronwall's inequality, we get

$$\|u_t - \widehat{w}\| \leqslant K e^{[\gamma - K b(R)]t} \|u_0 - \widehat{w}\| \leqslant K e^{(\gamma/2)t} \|u_0 - \widehat{w}\|, \, u_0 \in B_\varepsilon^X(\widehat{w})$$

which shows (6).

In order to show the decay estimate (7), we repeat the same arguments as before but now on the space Y. Then, from assumptions (H3) on Y and (H7), there exist δ_Y^F and δ_X^B such that

$$\begin{aligned}&B(z) = B(w) + DB(w)(z - w) + G^B(w, z) \quad \text{for any } z \in B_{\delta_X^B}(w), \\ &F(\phi) = F(\widehat{w}) + DF(\widehat{w})(\phi - \widehat{w}) + G^F(\widehat{w}, \phi) \quad \text{for any } \phi \in B^Y_{\delta_Y^F}(\widehat{w}),\end{aligned}$$

where now

$$\begin{aligned}&B_{\delta_X^B}(w) = \{z \in D_X(B) \cap D_Y(B); \ |w - z| < \delta_X^B\}, \\ &B_{\delta_Y^F}(\widehat{w}) = \{\phi \in C; \ \|\phi - \widehat{x}\|_Y < \delta_Y^F\}\end{aligned}$$

and, as before, $\|.\|_Y$ denotes the norm on the space $C_Y := C([-\tau, 0] : Y)$. Moreover, there exists two continuous increasing functions b_X^B and b_Y^F such that

$$|G^B(w, z)|_Y \leqslant b_X^B(|w - z|)|w - z| \quad \text{for any } z \in B_{\delta_X^B}(w), \tag{15}$$

$$|G^F(\widehat{w}, \phi)|_Y \leqslant b_Y^F(\|\phi - \widehat{w}\|_Y)\|\phi - \widehat{w}\|_Y \quad \text{for any } \phi \in B_{\delta_Y^F}(\widehat{w}). \tag{16}$$

Now

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) - \frac{\mathrm{d}w}{\mathrm{d}t} + Au(t) - Aw + \mathrm{D}B(w)(u(t) - w) - \mathrm{D}F(\widehat{w})(u_t - \widehat{w}) \ni G^B(w, u(t)) - G^F(\widehat{w}, u_t). \tag{17}$$

Thus, by using (H8) and arguing as in the first part we get that there exists a constant $K^* \geqslant 1$ such that

$$\|u_t - \widehat{w}\|_Y \leqslant K^* \mathrm{e}^{\gamma^* t}\|u_0 - \widehat{w}\|_Y + \int_0^t K^* \mathrm{e}^{\gamma^*(t-s)}(|G^B(w, u(s))|_Y + |G^F(\widehat{w}, u_s)|_Y)\,\mathrm{d}s \tag{18}$$

and then, by taking $\delta = \min(\delta_X^B, \delta_Y^F)$ and $R^* \in (0, \delta)$ such that

$$\max(b_X^B(R^*), b_Y^F(R^*)) < (-\gamma)/(4K), \tag{19}$$

we obtain that

$$\|u_t - \widehat{w}\|_Y \leqslant K^* \mathrm{e}^{\gamma^* t}\|u_0 - \widehat{w}\|_Y + K^* \int_0^t \mathrm{e}^{\gamma^*(t-s)}(b_X^B(R^*)\|u_s - \widehat{w}\|_X + b_Y^F(R^*)\|u_s - \widehat{w}\|_Y)\,\mathrm{d}s. \tag{20}$$

We define $\widetilde{R} = \min(R, R^*)$, $\widetilde{K} = \max(K, K^*)$, $\widetilde{\gamma} = \max(\gamma, \gamma^*) < 0$ and $\varepsilon^* = \min\{\widetilde{R}/(2\widetilde{K}), \delta\}$. Then, if $u_0 \in B_{\varepsilon^*}^{X \cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ and we assume, for instance, that $\widetilde{\gamma} = \gamma$, by adding (14) and (20) we deduce that

$$\begin{aligned}\|u_t - \widehat{w}\|_X + \|u_t - \widehat{w}\|_Y \leqslant{}& \widetilde{K}\mathrm{e}^{\widetilde{\gamma}t}(\|u_0 - \widehat{w}\|_X + \mathrm{e}^{(\gamma^* - \gamma)t}\|u_0 - \widehat{w}\|_Y) \\ &+ \widetilde{K}\int_0^t \mathrm{e}^{\widetilde{\gamma}(t-s)}[(b_X^F(\widetilde{R}) + b_X^B(\widetilde{R})\mathrm{e}^{(\gamma^* - \gamma)t})\|u_s - \widehat{w}\|_X \\ &+ b_Y^F(R^*)\mathrm{e}^{(\gamma^* - \gamma)t}\|u_s - \widehat{w}\|_Y]\,\mathrm{d}s.\end{aligned} \tag{21}$$

and estimate (7) follows, again, by Gronwall's inequality. □

Remark 2. It is not difficult to show that the assumption (H8) is implied (when A is linear) by the condition: "if $\lambda \in \mathbb{C}$ is given so that there exists $y \in D(B)\backslash\{0\}$ such that $Ay + \mathrm{D}B(w)y - \lambda y \ni \mathrm{D}F(\widehat{w})(\mathrm{e}^{\lambda \cdot} y)$ then $\mathrm{Re}\,\lambda > 0$". This allow to see Theorem 4.1 of Wu [22] (see also [19] and its references) as an special case of our abstract result with $B = 0$. In that case the "variation of the constants formula" can be used to get a different proof of the theorem since A is linear. Notice that if $B \neq 0$ and $D(B)\tilde{\mathbf{A}}X$ then the arguments of the proof of Wu [22] do not work (in spite of the claimed in the Example 4.8 given there).

Remark 3. When A is linear, as in the case without delay, assumption (H7) implies that the zero solution of the linearized problem $(\mathrm{d}U/\mathrm{d}t)(t)+AU(t)+\mathrm{D}B(w)U(t)$-$\mathrm{D}F(\widehat{w})U_t(.)=0$ in X, is locally asymptotically stable [22].

Remark 4. It is possible to prove the existence of global solutions for a general class of initial data (not necessarily near $\widehat{w}$) by using that $A+B \in \mathscr{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, some truncation of the nonlocal term $F(u_t)$ and passing to the limit by the compactness of the semigroup generated by A (see [21] for some related results).

An easy adaptation of the above proof leads to the following linearization result (now on a possibly smaller neighborhood of w) when A is differentiable.

Theorem 5. *The conclusion of the above result remains true if we assume, additionally, that condition* (H7) *also holds for A and we replace condition* (H8) *by*

(H9): The operator $y \to \mathrm{D}A(w)y + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})(\mathrm{e}^{\omega \cdot} y)$ belongs to $\mathscr{A}(\omega)$, for some $\omega \in \mathbb{C}$ with $\mathrm{Re}\, \omega = \gamma < 0$.

Remark 6. We claim that our arguments keeping A nonlinear after linearizing the rest of the terms (and in particular the way in which we apply Gronwall inequality) allow to extend, to the case of quasilinear equations, the so-called "method of quasilinearization" which, introduced by Bellman and Kalaba [7], we used to find solutions of a parabolic semilinear problem through the iteration of solutions of the linearized equation when starting in a super and a subsolution of the original semilinear problem (see, e.g., [10] and its references). This will be the subject of a future work by the authors.

3. Some examples

3.1. Example 1. The complex Ginzburg–Landau equation with a global delayed mechanism

Motivated by the special form of the nonlinear term of the equation in (P_3) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$ (notice that, in contrast with the case of scalar equations (see [19]) the space $\mathbf{L}^\infty(\Omega)$ is not suitable space to check assumption (H1): see [5]). A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature (see, for instance [3]). Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\varepsilon\Delta \\ \varepsilon\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\varepsilon \neq 0$ the diffusion matrix has a nonzero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t \geqslant 0}$ on X and the compactness of the semigroup is consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N=2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Concerning the rest of the terms of the equation in (P_3), we define $B\mathbf{u} = (1 + \mathrm{i}\beta)|\mathbf{u}|^2\mathbf{u}$ with $D(B)=\mathbf{L}^{12}(\Omega)$. By using the characterization of the semiinner braket [,] for the spaces $L^p(\Omega)$ (see, for instance [8]) it is easy to see that $\mathbf{B}$ verifies (H3). Moreover, by the results on the Frechet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [4]) we get that (H7) holds, with $\mathrm{D}B(\mathbf{y})\mathbf{v}=3(1+\mathrm{i}\beta)|\mathbf{y}|^2\mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. It can be found in the above-mentioned reference that assumption (H7) does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

The nonlocal term is defined, by

$$F(\mathbf{u}_t) = (1 + \mathrm{i}\theta)\mathbf{u}(t) + \mu \mathrm{e}^{\mathrm{i}\chi_0}[m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\overline{\mathbf{u}}(t-\tau))],$$

is locally Lipschitz continuous and its Frechet derivative is given by

$$\mathrm{D}F(\widehat{\mathbf{y}})\mathbf{v}(t) = -(1+\mathrm{i}\theta)\mathbf{v}(t) - \mu\mathrm{e}^{\mathrm{i}\chi_0}[m_1\mathbf{v}(t) + m_2\overline{\mathbf{v}}(t) - \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\overline{\mathbf{v}}(t-\tau))], \tag{22}$$

since for any $\phi \in C$, the nonlocal operator $\phi \to (1/|\Omega|)\int_\Omega \phi(s)\,\mathrm{d}x$ is linear and we can write $\mathrm{D}F(\widehat{\mathbf{y}})\phi = \int_{-\tau}^{0} \mathrm{d}\eta(s)\phi(s)$, with

$$\mathrm{d}\eta(s)\mathbf{v}(s) = \delta_0(s)(\mathbf{1}+\mathrm{i}\theta)\mathbf{v}(s) + \mu\mathrm{e}^{\mathrm{i}\chi_0}[\delta_0(s)(m_1\mathbf{v}(s) + m_2\overline{\mathbf{v}}(s)) + \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s) + m_4\overline{\mathbf{v}}(s))] \tag{23}$$

for any $\mathbf{v} \in C([-\tau,\infty) : \mathbf{L}^4(\Omega)$ and any $s \in [-\tau, \infty)$, where $\delta_0(s)$, $\delta_{-\tau}(s)$ denote the Dirac delta at the points $s = 0$ and $-\tau$, respectively. By well-known results, we have that $\eta : [-\tau, 0] \to B(X, X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analougouly replacing X by Y).

Finally, assumption (H6) can be read as a condition on the stationary state $\mathbf{y}$ (a study of the eigenvalue of operator A can be found, for instance, in [20]).

Remark 7. By introducing the representation operator $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho\mathrm{e}^{\mathrm{i}\phi}$ it is clear that the quasilinear operator $A\mathbf{P}(\mathbf{q})$ obtained from the operator $A\mathbf{u} = -(1 + \mathrm{i}\varepsilon)\Delta\mathbf{u}$ satisfies also condition $A \in \mathscr{A}(\omega)$ (since $\mathbf{P}$ is merely a change of variables). We point out that,

$$A\mathbf{P}(\mathbf{q}) = -(1+\mathrm{i}\varepsilon)[\Delta\rho - \rho|\nabla\phi|^2 + \mathrm{i}(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]\mathrm{e}^{\mathrm{i}\phi}.$$

Then, the “ formal linearization” of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ at $\mathbf{q}^*(x,y) := \mathbf{y} \equiv \rho_0$ becomes

$$D\mathbf{E}(\mathbf{q}^*)(\rho\mathrm{e}^{\mathrm{i}\phi}) = -(1+\mathrm{i}\varepsilon)[\Delta\rho + \mathrm{i}\rho_0\Delta\phi]\mathrm{e}^{\mathrm{i}\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ needs a slight modification of the above linear expression.

3.2. Example 2. Case in which A is nonlinear and nondifferentiable

It is not difficult to adapt the results of the first example to the case in which the vectorial operator is given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} A_1 & -\varepsilon\Delta \\ \varepsilon\Delta & A_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{24}$$

with $A_i : D(A_i) \to \mathscr{P}(L^4(\Omega))$ two (possibly different) m-accretive operators in $L^4(\Omega)$ as for instance,

$$\begin{cases} A_i u = -\mathrm{div}(|\nabla u|^{p_i-2}\nabla u) + \beta_i(u) \\ D(A_i) = \Big\{ u \in W^{1,1}(\Omega) \cap L^4(\Omega),\ u(x) \in D(\beta) \text{ a.e. } x \in \Omega,\ A_i u \in L^4(\Omega) \\ \qquad \text{and } - \left|\frac{\partial u}{\partial n}\right|^{p_i-2} \frac{\partial u}{\partial n} \in \gamma_i(u) \text{ on } \partial\Omega \Big\}, \end{cases}$$

where $p_i \in (1, +\infty)$ and β_i, γ_i are maximal monotone graphs of $\mathbf{R}^2$ (not necessarily associated to differentiable functions). We send the reader to Vrabie [21] (and its references) for the study of the assumptions (H1) and (H2) for each of the nonlinear operators A_i. We point out that the structure of the nonlinear diffusion operator (24) allows to guarantee that the diffusion operator is m-accretive on $\mathbf{L}^{4/3}(\Omega)$. For related works see Aftalion and Pacella [1,2]. The same holds also on $\mathbf{L}^{4/3}(\Omega)$.

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