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New Results on the Burgers and the Linear Heat Equations in Unbounded Domains

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Abstract. We consider the Burgers equation and prove a property which seems to have been unobserved until now: *there is no limitation on the growth of the nonnegative initial datum $u_0(x)$ at infinity when the problem is formulated on unbounded intervals, as, e.g. $(0+\infty)$, and the solution is unique without prescribing its behaviour at infinity*. We also consider the associate stationary problem. Finally, some applications to the linear heat equation with boundary conditions of Robin type are also given.

Nuevos resultados sobre las ecuaciones de Burgers y lineal del calor planteadas en dominios no acotados

Resumen. Mostramos una propiedad que parece no haber sido advertida anteriormente para las soluciones de la ecuación de Burgers: *no existe ninguna limitación en el crecimiento para el dato inicial $u_0(x)$ en el infinito cuando el problema se formula en intervalos no acotados como, por ejemplo, $(0,+\infty)$, y la solución es única*. Aplicamos este resultado al caso de condiciones de Robin para la ecuación lineal del calor. Consideramos también el problema de Burgers estacionario.

1 Introduction

Given $u_0 \in L^1_{\text{loc}}(0,+\infty)$, $u_0(x) \geq 0$ a.e. $x \in (0,+\infty)$, we consider the viscous Burgers' problem

$$(VBP)\begin{cases} u_t - u_{xx} + uu_x = 0 & x \in (0,+\infty),\ t>0, \\ u(0,t) = 0 & \liminf_{x\to\infty} u(t,x) \geq 0 t>0, \\ u(x,0) = u_0(x) & \text{on } (0,+\infty). \end{cases} \tag{1}$$

Our goal is to prove that there is no limitation on the growth of the nonnegative initial datum $u_0(x)$ at infinity in order to get a unique solution (i.e. without prescribing its behaviour at infinity). A related property was used in [7] for the study of the controllability question for this equation. This property contrasts with the pioneering results by A. N. Tychonov (1935) for the linear heat equation and its more recent generalizations by many authors. We prove that the property requires the nonnegativeness of the initial datum u_0. This contrasts also with the results on existence of solutions without growth conditions at infinity in the literature dealing with other classes of nonlinear parabolic and elliptic equations ([13, 5, 2, 10, 6, 3, 14]).

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In a second section, we shall show that the above results lead to some new (as far as we know) properties for the linear heat equation with a *radiation* Robin boundary condition at $x = 0$

$$(LHE : m) \begin{cases} v_t - v_{xx} = 0 & x \in (0, +\infty),\ t > 0, \\ v_x(t, 0) + mv(t, 0) = 0, & t > 0, \\ v(x, 0) = v_0(x) & \text{on } (0, +\infty). \end{cases} \tag{2}$$

We prove that starting without any limitation on the growth rate v_x/v at $t = 0$ a growth estimate (in terms of x/t) holds for any $t > 0$. In a final section, we consider the associated stationary (elliptic) Burgers problem

$$(SBP) \begin{cases} -u_{xx} + uu_x + \lambda u = f(x) & x \in (0, +\infty), \\ u(0) = 0, \quad \liminf_{x\to\infty} u(x) \geq 0 \end{cases}$$

with $\lambda \geq 0$ and $f \in L^1_{\text{loc}}(0, +\infty)$ ($f(x) \geq 0$ a.e. $x \in (0, +\infty)$). Problem (SBP) appears, for instance, in the time implicit semidiscretization of problem (VBP). The detailed proofs will be the object of a separeted article [9] (a previous presentation was made in the communication [8]).

2 On the viscous Burgers problem

It is well known that Burgers' equation plays a relevant role in many different areas of the mathematical physics, specially in Fluid Mechanics. Moreover the simplicity of its formulation, in contrast with the Navier-Stokes system, makes of the Burgers' equation a suitable model equation to test different numerical algorithms and results of a varied nature. The equation arises also in other contexts such as, e.g. cloud of electric ions and space charge repulsion ([12]).

The arguments for the elliptic problem can be adapted (in different ways) to be applied to the parabolic problem. Nevertheless other points of view are also possible. We start with some technical results which show (by some easy computations) the existence of some *universal solutions* (see also [1]).

Lemma 1 *The function $U^*(x,t) = \dfrac{x}{t}$ is an universal solution of (VBP) in the sense that*

$$\begin{cases} U^*_t - U^*_{xx} + U^* U^*_x = 0 & x \in (0, \infty),\ t > 0, \\ U^*(0, t) = 0, \quad U^*(x, t) \to +\infty \text{ as } x \to +\infty & t > 0, \\ U^*(x, 0) = +\infty & \text{on } (0, +\infty). \end{cases}$$

In particular, given $T > 0$ and $n > 0$ arbitrarily and given $u_0 \in L^1_{\text{loc}}(0, n)$ and $q \in C(0, T)$ with $u_0 \geq 0$ a.e. on $(0, n)$ and $0 \leq q(t) \leq t/n$ for any $t \in (0, T)$, then any weak solution u of the problem

$$(VBP)_{n,q} \begin{cases} u_t - u_{xx} + uu_x = 0 & x \in (0, n),\ t \in (0, T), \\ u(0, t) = 0, \quad u(t, n) = q(t), & t \in (0, T), \\ u(x, 0) = u_0(x) & \text{on } (0, n) \end{cases}$$

satisfies

$$0 \leq u(x, t) \leq \frac{x}{t} \qquad \text{on } (0, n) \times (0, T). \tag{3}$$

Lemma 2 *Given $n > 0$ arbitrarily, the function $U^*(x,t) = \frac{x}{t} + \frac{2}{n-x}$ satisfies*

$$\begin{cases} U^*_t - U^*_{xx} + U^* U^*_x \geq 0 & x \in (0, n),\ t > 0, \\ U^*(0, t) = \frac{2}{n}, \quad U^*(n, t) = +\infty & t > 0, \\ U^*(x, 0) = +\infty & \text{on } (0, n). \end{cases}$$

In particular, given $T > 0$ and $n > 0$ arbitrarily and given $u_0 \in L^1_{\text{loc}}(0,n)$ and $q \in C(0,T)$ with $u_0 \geq 0$, a.e. on $(0,n)$ and $q(t) \geq 0$ for any $t \in (0,T)$ then any weak solution u of the problem $(VBP)_{n,q}$ satisfies

$$0 \leq u(x,t) \leq \frac{x}{t} + \frac{2}{n-x} \qquad \text{on } (0,n) \times (0,T). \tag{4}$$

Remark 1 *In a recent paper ([17]) K. Yamada studies the Cauchy problem on $\mathbb{R}^N$ associated to $u_t - \Delta u + \operatorname{div} \mathbf{G}(u) = 0$ when $|\mathbf{G}'(u)| \leq C$ and under the growth $|u_0(x)| \leq c\,|x|$ for $|x| \to \infty$ (the case of u_0 without any limitation on the growth is not considered there).*

By using some not difficult arguments it is possible to construct a monotone sequence of approximate solutions $\{u_n\}$ which, passing to the limit, leads to the following result:

Theorem 1 ([9]) *Assume $u_0 \in L^1_{\text{loc}}(0,+\infty)$ and $u_0(x) \geq 0$ a.e. $x \in (0,+\infty)$. Then there exists a very weak solution u of (VBP). Moreover, if $u_0 \in L^2_{\text{loc}}(0,+\infty)$ the solution u belongs to $C([0,T] : L^2_{loc}(0,+\infty))$, for any $T > 0$, and it is the unique solution in this class of functions.*

IDEA OF THE PROOF OF UNIQUENESS. We can prove that, for given $T > 0$, $k \geq 0$ and $N > 2 + k$, there exist a positive constant K such that

$$\frac{d}{dt}\left(\int_0^n (n-x)^N u(x,t)^{k+1}\,dx\right) \leq K n^{N+2-k}. \tag{5}$$

This shows the equicontinuity in the approximating arguments and, by Ascoli-Arzela theorem, it leads to that the limit function $u \in C([0,T] : L^2_{\text{loc}}(0,+\infty))$. Now, due to the estimate (3) we can argue as in the proof of Theorem 4.3 of [17] and using the equicontinuity $C([0,T] : L^2_{loc}(0,+\infty))$ we get the result. □

Remark 2 *It is easy to construct counterexamples showing that the condition $u_0(x) \geq 0$ and/or the fact that the spatial domain be bounded from above, as it is the case of $(0,+\infty)$, are necessary conditions to get the conclusion of Theorem 1 (e.g., $u(x,t) = (-x)/(1-t)$ is a solution of the equation).*

Remark 3 *Although estimates (3) and (4) are universal (i.e. independent of any $u_0 \in L^1_{\text{loc}}(0,+\infty)$), it is possible to get some of localizing estimates (in the spirit of the ones which will be mentioned for the ellliptic problem). For instance, from (5), for a given $T > 0$ there exist two positive constants C_1, C_2 such that any solution u of the problem $(VBP)_{n,q}$ must satisfy the estimate*

$$\int_0^{n/2} u(x,t)^{k+1}\,dx \leq C_1 \int_0^n u_0(x)^{k+1}\,dx + \frac{C_2}{n^{k-2}}\,t, \qquad \text{for any } t \in (0,T), \tag{6}$$

for any $n > 0$, any $u_0 \in L^{k+1}_{\text{loc}}(0,+\infty)$ and any $q \in C(0,T)$ with $u_0 \geq 0$ and $q(t) \geq 0$.

Remark 4 *Theorem 1 can be extended to non homogeneous boundary conditions since, for any $k > 0$ the function $U^{\#}(x,t) = \frac{x}{t} + k$ is a supersolution of the problem*

$$(VBP:k)\begin{cases} u_t - u_{xx} + uu_x = 0 & x \in (0,+\infty),\ t \in (0,T), \\ u(0,t) = k, \quad \liminf_{x\to\infty} u(t,x) \geq 0 & t \in (0,T), \\ u(x,0) = u_0(x) & \text{on } (0,+\infty). \end{cases}$$

Remark 5 *The application of this type of arguments to other equations as, for instance, $u_t - (u^m)_{xx} + (u^\lambda)_x = 0$, with $\lambda > m \geq 0$, or the nonviscous Burgers equation $u_t - uu_x = 0$, is in progress and will be the object of some future publications.*

3 Applications to the linear heat equation with Robin boundary conditions

We consider now the linear heat equation with Robin boundary conditions $(LHE : m)$. It is well known (see, for instance [15]) that the sign of the coefficient m leads to very different behaviors of the respective solutions. Here we shall deal with the case $m \geq 0$ (sometimes called *radiation Robin boundary conditions*). Our goal is to study the growth of the rate v_x/v for large values of x.

Theorem 2 *Let $m \geq 0$ and let $v_0 \in L^1(0,+\infty) \cap W^{1,1}_{loc}(0,+\infty)$ such that $\frac{v_{0x}}{v_0} \in L^1_{\text{loc}}(0,+\infty)$, $\frac{v_{0x}(x)}{v_0(x)} \leq 0$ a.e. $x \in (0,+\infty)$ and such that $\liminf_{x\to\infty} \frac{v_x(t,x)}{v(t,x)} \leq 0$ for any $t \geq 0$, where v is the solution of* (2)*. Then necessarily*

$$\frac{2v_x(t,x)}{v(t,x)} \geq -\frac{x}{t} - 2m \qquad \text{for any } t > 0 \text{ and any } x \in (0,+\infty). \tag{7}$$

PROOF. By the Hopf-Cole transformation (see, e.g., [16]) we know that the solutions of (1) and (2) can be connected by the expression

$$u(t,x) = -\frac{2v_x(t,x)}{v(t,x)}.$$

Theorem 2 is, then, a consequence of Theorem 1. ■

Corollary 1 *Under the assumptions of Theorem* 2 *we get that, if $v_0 \geq 0$ a.e. on $(0,+\infty)$ then, given $x_0 \geq 0$*

$$v(t,x) \geq v(t,x_0)e^{-(\frac{x^2}{4t}+mx)} \qquad \text{for any } t > 0 \text{ and for any } x \geq x_0. \tag{8}$$

Remark 6 *Notice that although we can deduce, from the strong maximum principle, that $v(t,x) > 0$ for any $t > 0$ and for any $x \geq x_0$ (thanks to the assumption $v_0 \geq 0$ a.e. on $(0,+\infty)$) the estimate (8) is more precise and contains some global information that can not be deduced directly from the maximum principle. Moreover it has an universal nature in the sense that the decay rate is independent of the initial datum.*

4 The stationary problem

We shall prove the existence and uniqueness of the solution of the (SBP) problem with $\lambda \geq 0$ and

$$f \in L^1_{\text{loc}}(0,+\infty) \quad \text{and } f(x) \geq 0 \quad \text{a.e. } x \in (0,+\infty). \tag{9}$$

This problem is sometimes named elliptic Burgers-Sivashinsky problem (see Brauner [4]).

Definition 1 *A function $u \in L^2_{\text{loc}}(0,+\infty)$ is said to be a "very weak solution" of problem SBP if $\liminf_{x\to\infty} u(x) \geq 0$ and*

$$\int_0^\infty \left(-u\zeta_{xx} - \frac{u^2}{2}\zeta_x + \lambda u\zeta\right) dx = \int_0^\infty f\zeta \, dx \qquad \forall \zeta \in W^{2,\infty}(0,+\infty) \text{ with compact support.}$$

It is easy to see that any very weak solution u must satisfy some additional regularity. For instance, necessarily $u \in C[0,+\infty)$, $u(0) = 0$ and u is a *strong solution* in the sense that $(u_x - (\frac{u^2}{2}))_x \in L^1_{\text{loc}}(0,+\infty)$. Moreover, since $f(x) \geq 0$ a.e. $x \in (0,+\infty)$ (and $\liminf_{x\to\infty} u(x) \geq 0$) we get $u(x) \geq 0$.

Theorem 3 *Assume* (9)*. Then for any $\lambda \geq 0$ there exists a unique very weak solution u of (SBP).*

The proof will be divided in different steps. Let us start by proving the existence of a very weak solution. We shall follow a methodology which seems to be quite general and allows to connect two qualitative properties apparently disconnected: the existence of the so called *large solutions* on bounded domains and the existence of solutions on unbounded domains *without prescribing the behaviour at infinity*. To be more precise, as we shall see later (and as in [10]), it will be useful to work with a slightly more general framework (in particular to get an easy proof of the uniqueness of solutions of (SBP)). Let $n > 0$. Given

$$A \in L^{\infty}_{\mathrm{loc}}(0,n), \quad A(x) \geq A_0 \quad \text{a.e.} \quad x \in (0,n), \text{ for some } A_0 > 0, \tag{10}$$

we shall prove a *localizing property* which contains some similitudes with the one used as key idea in the pioneering paper [5] on the study of semilinear equations in $\mathbb{R}^N$ but with an entirely different proof.

Lemma 3 *Let $n > 0$ and $f \in L^1(0,n)$, $f \geq 0$, a.e. on $(0,n)$. Let $u \in L^2_{\mathrm{loc}}(0,n)$, $u \geq 0$, satisfying (weakly)*

$$(SBP)_n \begin{cases} -u_{xx} + (A(x)u^2)_x + \lambda u = f(x) & x \in (0,n), \\ u(0) = 0 \end{cases}$$

For any $k \geq 0$, let $\psi_k \colon (k,+\infty) \to (0,+\infty)$ be the function defined by $\psi_k(s) = \displaystyle\int_s^{+\infty} \frac{dr}{A_0 r^2 - k}$. Then, if $k = \|f\|_{L^1(0,n)}$, we get

$$u(x) \leq \frac{1}{n}\left(n\sqrt{\frac{k}{A_0}}\chi_{\left\{x\in(0,n):u(x)\leq n\sqrt{\frac{k}{A_0}}\right\}} + (\psi_{n^2k})^{-1}(1-x)\chi_{\left\{x\in(0,n):u(x)>n\sqrt{\frac{k}{A_0}}\right\}}\right) \text{ a.e. } x \in (0,n). \tag{11}$$

Moreover, if $\liminf_{x\to n} u(x) \geq 0$ then $u(x) \geq 0$ a.e. $x \in (0,n)$.

PROOF. Assume for the moment that $n = 1$. Integrating the equation from 0 to x we get that $-u_x(x) + A_0 u(x)^2 \leq k$, where we use the fact that $u(x) \geq 0$, that $u_x(0) \geq 0$ and (10). Then, if v satisfies

$$\begin{cases} -v_x(x) + A_0 v(x)^2 = k, \\ v(1) = +\infty, \end{cases} \tag{12}$$

we deduce that $u(x) \leq v(x)$ at least on the set where $v(x) \geq \sqrt{k/A_0}$. Since the above equation has separated variables, the (unique) solution of (12) is given by $v(x) = (\psi_k)^{-1}(1-x)$ and we get the estimate on the set $(x_k, 1)$ where $x_k \in [0,1)$ is such that $(\psi_{k,1})^{-1}(x_k) \geq \sqrt{k/A_0}$. Since, necessarily $\left\{x \in (0,n) : u(x) > \sqrt{k/A_0}\right\} \subset (x_k, 1)$ we get the conclusion. Once proved (11) for $n = 1$, we introduce the change of variable $x = nx'$ and the change of unknown $u(x) = hw(nx')$. Then, if u satisfies $(SBP)_n$ and we take $h = 1/n$ we get that w satisfies $(SBP)_1$ but replacing λ by λn^2 and f by $n^3 f(nx')$. Since $\int_0^1 f(nx')\,dx' = \frac{1}{n}\int_0^n f(x)\,dx$ we get the conclusion through the proof of the case $n = 1$. The nonnegativeness of u, once we assume that $\liminf_{x\to n} u(x) \geq 0$ is consequence of the maximum principle. ■

Remark 7 *Notice that the first estimate does not require any information on the (nonnegative) boundary value $u(n)$ and that the dependence on f is merely through the global information given by $f \geq 0$ and $\|f\|_{L^1(0,n)}$. Moreover, as in* [10], *the estimate allow to get some result on the asymptotic behaviour of solutions when $x \to +\infty$ (see* [9] *for more details).*

PROOF OF THE EXISTENCE OF SOLUTIONS OF THEOREM 3. We consider the problem:

$$(SBP^n_\infty) \begin{cases} -u_{xx} + (A(x)u^2)_x + \lambda u = f(x) & \text{in } (0,n), \\ u(0) = 0, \; u(n) = +\infty. \end{cases}$$

Lemma 4 ([9]) *Assume* (9) *and* (10). *Then for any* $\lambda \geq 0$ *there exists at least a weak solution* u *of* (SBP^n_∞).

Then, if we consider the sequence $\{u_n\}$ formed by solutions u_n of (SBP^n_∞) we get that $\{u_n\}$ is decreasing with n (in the sense that $u_{n-1}(x) \leq u_n(x)$ a.e. $x \in (0, n)$). It is now an easy task to prove that the function $u(x)$ defined through the pointwise limit of $\{u_n(x)\}$ is a weak solution of (SBP) (see [9] for details). ■

PROOF OF THE UNIQUENESS OF SOLUTIONS OF THEOREM 3. We follow some arguments introduced in [10] for other superlinear problem. Let u_1, u_2 be two possible (nonnegative) weak solutions of (SBP). Let $v = u_1 - u_2$. Then $v(0) = 0$ and v satisfies $-v_{xx} + (A(x)v^2)_x + \lambda v = 0$ on $(0, +\infty)$ with $A(x) = \frac{u_1(x)^2 - u_2(x)^2}{(u_1(x) - u_2(x))^2}$. Notice that such a function satisfies (10) for any $A_0 \in (0, 1)$. Then we can apply estimate (11) with $k = 0$. Since $\psi_0(s) = \int_s^{+\infty} \frac{dr}{A_0 r^2} = \frac{1}{A_0 s}$ for any $s > 0$, we get that $0 \leq v(x) \leq \frac{1}{n}(\frac{1}{A_0(1-x)})$ a.e. $x \in (0, n)$. Finally, since n is here arbitrary, we get that $v \equiv 0$. ■

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