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An energy balance climate model with hysteresis

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Abstract

Energy balance climate models of Budyko type lead to reaction–diffusion equations with slow diffusion and memory on the 2-sphere. The reaction part exhibits a jump discontinuity (at the snow line). Here we introduce a Babuška–Duhem hysteresis in order to account for a frequent repetition of sudden and fast warming followed by much slower cooling as observed from paleoclimate proxy data. Existence of global solutions and of a trajectory attractor will be established for the resulting system of a parabolic differential inclusion and an ode.

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1. Introduction

Throughout most of the paleoclimate proxy data there is a frequent repetition of sudden and fast warming followed by much slower cooling. We refer to [30] and the references therein for more details.

In particular, the advance of ice-sheets at the beginning of an ice-age occurs at a rate which is considerably slower than the rate by which they retreat during the transition to an

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interglacial period. In order to account for such an effect in an energy balance climate model, one introduces a hysteresis functional for the latent energy flux density of ice formation. This leads to a functional reaction–diffusion problem of the form

$$\begin{cases} \partial_t [c(x)u + w] - \nabla \cdot [k(x)|\nabla u|^{p-2}\nabla u] + g(u) = f(t, x, u, v), & x \in M, t > 0, \\ w(t, x) = \mathcal{W}_x[v(\cdot, x)](t), & t > 0, x \in M, \\ v(t, x) := \int_{-T}^0 \beta(s, x)u(t + s, x) ds, & t > 0, x \in M, \\ u(s, x) = u_0(s, x), & -T \leq s \leq 0, x \in M, \end{cases} \quad (1)$$

on the two-sphere M (or, more general, a closed, oriented surface) with linear or slow diffusion ($p \geq 2$). Global existence and boundedness of solutions were obtained for a corresponding problem without hysteresis [16]. An extension of these results will be presented here in case that \mathcal{W} is a Duhem hysteresis operator (cf. [36, Chapter V] for some background). In general, ∂_t and \mathcal{W}_x do not commute, but this is the case for Babuška's version of the Duhem hysteresis operator, where the derivative of the output function w is given by

$$\begin{aligned} \partial_t w(t, x) &= h_I(t, x, v(t, x), w(t, x))[\partial_t v(t, x)]^+ \\ &\quad - h_D(t, x, v(t, x), w(t, x))[\partial_t v(t, x)]^- \end{aligned} \quad (2)$$

with $v(t, x) := \int_{-T}^0 \beta(s, x)u(t + s, x) ds$ for $(t, x) \in \mathbf{R}_+ \times M$. As for the climate model, one expects that h_I and h_D are bounded, nonnegative functions, which decrease in v on a climatologically relevant interval, and have compact support. Moreover, h_I is much smaller than h_D near ice-age conditions (cf. the end of the next section for a simple heuristic consideration).

Throughout, we are going to focus on so-called Budyko-type energy balance models, which are characterized by the occurrence of a jump discontinuity in the albedo along the snow line. Mathematically, we handle this discontinuity by understanding the first equation under (1) as a partial functional differential inclusion (cf. (10) below). Our approach for treating (10) follows [16], where the approximate selection theorem for upper semi-continuous multi-valued mappings was used in order to establish approximate solutions, an approach, which is climatologically meaningful, too, since the resulting equations are (up to parameter selection) those obtained from so-called Sellers-type energy balance models.

Section 2 contains a brief outline of the climatological background, existence of local and global solutions are established for Sellers-type equations in Sections 3 and 4, respectively, whereas Section 5 is devoted to global solvability in case of Budyko-type models. Finally, we prove the existence of a trajectory attractor in a Budyko-type setting in Section 6 under an additional hypothesis which restricts the class of hysteresis operators.

2. Climatological background and hypotheses

Eq. (1) arises from a one-layer energy balance climate model (EBM) which is to predict the evolution of a, say, 10-year mean of atmospheric temperature $u = u(t, x)$ at sea level. Here x denotes the position on the earth’s surface M and t time in years. Starting point is the balance equation of energy which can be written in the form

$$\partial_t e(t, x) - \nabla \cdot \vec{\mathbf{j}}(t, x) = R_{\text{net}}(t, x), \tag{3}$$

where e , $\vec{\mathbf{j}}$ and R_{net} denote the internal energy flux, the flux due to horizontal heat transport and the net radiation flux, respectively. EBMs are designed by heuristically deriving functional expressions for e , $\vec{\mathbf{j}}$ and R_{net} in terms of u in such a way that the (in the model context) fundamental feedback mechanisms are accounted for. This approach goes back to Budyko [8] and Sellers [33] who proposed independently two such models in 1969. The interested reader is referred to [19] or [21] for an introduction and to the articles in [14] on EBMs for more recent developments.

Let us briefly discuss the (for us) significant qualitative features of the three flux terms beginning with the net radiation flux. R_{net} is equal to the difference between the absorbed and emitted radiation fluxes. The absorbed radiation flux is given by $Q(t, x)[1 - \alpha(x, u, v)]$ with Q the (to u) corresponding 10-year average of the incoming solar radiation flux and α the albedo, i.e. the relative portion of the incoming flux reflected to space. The function α depends on x due to land-water distribution, orography and vegetations zones, but more importantly on temperature which is utilized as an indicator for ice- and snow cover. Budyko and later North and collaborators modeled the temperature dependence by means of a step function, which assigns a high value for the albedo to all temperatures below a certain threshold value, usually -10°C (ice- or snow cover occurs in that case), and a lower value, if u lies above that threshold. Sellers and Ghil used, roughly speaking, a continuous interpolation of such a step function. In linking the albedo to u alone, one neglects the very important long response times the cryosphere exhibits. E.g. the expansion or the retreat of the huge continental ice sheets occurs with response times of thousands of years, a feature, which Bhattacharya et al. [5] proposed to incorporate by substituting u by a long term average of u , here called v , e.g. $v(t, x) := \int_{-T}^0 \beta(s, x)u(t + s, x) ds$ with $T \approx 10^4$ years. Of course, one can refine this procedure by having independent ice- and snow lines. In that case, one understands ice-lines as the boundaries of regions that are covered by continental ice-sheets or huge glaciers (slow response times in comparison with the 10-year mean), whereas snow lines refer to boundaries of regions where the variations in ice- or snow cover occur on the time scale of u . This approach was chosen in [25,23,24] and will be employed here, too. The emitted radiation flux is modelled either according to the Stefan-Boltzmann law with temperature in Kelvin (Sellers) or by a first order approximation of empirical radiation data (Budyko). Our qualitative setting, a strictly increasing function g with $|g(y)| \rightarrow \infty$ as $|y| \rightarrow \infty$, comprises both cases.

The 10-year mean of the horizontal heat flux is described in EBMs by a diffusive approximation. Most papers use a linear diffusion operator, however following a suggestion of Stone [35], some authors have also considered “slow diffusion”, i.e. $p > 2$ (cf. [13,14,18,15]) which leads to the generalized p -Laplacian appearing in (1).

Finally, as long as one disregards the latent energy stored in continental ice sheets and glaciers, the internal energy flux e is given by $c(x)u(t, x)$ with c the heat capacity which varies considerably with x due to the land–water distribution. However, a more accurate modeling suggests to set $e(t, x) = c(x)u(t, x) + \Xi(v(t, x))$ where c denotes the thermal inertia and $\Xi(v(t, x))$ stands for the latent energy density due to huge ice accumulations. This approach is closely related to the one for the Stefan problem (cf. [29] e.g.) with the obvious change that Ξ should depend on the long-term temperature mean v rather than on u in view of the long time scale of the latent fluxes. Ξ is a nonnegative bounded decreasing function with derivative ξ having compact support. This modeling idea underlies [16], where existence and boundedness of global solutions for the resulting problem were investigated. Clearly, one cannot reproduce another well-known phenomenon in this way, namely the fact that the transition from an ice-age to an interglacial period occurs considerably faster than the one from a moderate climate to an ice-age. There is no complete understanding of this phenomenon so far, but variations in the paths of ocean currents and in vegetation growth are surely among the causes. In order to incorporate such a feature into an energy balance climate model, we choose here to replace $\Xi(v(t, x))$ by the output function w of a Duhem–Babuška hysteresis operator as given in (2), i.e. one sets now $e(t, x) = c(x)u(t, x) + w(t, x)$. Hysteresis phenomena have been widely studied in the mathematical literature, and we refer to [7,28,36,38], or [27] for differential models, and to [34], where an m-accretive setting is explored. It is to be noted, though, that the resulting problem, here, is very different from the typical parabolic (semi-group) setting of the literature mentioned before, since the hysteresis effect appears on much slower time scales than the ones resolved by the model. These scales are represented by memory terms or mathematically, an average v of the unknown function u replaces u as input of the hysteresis operator.

Observing for sufficiently smooth u that $\partial_t v(t, x) = \int_{-T}^0 \beta(s, x) \partial_t u(t + s, x) ds$, one obtains

$$\begin{aligned} \partial_t w(t, x) &= h_I(t, x, v(t, x), w(t, x)) [\partial_t v(t, x)]^+ \\ &\quad - h_D(t, x, v(t, x), w(t, x)) [\partial_t v(t, x)]^- \\ &= h_I(t, x, v(t, x), w(t, x)) \\ &\quad \times \left[\beta(0, x) u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x) u(t + s, x)] ds \right]^+ \\ &\quad - h_D(t, x, v(t, x), w(t, x)) \\ &\quad \times \left[\beta(0, x) u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x) u(t + s, x)] ds \right]^- \end{aligned}$$

in view of $\beta(-T, \cdot) \equiv 0$ with T the memory span of the system, e.g. $T \approx 10^4$ years. Clearly, w should be nonnegative and bounded. This can be achieved by assuming that $h_J(t, x, v, w) = 0$ for $J \in \{I, D\}$ whenever $w \notin [\underline{w}, \bar{w}]$, where $0 < \underline{w} < \bar{w} < \infty$.

Collecting all terms apart from that for the absorbed radiation flux and a technical correction term in case of Budyko–North-type models on the left-hand side, one is led to

$$\left\{ \begin{array}{l}
 c(x)\partial_t u(t, x) - \nabla \cdot [k(\cdot)|\nabla u(t, \cdot)|^{p-2}\nabla u(t, \cdot)](x) \\
 + h_I(t, x, v(t, x), w(t, x)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\
 - h_D(t, x, v(t, x), w(t, x)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^- \\
 + g(u(t, x)) = f(t, x, u(t, x), v(t, x)), \quad t > 0, \quad x \in M \text{ a.e.}, \\
 v(t, x) = \int_{-T}^0 \beta(s, x)u(t + s, x) ds, \quad t > 0, \quad x \in M, \\
 \partial_t w(t, x) = h_I(t, x, v(t, x), w(t, x)) \\
 \quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\
 \quad - h_D(t, x, v(t, x), w(t, x)) \\
 \quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^- \\
 \quad t > 0, \quad x \in M, \\
 u(s, x) = u_0(s, x), \quad s \in [-T, 0], \quad x \in M, \\
 w(0, x) = w_0(x), \quad x \in M.
 \end{array} \right. \tag{4}$$

Throughout we will employ the following basic hypotheses which cover the various model settings.

- (H0) $M \subset \mathbf{R}^2$ two-dimensional, compact, oriented Riemannian manifold without boundary; $T > 0$;
- (H1) $c, k \in C^2(M)$ positive, $p \geq 2, \beta \in C^1([-T, 0] \times M, \mathbf{R}_+), \beta(-T, \cdot) \equiv 0, \beta(s, x) > 0$ for $s \in (-T, 0]$ and $x \in M, \int_{-T}^0 \beta(s, x) ds = 1$ for $x \in M$;
- (H2) (a) $h_J \in C^1(\mathbf{R}_+ \times M \times \mathbf{R}^2, \mathbf{R})$ bounded for $J \in \{I, D\}$,
 (b) $g \in C^1(\mathbf{R}), g(0) = 0, g$ strictly increasing and odd, $\lim_{y \rightarrow \infty} g(y) = \infty$, additionally, if $p = 2$, there exists a $r > 0$ with $\lim_{|y| \rightarrow \infty} g(y)/|y|^r = 0$;
 (c) $f: \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ bounded.

Of course, we have to supply regularity hypotheses for f , e.g. one can consider f to be C^1 in case of a Sellers-type model.

Finally, let us mention a simple-minded heuristic consideration for a diffusion problem $\partial_t u + h_I[\partial_t v]^+ - h_D[\partial_t v]^- - \Delta u = 0$, which indicates the effect of the hysteresis term. If $h_I = h_D$, then we are in a situation without hysteresis effect, which corresponds to problem studied in [16]. Comparison with $\partial_t u - \Delta u = 0$ shows that $h_I \geq 0$ has the effect of slowing down the temperature increase when $\partial_t v > 0$ (leaving an ice-age regime) as well as the temperature decrease when entering such a regime $\partial_t v < 0$. Clearly, the global sign condition for $\partial_t v$ is unrealistic for both cases, but hopefully instructive. Next, fix a function $\gamma = \gamma(t, x) > 0$ and let u, \underline{u} , and \bar{u} be solutions of $\partial_t u - \Delta u = 0, \partial_t u - h_D \gamma(t, x) - \Delta u = 0$, and $\partial_t u + h_I \gamma(t, x) - \Delta u = 0$, respectively, which are, all, equal at $t = 0$. Then $u - \bar{u}$ fulfils

$\partial \tilde{u} - \Delta \tilde{u} = h_I \gamma$, whereas $\underline{u} - u$ satisfies $\partial \tilde{u} - \Delta \tilde{u} = h_D \gamma$. Thus, $0 < h_I < h_D$ would result in a slow decrease of temperature, but more rapid increase.

3. Existence of local solutions for Sellers-type problems

Sellers-type refers to models with “Lipschitz-smooth” albedo functions. Here, we assume (H0)–(H2) and that

(H3) $f \in C_{\text{loc}}^{1-}(\mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$.

We follow the approach of Section 2.2. of [16] and realize (4) as an functional evolution equation in the phase space $C := C([-T, 0], C(M))$. To this end, let us recall some well-known facts and concepts.

3.1. The degenerate diffusion operator in $C(M)$

Let $J_p(\varphi) := \frac{1}{p} \int_M k |\nabla \varphi|^p$ for $\varphi \in W^{1,p}(M)$ and $p \in (2, \infty)$. We leave the simpler case $p = 2$ (linear diffusion, analytic semi-groups) to the reader and refer to [4] for Sobolev spaces on manifolds. J_p is Gateaux differentiable with $J'_p(\varphi)(\psi) = \int_M k(\cdot) |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \psi$ for $\varphi, \psi \in W^{1,p}(M)$. Setting $J_p \equiv \infty$ on $L^2(M) \setminus W^{1,p}(M)$ one can extend J_p to a proper convex lower semi-continuous functional \bar{J}_p on $L^2(M)$. The subdifferential $\partial \bar{J}_p(\varphi)$ of \bar{J}_p at $\varphi \in L^2(M)$ is nonempty, iff there exists a $\gamma \in L^2(M)$ with $\int_M k(\cdot) |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \psi = \int_M \gamma \psi$ for all $\psi \in W^{1,p}(M)$. Then $\partial \bar{J}_p(\varphi) = \{\gamma\}$. Set $\tilde{A}_p \varphi = \gamma$. It is well known that the closure of the domain of the subdifferential is equal to the closure of the effective domain of the functional, thus $\text{dom}(\tilde{A}_p)$ is dense in $L^2(M)$ and of course a subset of $W^{1,p}(M)$. Define A_p by $\text{dom}(A_p) := \{\psi \in \text{dom}(\tilde{A}_p) : \tilde{A}_p \psi \in C(M)\}$ and $A_p \psi = \tilde{A}_p \psi$ for $\psi \in \text{dom}(A_p)$. It is well known that A_p is m-accretive and generates a compact semi-group of nonexpansive mappings on $C(M)$.

3.2. Mild solvability

Next, recall the concept of a mild solution U of the nonhomogeneous evolution equation

$$\dot{U}(t) + A U(t) = z(t) \quad (5)$$

on a nondegenerate interval $I := [a, b] \subset \mathbf{R}$, where $A : D \rightarrow X$ is an m-accretive operator in a Banach space X and $z \in L^1(I, X)$. We refer to Vrabie’s monograph [37] for details on this concept and henceforth for results from the theory of m-accretive operator which will be employed freely throughout. Let $\varepsilon > 0$. $\mathcal{D}(\varepsilon, t_0, t_1, \dots, t_n, z_1, \dots, z_n)$ is called an ε -discretization of z on $[a, b]$ iff

- $a \leq t_0 \leq t_1, \dots, t_n \leq b, t_0 - a \leq \varepsilon, t_j - t_{j-1} \leq \varepsilon$ for $1 \leq j \leq n, b - t_n \leq \varepsilon$;
- $z_1, \dots, z_n \in X$;
- $\sum_{1 \leq j \leq n} \int_{t_{j-1}}^{t_j} \|z(s) - z_j\|_{\infty} ds \leq \varepsilon$.

Note, the symbol $\|\cdot\|_{\infty}$ is indiscriminately used for the maximum norm on various spaces, e.g., on $C(M)$ and $C(I, X)$, throughout the paper. Next, we fix an ε -discretization

$\mathcal{D}(\varepsilon, t_0, t_1, \dots, t_n, z_1, \dots, z_n)$ of z on $[a, b]$. A step function $v: [t_0, t_n] \rightarrow X$ which satisfies $v|_{(t_{j-1}, t_j]}$ to be constant for $1 \leq j \leq n$, is called an approximate solution of (5) on $[a, b]$ with respect to $\mathcal{D}(\varepsilon, t_0, t_1, \dots, t_n, z_1, \dots, z_n)$ iff $\text{ran}(v) \subset \text{dom}(A)$ and

$$\frac{v(t_j) - v(t_{j-1})}{t_j - t_{j-1}} + Av_j = z_j, \quad 1 \leq j \leq n.$$

Finally, a function $U \in C(I, X)$ is said to be a mild solution of (5), iff for each $\varepsilon > 0$ there is an ε -discretization $\mathcal{D}(\varepsilon, t_0, t_1, \dots, t_n, z_1, \dots, z_n)$ of z on $[a, b]$ and an approximate solution v of (5) on $[a, b]$ with respect to $\mathcal{D}(\varepsilon, t_0, t_1, \dots, t_n, z_1, \dots, z_n)$ such that $\|U(t) - v(t)\|_X < \varepsilon$ for all $t \in [t_0, t_n]$.

Results by Benilan, Crandall-Evans or Kobayashi guarantee a unique mild solution $U = U(t; a, U_0, z)$ of (5) on $[a, b]$ in case that $U_0 \in X$ and $z \in L^1((a, b), X)$. Noting that a mild solution of (5) is also an integral solution of (5), one obtains thanks to a theorem of Benilan that

$$\begin{aligned} & \|U(t; a, U_0, z) - U(t; a, \check{U}_0, \check{z})\|_\infty \\ & \leq \|U(s; a, U_0, z) - U(s; a, \check{U}_0, \check{z})\|_\infty + \\ & + \int_s^t [U(\tau; a, U_0, z) - U(\tau; a, \check{U}_0, \check{z}), z(\tau) - \check{z}(\tau)]_+ d\tau \end{aligned} \tag{6}$$

for $a \leq s \leq t \leq b$, $U_0, \check{U}_0 \in X$ and $z, \check{z} \in L^1((a, b), X)$, where $[\cdot, \cdot]_+ : X \times X \rightarrow \mathbf{R}$ denotes the normalized upper semi-inner product given by

$$[x, y]_+ := \lim_{h \rightarrow 0^+} \frac{\|x + hy\| - \|x\|}{h}. \tag{7}$$

3.3. A functional evolution equation induced by (4)

Let $b > 0$, $w_0 \in C(M)$, $u \in C([-T, b], C(M))$ and $v(t, x) := \int_{-T}^0 \beta(s, x)u(t + s, x) ds$. Fix $x \in M$, and consider the ordinary differential equation

$$\left\{ \begin{aligned} \partial_t w(t, x) &= h_I(t, x, v(t, x), w(t, x)) \\ &\quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\ &\quad - h_D(t, x, v(t, x), w(t, x)) \\ &\quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^-, \\ w(0, x) &= w_0(x). \end{aligned} \right. \tag{8}$$

Hypothesis (a) of (H2), the uniform boundedness of u and v , and standard existence and uniqueness results imply that (8) has a global solution $W = W(t, x; w_0, u)$ on $[0, b]$. More precisely, the right-hand side of (8) satisfies a linear growth estimate of the form $\leq \text{const}[1 + \sup_{\tau \in [0, b]} \|u(\tau, \cdot)\|_\infty + |w|][1 + \|u(\tau, \cdot)\|_\infty]$. Employing the following elementary Gronwall’s inequality:

3.3.1. Let $\varphi : [0, b] \rightarrow \mathbf{R}$ be continuous and piecewise continuously differentiable. If there are nonnegative constants μ, ν with $|\dot{\varphi}(t)| \leq \mu |\varphi(t)| + \nu$ for $t \in [0, b]$, then $|\varphi(t)| \leq \exp(\mu t) (|\varphi(0)| + \nu t)$ for $t \in [0, b]$.

One has that $\sup_{t \in [0, b]} \sup_{x \in M} |W(t, x; w_0, u)| < \infty$, uniformly for w_0 in bounded subsets of $C(M)$ and u in bounded subsets of $C([T, b], C(M))$. Also, if one assumes $u \in C_b([-T, \infty), C(M))$ (Banach space of uniformly bounded continuous functions from $[-T, \infty)$ into $C(M)$ under the supremum norm), this linear growth estimate guarantees global solvability on \mathbf{R}_+ of (8). There are standard techniques to utilize 3.3.1. or Gronwall's lemma in order to derive "Lipschitz-dependence" of solutions on parameters assuming Lipschitz-dependence of the "right-hand side" of ode-systems. Here we also need

Claim 1. $W(t, \cdot; w_0, u)$ is continuous.

Fix $w_0 \in C(M)$ and $u \in C([-T, b], C(M))$. One has for $t \in [0, b]$ and $x, \hat{x} \in M$:

$$\begin{aligned} & |\partial_t[W(t, x; w_0, u) - W(t, \hat{x}; w_0, u)]| \\ & \leq \left| h_I(t, x, v(t, x), W(t, x; w_0, u)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t+s, x)] ds \right]^+ \right. \\ & \quad \left. - h_I(t, \hat{x}, v(t, \hat{x}), W(t, \hat{x}; w_0, u)) \left[\beta(0, \hat{x})u(t, \hat{x}) - \int_{-T}^0 [\partial_s \beta(s, \hat{x})u(t+s, \hat{x})] ds \right]^+ \right| \\ & \quad + \left| h_D(t, x, v(t, x), W(t, x; w_0, u)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t+s, x)] ds \right]^- \right. \\ & \quad \left. - h_D(t, \hat{x}, v(t, \hat{x}), W(t, \hat{x}; w_0, u)) \left[\beta(0, \hat{x})u(t, \hat{x}) - \int_{-T}^0 [\partial_s \beta(s, \hat{x})u(t+s, \hat{x})] ds \right]^- \right|. \end{aligned}$$

Consider the first term:

$$\begin{aligned} & \left| h_I(t, x, v(t, x), W(t, x; w_0, u)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t+s, x)] ds \right]^+ \right. \\ & \quad \left. - h_I(t, \hat{x}, v(t, \hat{x}), W(t, \hat{x}; w_0, u)) \left[\beta(0, \hat{x})u(t, \hat{x}) - \int_{-T}^0 [\partial_s \beta(s, \hat{x})u(t+s, \hat{x})] ds \right]^+ \right| \\ & \leq \mathcal{C} \left(\|\beta\|_{\infty, 1}, \sup_{\tau \in [-T, b]} \|u(\tau, \cdot)\|_{\infty} \right) |h_I(t, x, v(t, x), W(t, x; w_0, u)) \\ & \quad - h_I(t, \hat{x}, v(t, \hat{x}), W(t, \hat{x}; w_0, u))| \\ & \quad + \sup_{\tau \in [0, b]} \sup_{z \in M} |h_I(\tau, z, v(\tau, z), W(\tau, z; w_0, u))| \\ & \quad \times \left[|\beta(0, x)u(t, x) - \beta(0, \hat{x})u(t, \hat{x})| \right. \\ & \quad \left. + \left| \int_{-T}^0 [\partial_s \beta(s, x)u(t+s, x)] ds - \int_{-T}^0 [\partial_s \beta(s, \hat{x})u(t+s, \hat{x})] ds \right| \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \mathcal{C} \left(\|\beta\|_{\infty,1}, \sup_{\tau \in [-T,b]} \|u(\tau, \cdot)\|_{\infty} \right) \\
 &\quad \times \text{Lip}(h_I) \left[\text{dist}_M(x, \hat{x}) + |v(t, x) - v(t, \hat{x})| \right. \\
 &\quad \left. + |W(t, x, w_0, u) - W(t, \hat{x}, w_0, u)| \right] \\
 &\quad + \sup_{\tau \in [0,b]} \sup_{z \in M} |h_I(\tau, z, v(\tau, z), W(\tau, z; w_0, u))| \\
 &\quad \times \left[\sup_{\tau \in [-T,b]} \|u(\tau, \cdot)\|_{\infty} |\beta(0, x) - \beta(0, \hat{x})| \right. \\
 &\quad \left. + \|\beta(0, \cdot)\|_{\infty} |u(t, x) - u(t, \hat{x})| \right. \\
 &\quad \left. + T \sup_{s \in [-T,0]} \|\partial_1 \beta(s, \cdot)\|_{\infty} \sup_{s \in [-T,b]} \|u(s, x) - u(s, \hat{x})\|_{\infty} \right. \\
 &\quad \left. + \sup_{\tau \in [-T,b]} \|u(\tau, \cdot)\|_{\infty} \right. \\
 &\quad \left. \times \|\partial_1 \beta(\cdot, x) - \partial_1 \beta(\cdot, \hat{x})\|_{L^1} \right].
 \end{aligned}$$

A quite similar estimate can be established for the second term. Thus, if h_I and h_D are bounded, one finds constants $\mathcal{C}_j, j=1, 2$, only depending on $\|\beta\|_{\infty,1}, \sup_{\tau \in [-T,b]} \|u(\tau, \cdot)\|_{\infty}$ and the Lipschitz constants of the $h_J, J \in \{I, D\}$, such that

$$\begin{aligned}
 &|\partial_t [W(t, x; w_0, u) - W(t, \hat{x}; w_0, u)]| \\
 &\leq \mathcal{C}_1 [\text{dist}_M(x, \hat{x}) + |u(t, x) - u(t, \hat{x})| \\
 &\quad + |v(t, x) - v(t, \hat{x})| + \sup_{s \in [-T,b]} |u(s, x) - u(s, \hat{x})| + |\beta(0, x) - \beta(0, \hat{x})| \\
 &\quad + \|\partial_1 \beta(\cdot, x) - \partial_1 \beta(\cdot, \hat{x})\|_{L^1}] + \mathcal{C}_2 |W(t, x, w_0, u) - W(t, \hat{x}, w_0, u)|.
 \end{aligned}$$

Let $\varepsilon > 0$. Since $u \in C([-T, b], C(M))$ is also uniformly continuous as a function on $[-T, b] \times M$ and the same holds for v and $\partial_1 \beta$ as function on $[-T, 0] \times M$, one finds a $\delta \in (0, \varepsilon)$ such that the first term of the last inequality is less than ε , whenever $\text{dist}_M(x, \hat{x}) < \delta$. Observing that $W(t, x; w_0, u) - W(t, \hat{x}; w_0, u) = 0$ at $t = 0$, one obtains from 3.3.1. that $|W(t, x; w_0, u) - W(t, \hat{x}; w_0, u)| \leq \varepsilon t \exp(\mathcal{C}_2 t)$, which guarantees the uniform continuity of $W(t, \cdot; w_0, u)$ on M .

We note in passing that $t \mapsto W(t, \cdot; w_0, u) \in C([0, b], C(M))$.

In fact, we see from what was stated before, there is a $\hat{\mathcal{C}} \in (0, \infty)$ with

$$|W(t, x; w_0, u) - W(\hat{t}, x; w_0, u)| \leq \hat{\mathcal{C}} |t - \hat{t}|$$

for $t, \hat{t} \in [0, b]$ and all $x \in M$. This inequality holds uniformly for w_0 and u in bounded sets.

Claim 2. $w \in C^1((0, \infty), C(M))$,

$$\begin{aligned}
 & |\partial_t[W(t, x; w_0, u) - W(\tilde{t}, x; w_0, u)]| \\
 & \leq |h_I(t, x, v(t, x), W(t, x; w_0, u)) \\
 & \quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\
 & \quad - h_I(\tilde{t}, x, v(\tilde{t}, x), W(\tilde{t}, x; w_0, u)) \\
 & \quad \times \left[\beta(0, x)u(\tilde{t}, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(\tilde{t} + s, x)] ds \right]^+ \\
 & \quad + |h_D(t, x, v(t, x), W(t, x; w_0, u)) \\
 & \quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^- \\
 & \quad - h_D(\tilde{t}, x, v(\tilde{t}, x), W(\tilde{t}, x; w_0, u)) \\
 & \quad \times \left[\beta(0, x)u(\tilde{t}, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(\tilde{t} + s, x)] ds \right]^- \Big|.
 \end{aligned}$$

The first term of the right-hand side can be estimated as being

$$\begin{aligned}
 & \leq \mathcal{C} \left(\|\beta\|_{\infty, 1}, \sup_{\tau \in [-T, b]} \|u(\tau, \cdot)\|_{\infty} \right) |h_I(t, x, v(t, x), W(t, x; w_0, u)) \\
 & \quad - h_I(\tilde{t}, x, v(\tilde{t}, x), W(\tilde{t}, x; w_0, u))| \\
 & \quad + \sup_{\tau \in [0, b]} \sup_{z \in M} |h_I(\tau, z, v(\tau, z), W(\tau, z; w_0, u))| \left[|\beta(0, x)u(t, x) \right. \\
 & \quad - \beta(0, x)u(\tilde{t}, x)| + \left| \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right. \\
 & \quad \left. - \int_{-T}^0 [\partial_s \beta(s, x)u(\tilde{t} + s, x)] ds \right| \\
 & \leq \mathcal{C} \left(\|\beta\|_{\infty, 1}, \sup_{\tau \in [-T, b]} \|u(\tau, \cdot)\|_{\infty} \right) \text{Lip}(h_I)[|t - \tilde{t}| + |v(t, x) - v(\tilde{t}, x)| \\
 & \quad + |W(t, x, w_0, u) - W(\tilde{t}, x, w_0, u)|] \\
 & \quad + \sup_{\tau \in [0, b]} \sup_{z \in M} |h_I(\tau, z, v(\tau, z), W(\tau, z; w_0, u))| \\
 & \quad \times \left[\|\beta(0, \cdot)\|_{\infty} |u(t, x) - u(\tilde{t}, x)| + T \sup_{s \in [-T, 0]} \|\partial_1 \beta(s, \cdot)\|_{\infty} \right. \\
 & \quad \left. \times \sup_{s \in [-T, 0]} \|u(t + s, \cdot) - u(\tilde{t} + s, \cdot)\|_{\infty} \right].
 \end{aligned}$$

Now, arguing like in Claim 1 we get that for every $\varepsilon > 0$ there exists δ such that if $|t - \tilde{t}| < \delta$ then

$$\sup_{x \in M} |\partial_t [W(t, x; w_0, u(t, x)) - W(\tilde{t}, x; w_0, u(\tilde{t}, x))]| < \varepsilon.$$

We have also used that u is uniformly continuous on every compact set of $(-T, \infty)$ to estimate the term $|u(t + s, x) - u(\tilde{t} + s, x)|$.

We also have “Lipschitz-dependence” on the various parameters, which, again, can be derived by means of 3.3.1. We only mention that

Claim 3. *Let $\mathcal{B} \subset C([-T, b], C(M))$ be bounded. Then there exists $\kappa \in (0, \infty)$ with*

$$\|W(t, \cdot; w_0, u) - W(t, \cdot; w_0, \hat{u})\|_\infty \leq \kappa \sup_{\tau \in [-T, b]} \|u(\tau) - \hat{u}(\tau)\|_\infty$$

for all $t \in [0, b]$ and all $u, \hat{u} \in \mathcal{B}$.

Finally, the family $(t, u) \mapsto W(t, \cdot; w_0, u)$ has the *Volterra-property*, i.e., it follows for $t \in (0, \infty)$ and $u, \hat{u} \in C_b([-T, \infty), C(M))$ with $u|_{[-T, t]} = \hat{u}|_{[-T, t]}$ that $W(t, \cdot; w_0, u) = W(t, \cdot; w_0, \hat{u})$.

Now, we can rewrite (4) by introducing the following mappings: $A\psi := \frac{1}{c(\cdot)} A_p \psi$ for $\psi \in \text{dom}(A_p)$; $A : C(M) \supset \text{dom}(A) \rightarrow C(M)$ is m -accretive.

$$\begin{aligned} R(t, \varphi, \psi, \vartheta)(x) := & \frac{1}{c(x)} \left[f \left(t, x, \varphi(x), \int_{-T}^0 \beta(s, x) \vartheta(s, x) ds \right) - g(\varphi(x)) \right. \\ & - h_I \left(t, x, \int_{-T}^0 \beta(s, x) \vartheta(s, x) ds, \psi(x) \right) \\ & \times \left(\beta(0, x) \vartheta(0, x) - \int_{-T}^0 \partial_1 \beta(s, x) \vartheta(s, x) ds \right) \\ & + h_D \left(t, x, \int_{-T}^0 \beta(s, x) \vartheta(s, x) ds, \psi(x) \right) \\ & \left. \times \left(\beta(0, x) \vartheta(0, x) - \int_{-T}^0 \partial_1 \beta(s, x) \vartheta(s, x) ds \right) \right] \end{aligned}$$

for $t \in [0, b], x \in M, \varphi, \psi \in C(M)$ and $\vartheta \in C([-T, 0], C(M))$. Clearly, (H1)–(H3) imply that $R \in C^{1-}([0, b] \times C(M)^2 \times C([-T, 0], C(M)), C(M))$. Employing the symbol u_t for $s \mapsto u(t + s)$ for $t \in [0, \infty)$ and $s \in [-T, 0]$, (4) can be rewritten as a functional evolution equation

$$\begin{cases} \dot{u}(t) + A u(t) = R(t, u(t), W(t, \cdot; w_0, u), u_t) & t > 0, \\ u(t) = u_0(t) & t \in [-T, 0]. \end{cases} \tag{9}$$

A *mild solution of (9) on $[0, b]$* is a $u \in C([-T, 0], C(M))$ which solves (5) with $z(t) = R(t, u(t), W(t, \cdot; w_0, u), u_t)$ for $t \in [0, b]$ and satisfies $u(t) = u_0(t)$ for $t \in [-T, 0]$.

3.4. Existence of local solutions of (9)

If it would not be for the term $W(t, x; w_0, u)$, one could just appeal to Theorem 5.3.1. in [37] for the existence of a local mild solution of (9). However, one realizes by inspection of the proof that the very same techniques cover the slightly more general case here. Also, Theorem 5.3.2. of that monograph carries over resulting in the following:

Theorem 3.1. *Let (H0)–(H3) be satisfied, $u_0 \in C([-T, 0], C(M))$, and $w_0 \in C(M)$. Then there exists a noncontinuable mild solution of (9). Moreover, if u is such a noncontinuable solution of (9) and $\bar{t} := \sup\{t : t \in \text{dom}(u)\} < \infty$, then $\lim_{t \rightarrow \bar{t}^-} \|u_t\|_\infty = \infty$.*

3.5. Uniqueness for (9)

Pretty much standard methods allow to establish unique mild solvability of (9) thanks to the Volterra property of W .

Theorem 3.2. *Let (H0)–(H3) be satisfied, $u_0 \in C([-T, 0], C(M))$, and $w_0 \in C(M)$. Then there exists at most one noncontinuable mild solution of (9).*

Proof. Recall that the normalized upper semi-inner product on $C(M)$ is given by $[\varphi, \psi]_+ := \max\{\psi(x)\text{sgn}(\varphi(x)) : x \in M, |\varphi(x)| = \|\varphi\|_\infty\}$ for $\varphi, \psi \in C(M)$ and $\varphi \neq \mathbf{0}$, and that $[\mathbf{0}, \psi]_+ = \|\psi\|_\infty$ for $\psi \in C(M)$. Now, let u_1 and u_2 be noncontinuable mild solution of (9) for a given pair $(u_0, w_0) \in C([-T, 0], C(M)) \times C(M)$. Let $\hat{t} := \sup\{t \in \text{dom}(u_1) \cap \text{dom}(u_2) : u_1|_{[-T, t]} = u_2|_{[-T, t]}\}$. Note that the Volterra property implies $W(t, x; w_0, u_1) = W(t, x; w_0, u_2)$ for all $t \in [0, \hat{t}]$. If \hat{t} is not equal to the positive exit time of u_1 (and therefore also of u_2), we find a $b > \hat{t}$ with $[-T, b] \subset \text{dom}(u_1) \cap \text{dom}(u_2)$. Employing Eq. (6) one obtains that $\|u_1(t) - u_2(t)\|_\infty \leq \|u_1(\hat{t}) - u_2(\hat{t})\|_\infty + \int_{\hat{t}}^t [u_1(\tau) - u_2(\tau), R(\tau, u_1(\tau), W(\tau, \cdot; w_0, u_1), u_1\tau) - R(\tau, u_2(\tau), W(\tau, \cdot; w_0, u_2), u_2\tau)]_+ d\tau$ for $\hat{t} \leq t \leq b$, which yields

$$\begin{aligned} & \|u_1(t) - u_2(t)\|_\infty \\ & \leq \int_{\hat{t}}^t \left\| \frac{1}{c(\cdot)} \left[f(\tau, \cdot, u_1(\tau), \int_{-T}^0 \beta(s, \cdot) u_1(\tau + s, \cdot) ds) \right. \right. \\ & \quad \left. \left. - f(\tau, \cdot, u_2(\tau), \int_{-T}^0 \beta(s, \cdot) u_2(\tau + s, \cdot) ds) \right] \right\|_\infty d\tau \\ & \quad + \int_{\hat{t}}^t \left\| \frac{1}{c(\cdot)} \left[h_I \left(\tau, \cdot, \int_{-T}^0 \beta(s, \cdot) u_1(\tau + s, \cdot) ds, W(\tau, \cdot; w_0, u_1) \right) \right. \right. \\ & \quad \times \left(\beta(0, \cdot) u_1(\tau) - \int_{-T}^0 \partial_1 \beta(s, \cdot) u_1(\tau + s) ds \right) \\ & \quad \left. - h_I \left(\tau, \cdot, \int_{-T}^0 \beta(s, \cdot) u_2(\tau + s, \cdot) ds, W(\tau, \cdot; w_0, u_2) \right) \right. \\ & \quad \left. \times \left(\beta(0, \cdot) u_2(\tau) - \int_{-T}^0 \partial_1 \beta(s, \cdot) u_2(\tau + s) ds \right) \right\|_\infty d\tau \\ & \quad + \int_{\hat{t}}^t \left\| \frac{1}{c(\cdot)} \left[h_D \left(\tau, \cdot, \int_{-T}^0 \beta(s, \cdot) u_1(\tau + s, \cdot) ds, W(\tau, \cdot; w_0, u_1) \right) \right. \right. \end{aligned}$$

$$\begin{aligned} & \times \left(\beta(0, \cdot)u_1(\tau) - \int_{-T}^0 \partial_1 \beta(s, \cdot)u_1(\tau + s) \, ds \right) \\ & - h_D \left(\tau, \cdot, \int_{-T}^0 \beta(s, \cdot)u_2(\tau + s, \cdot) \, ds, W(\tau, \cdot; w_0, u_2) \right) \\ & \times \left(\beta(0, \cdot)u_2(\tau) - \int_{-T}^0 \partial_1 \beta(s, \cdot)u_2(\tau + s) \, ds \right) \Big] \Big\|_{\infty} \, dt \quad \hat{t} \leq t \leq b \end{aligned}$$

Now, one can employ in the usual way the various local Lipschitz-conditions, the uniform boundedness of the solutions u and v on $[\hat{t}, b]$ and the Volterra property of W , and conclude that there exists a $C > 0$ with $\|u_1(t) - u_2(t)\|_{\infty} \leq C(t - \hat{t}) \sup_{\tau \in [\hat{t}, t]} \|u_1(\tau) - u_2(\tau)\|_{\infty}$ for all $t \in [\hat{t}, b]$. Since $C(t - \hat{t}) \rightarrow 0$ as $t \rightarrow \hat{t}$, there exists a $\delta > 0$ with $u_1(t) = u_2(t)$ for $t \in (\hat{t}, \hat{t} + \delta]$. This shows that \hat{t} cannot be smaller than the positive exit time of u_1 and of u_2 , hence $u_1 = u_2$. \square

4. Global solvability and boundedness of solutions for Seller-type problems

In view of Theorem 3.1, it suffices to establish $\|\cdot\|_{\infty}$ -boundedness on finite intervals for noncontinuable solutions of (9) in order to obtain global solvability. In fact, the noncontinuable solutions will possess bounds which only depend on the $\|\cdot\|_{\infty}$ of the initial condition and on the data of (4), if we add the following hypothesis:

$$(H4) \quad \limsup_{y \rightarrow \infty} \frac{g(y)}{y} > \max\{\|h_I\|_{\infty}, \|h_D\|_{\infty}\} \left[\|\beta(0, \cdot)\|_{\infty} + \sup_{x \in M} \int_{-T}^0 |\partial_1 \beta(s, x)| \, ds \right].$$

Theorem 4.1. *Let (H0)–(H4) be satisfied. Then, every noncontinuable solution of (9) is global, i.e. its domain is equal to $[-T, \infty)$, and uniformly bounded. The bound is uniform for initial conditions u_0 in bounded subset of $C([-T, 0], C(M))$.*

Proof. Let $r \in (0, \infty)$ so large that

$$\begin{aligned} & g(y) - \max\{\|h_I\|_{\infty}, \|h_D\|_{\infty}\} \left[\|\beta(0, \cdot)\|_{\infty} + \sup_{x \in M} \int_{-T}^0 |\partial_1 \beta(s, x)| \, ds \right] \\ & \times (y + 1) - \|f\|_{\infty} \geq \|c\|_{\infty} \end{aligned}$$

for all $y \in (r, \infty)$. Consider $u_0 \in C([-T, 0], C(M))$ with $\|u_0\|_{\infty} \leq r$. If u is a solution of (9) with $u(t) = u_0(t)$ for $t \in [-T, 0]$, then we claim that $\|u\|_{\infty} \leq r$. Otherwise, there exists an interval $[t_0, t_1]$ ($0 \leq t_0 < t_1 < \infty$) with $\|u(t_0)\|_{\infty} = r$ and $\|u(t)\|_{\infty} > r$ for $t \in (t_0, t_1]$. Setting $z(t) := R(t, u(t), W(t, \cdot; w_0, u), u_t)$ for $t \in [t_0, t_1]$, one concludes that $u|_{[t_0, t_1]}$ is a mild solution of $\dot{u} + Au(t) = z(t)$, hence an integral solution (cf. [37, Chapter 1.7]). Therefore

$$\|u(t)\|_{\infty} = \|u_{t_0}\|_{\infty} + \int_{t_0}^t [u(\tau), R(\tau, u(\tau), W(\tau, \cdot; w_0, u), u_{\tau})]_+ \, d\tau$$

for $t \in [t_0, t_1]$. Our choice of r together with (H4) guarantees that the integrand is negative on $(t_0, t_1]$, which yields $\|u(t)\|_{\infty} < r$ for $t \in (t_0, t_1]$ a contradiction. Note that $[u(\tau), z(\tau)]_+ =$

$\max\{z(\tau)(x)\operatorname{sgn}(u(\tau)(x)) : x \in M, u(\tau)(x) = \|u(\tau)\|_\infty\}$ and that $-g(u(\tau)(x))\operatorname{sgn}(u(\tau)(x)) < 0$ and $\operatorname{sgn}(z(\tau)(x)) = -\operatorname{sgn}(g(u(\tau)(x)))$ hold for all $\tau \in [t_0, t_1]$ and all $x \in M$ with $u(\tau)(x) = \|u(\tau)\|_\infty$. This a priori estimate implies first existence of a global solution and then uniform boundedness. \square

Remark 4.2. The proof actually shows that there exists an absorbing ball \mathcal{B} for (9) in $C([-T, 0], C(M))$ in the sense that for each bounded subset $B \in C([-T, 0], C(M))$ and each $w_0 \in C(M)$ there exist a time $t_B > 0$ such that $[U(t; u_0, w_0)] \in \mathcal{B}$ for $t > t_B$. Here $U(\cdot; u_0, w_0)$ denotes the unique solution of (9).

Let us reformulate the results we have established so far in terms of Eq. (4).

Definition 4.3. Let (H1)–(H4) be fulfilled. We call $(u, w) \in C([-T, \infty), C(M)) \times C([0, \infty), C(M))$ a global $(C(M)$ -)mild solution of (4), iff $w = W(t, \cdot; w_0, u)$ and u satisfies (9).

Corollary 4.4. Let (H1)–(H4) be fulfilled. Then (4) has a unique global mild solution (u, w) . There exists an absorbing ball in $C([-T, 0], C(M))$ for u with uniform absorbing time for initial data in bounded subsets of $C([-T, 0], C(M))$. Moreover, w grows at most linearly at ∞ , and if w is actually unbounded for one initial pair (u_0, w_0) , then this is true for all solutions (u, w) .

Note, Claim 3 of Section 3 implies the last assertion.

5. Existence of global solutions for Budyko-type models

We adapt the approximation procedure from [16], which has been refined in [26]. From a mathematical point of view, the distinct feature of a Budyko-type model is a jump-discontinuity of f which reflects the rapid motion of the snow line relative to the 10-year mean of temperature. Replacing f by an upper semi-continuous interval-valued function F , one is led to

$$\left\{ \begin{array}{l} c(x)\partial_t u(t, x) - \nabla \cdot [k(\cdot)|\nabla u(t, \cdot)|^{p-2}\nabla u(t, \cdot)](x) \\ \quad + h_I(t, x, v(t, x), w(t, x)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\ \quad - h_D(t, x, v(t, x), w(t, x)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^- \\ \quad + g(u(t, x)) \in F(t, x, u(t, x), v(t, x)) \quad t > 0, x \in M \text{ a.e.,} \\ v(t, x) = \int_{-T}^0 \beta(s, x)u(t + s, x) ds \quad t > 0, x \in M, \\ \partial_t w(t, x) = h_I(t, x, v(t, x), w(t, x)) \\ \quad \times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\ \quad - h_D(t, x, v(t, x), w(t, x)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^- \\ \quad t > 0, x \in M, \\ u(s, x) = u_0(s, x), \quad s \in [-T, 0], x \in M, \\ w(0, x) = w_0(x), \quad x \in M. \end{array} \right. \tag{10}$$

As for F , we assume

(H5) $F : \mathbf{R} \times M \times \mathbf{R} \times \mathbf{R} \rightarrow 2^{\mathbf{R}}$ is bounded and upper semi-continuous. Moreover, $F(t, x, u, v)$ is a nonempty, compact intervals for every $(t, x, u, v) \in \mathbf{R} \times M \times \mathbf{R} \times \mathbf{R}$.

We need the following fact about such “multis”, which can be found in [2,3,12].

5.1. Envelops

Let (H5) be satisfied. Then there exist an upper semi-continuous function $\bar{f} : \mathbf{R} \times M \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ and a lower semi-continuous function $\underline{f} : \mathbf{R} \times M \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ with $F(t, x, u, v) = [\underline{f}(t, x, u, v), \bar{f}(t, x, u, v)]$ for all $(t, x, u, v) \in \mathbf{R} \times M \times \mathbf{R} \times \mathbf{R}$.

5.2. Approximate selection theorem

Let X be a metric space, Z be a Banach space and $G : X \rightarrow 2^Z$ be upper semi-continuous with $G(m)$ nonempty and convex for $m \in X$. Then, given $\varepsilon > 0$, there exists a locally Lipschitz function $g_\varepsilon : X \rightarrow Z$ such that the range of g_ε is contained in the convex hull of the range of G and the graph of g_ε is contained in an ε -neighborhood of the graph of G .

5.3. The diffusion operator in $L^2(M)$

We cannot expect $C(M)$ -mild solution of (10) in view of the right-hand side of the first equation under (10). Therefore, we employ the following realization of the slow-diffusion operator. Let $H := L^2(M)$ be equipped with the inner product $\langle \varphi, \psi \rangle := \int_M \varphi \psi c$ for $\varphi, \psi \in H$. Denote by G the antiderivative of g with $G(0) = 0$. Using the notations of 3.1, we can define a proper lower semicontinuous functional \tilde{J}_p on H by setting $\tilde{J}_p(\varphi) := J_p(\varphi) + \int_M G \circ \varphi$ for $\varphi \in W^{1,p}(M)$ and $\tilde{J}_p(\varphi) := \infty$ for $\varphi \in H \setminus W^{1,p}(M)$. The subdifferential A_g of \tilde{J}_p is an m -accretive operator in H , and a solution u of the differential equation $\dot{u} + A_g u = z(t)$ ($z \in L^1((0, b), H)$, $b > 0$) corresponds to a mild solution of $c(x)u_t - \nabla \cdot [k(x)|\nabla u|^{p-2}\nabla u] + g(u) = c(x)z(t)(x)$.

5.4. Existence

We are going to assume throughout this subsection that (H0)–(H2), (H4), and (H5) are satisfied.

Definition 5.1. Given $b \in (0, \infty]$, $w_0 \in L^2(M)$ and $u_0 \in C([-T, 0], L^2(M))$. One calls $(u, w) \in C([-T, b), L^2(M)) \times C([0, b), L^2(M))$ an (L^2) -mild solution of (10), iff

$u|_{[-T,0]} = u_0, w(0) = w_0$ and the following conditions hold:

- $w(\cdot, x) \in W_{loc}^{1,2}([0, b), \mathbf{R})$ for a.e. $x \in M$ and satisfies

$$\begin{aligned} \partial_t w(t, x) &= h_I(t, x, v(t, x), w(t, x)) \\ &\times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^+ \\ &- h_D(t, x, v(t, x), w(t, x)) \\ &\times \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds \right]^- \\ &t \in (0, b), x \in M \text{ a.e.} \end{aligned}$$

- there exists $z \in L^1((0, b), H)$ such that $u|_{[0,b]}$ is a mild solution of $\dot{u} + A_g u = \frac{z(t)}{c}$ and $z(t)(x) + \partial_t w(t, x) \in F(t, x, u(t, x), \int_{-T}^0 \beta(s, x)u(t + s, x) ds)$ for a.a. $t \in (0, b)$ and $x \in M$.

Theorem 5.2. *Let (H0)–(H2), (H4), (H5) be satisfied, $u_0 \in C([-T, 0], C(M))$ and $w_0 \in C(M)$. Then (10) has a global mild solution (u, w) with $u \in C([-T, \infty), C(M))$ uniformly bounded and $w \in C(\mathbf{R}_+, C(M))$.*

Proof. Invoking 5.2, one finds a uniformly bounded sequence (f_j) of locally Lipschitz-functions $f_j : \mathbf{R} \times M \times \mathbf{R}^2 \rightarrow \mathbf{R}$ with $\text{dist}(\text{Graph}(f_j), \text{Graph}(F)) \rightarrow 0$, where dist denotes a product metric on $\mathbf{R} \times M \times \mathbf{R}^3$ (natural identifications). Corollary 4.4 guarantees a unique global $C(M)$ -mild solutions (u_j, w_j) of (4) with $f = f_j$ for $j \in \mathbf{N}$. Moreover, one has $\sup_{t \in \mathbf{R}} \|u(t)\|_\infty < \infty$. Using the notations from Section 3, the u_j are solutions of (9) with R replaced by a suitable R_j . Clearly, setting $z_j(t) := R_j(t, u_j, W(t, \cdot; w_0, u_j)(u_j)_t)$, u_j satisfies

$$\begin{cases} \dot{u} + A u = z_j & \text{on } (0, \infty), \\ u(0) = u_0(0) \end{cases} \tag{11}$$

in the mild sense. Since the (z_j) are uniformly bounded, we can argue as in the proof of Theorem 3.1 of [16] utilizing a theorem of Baras (cf. [37, Theorem 2.3.2]) and Cantor’s diagonal argument. We obtain that $(u_j|_{\mathbf{R}_+})$ possesses a subsequence which converges uniformly on compact subsets of \mathbf{R}_+ , hence there exists a subsequence (u_{l_j}) of (u_j) and a $u_\infty \in C([-T, 0], C(M))$ such that $\sup_{t \in [-T, n]} \|u_{l_j}(t) - u_\infty(t)\|_\infty \rightarrow 0$ as $j \rightarrow \infty$ for all $n \in \mathbf{N}$. In particular, u_∞ is uniformly bounded. By passing to another subsequence, if necessary, and employing once more Cantor’s diagonal procedure, we can assume that there is a $z_\infty : \mathbf{R}_+ \times M \rightarrow \mathbf{R}$ with $z_{l_j}|_{[0, n] \times M} \rightharpoonup z_\infty|_{[0, n] \times M}$ (weak convergence) in $L^r((0, n) \times M)$ for all $n \in \mathbf{N}$ and $r \in [1, \infty)$. Noting that $w_{l_j}(t)(x) = W(t, x; w_0, u_{l_j})$ for $j \in \mathbf{N}$, one concludes by means of Claim 3 of section 3 that (w_{l_j}) is a Cauchy sequence in $C(\mathbf{R}_+, C(M))$ with respect to uniform convergence on compacta, hence has a limit $w_\infty \in C(\mathbf{R}_+, C(M))$ in that topology. Rewriting (8) with $w = w_{l_j}$ as an Volterra integral equation and passing to the limit as $j \rightarrow \infty$ shows that w_∞ is equal to the solution $(t, x) \mapsto W(t, x; w_0, u_\infty)$. Therefore, in order to establish that (u_∞, w_∞) is an L^2 -mild solution of (10), it remain to

conclude that $z_\infty(t, x) + \hat{\partial}_t w_\infty(t, x) + g(u_\infty(t, x)) \in F(t, x, u_\infty(t, x), (u_\infty)_t)$ for almost all $(t, x) \in \mathbf{R} \times M$. But, this follows by quite the same reasoning as in the proof of the corresponding part of Theorem 3.1 in [16] employing the envelopes guaranteed by 5.1. \square

Remark 5.3. (i) One cannot expect a uniqueness result as examples by Díaz and Tello in special cases show. (ii) Obviously, if (u, w) is a solution of (10) as guaranteed in the proof of Theorem 5.2, one obtains that $\|w(t)\|_\infty$ grows at most linearly at ∞ . Also, there exist a ball \mathcal{B} in $C(M)$ and a function $\underline{t} : (0, \infty) \rightarrow (0, \infty)$ such that $u(t) \in \mathcal{B}$ for $t \geq \underline{t}(\rho)$ whenever $\|u_0\|_\infty \leq \rho$. (iii) Quite the same reasoning as in Lemma 3.1 of [26] shows that every L^2 -mild solution of (10) belongs to $C([-T, \infty), C(M)) \times C([0, \infty), C(M))$. Note that if (u, w) is a mild solution of (10), u satisfies an evolution equation of the form

$$\begin{cases} \dot{u} + A_g u = z & \text{on } (0, \infty), \\ u(0) = u_0(0), \end{cases} \tag{12}$$

with $z \in L^1_{\text{loc}}(\mathbf{R}_+, L^2(M))$.

5.5. Regularity

Again, let (H0)–(H2), (H4), and (H5) be satisfied throughout, and assume that $U = (u, w)$ is a mild solution of (10) with $u_0 \in C([-T, 0], C(M))$ and $w(0) \in C(M)$. Then u satisfies (12) with $z \in L^2_{\text{loc}}(\mathbf{R}_+, L^2(M))$, thus, a result of Brezis (cf. [6, Théorème 3.6] or [37, Theorem 1.9.3]) yields:

Remark 5.4. u is a strong solution of (12) on \mathbf{R}_+ , i.e., $u(t) \in \text{dom}(A_g) \subset W^{1,p}(M)$ for a.e. $t \in (0, \infty)$, $u|_{(0,\infty)} \in W^{1,1}_{\text{loc}}((0, \infty), L^2(M))$ and (12) holds pointwise a.e. in $(0, \infty)$. Moreover, $t \mapsto \bar{J}_p(u(t))$ belongs to $L^1((0, b)) \cap AC([\delta, b])$ for $0 < \delta < b < \infty$, hence u is a weak solution of $c(x)u_t - \nabla \cdot [k(x)|\nabla u|^{p-2}\nabla u] + g(u) = \zeta(t, x) - \hat{\partial}_t w(t, x)$ with $\zeta(t, x) \in F(t, x, u(t, x), v(t, x))$ for $(t, x) \in (0, \infty) \times M$ a.e. in the sense of Chapter II, Section 1 in [17].

We need the following version of Theorem 1.1, Chapter 3 in [17] which requires the discussion on “local thrust” in that section.

Theorem 5.5. *Let (H0) and (H1) be satisfied, $p > 2, \sigma \in (0, 1), \bar{t} \in (0, \infty), b \in L^\infty((0, \bar{t}) \times M), u_0 \in C^\sigma(M)$, and $u : [0, \bar{t}] \times M \rightarrow \mathbf{R}$ be a bounded local weak solution of $c(x)u_t - \nabla \cdot [k(x)|\nabla u|^{p-2}\nabla u] = b(t, x)$. Then there exist $\hat{\sigma} \in (0, \sigma)$ and $\gamma > 0$, which depend on σ and \bar{t} , but not on u such that*

$$|u(t_1, x_1) - u(t_2, x_2)| \leq \gamma \|u\|_\infty [\text{dist}_M(x_1, x_2) + |t_1 - t_2|^{\frac{p-2}{p}}] \hat{\sigma}$$

for all $t_1, t_2 \in [0, \bar{t}]$ and $x_1, x_2 \in M$.

Proof. Assume that $c \equiv 1$. Since M is a compact manifold without boundary, interior estimates (suitable chart-domains in M as bases of time–space cylinders which cover $[0, \bar{t}] \times M$) suffice. As for reaching the hyperplane $t = 0$, one consults the last remark of Section 3.1-(i) in [17]. In order to allow for c as under (H1), one observes that

$u_t - \nabla \cdot \left[\frac{k(x)}{c(x)} |\nabla u|^{p-2} \nabla u \right] - \nabla \cdot \frac{1}{c(x)} \cdot [|\nabla u|^{p-2} \nabla u] = \frac{b(t,x)}{c(x)}$ satisfies the structural conditions of Chapter 2.1. in [17] when considered in the framework of charts. We call the reader’s attention to the fact that “ \cdot ” in the expression $\nabla \frac{1}{c(x)} \cdot [|\nabla u|^{p-2} \nabla u]$ denotes the action of the Riemannian metric (inner product on the tangent spaces). The identity $\nabla \cdot [g \cdot Y] = g \nabla \cdot Y + \nabla g \cdot X$ for $g \in W^{1,1}(M)$ and $X \in W^{1,1}(M, T(M))$, $T(M)$ the tangent bundle of M , substitutes for the traditional identity in \mathbf{R}^n . \square

We need a version of the following result which is derived in the case $p > 2$ from the previous theorem by employing the compact embedding of $W^{1,p}(M) \hookrightarrow C^\sigma(M)$ for $\sigma \in (0, 1 - \frac{2}{p})$ and Remark 5.4., which guarantees that one can find a $\tilde{t} \in (0, \underline{t}]$ such that $u(\tilde{t}) \in C^\sigma(M)$.

Corollary 5.6. *Let the hypotheses stated in the beginning of this subsection be satisfied, $p > 2$ and $\sigma \in (0, 1 - \frac{2}{p})$. Then for each $\underline{t} > 0$ and $b > \underline{t}$ one has $u|_{[\underline{t}, b]} \in C^{\frac{\sigma(p-2)}{p}}([\underline{t}, b], C^\sigma(M))$.*

We note that the Hölder norm estimate is uniform for $\|\cdot\|_\infty$ -bounded set of functions u if the $U = (u, w)$ are mild solutions of (10). As for $p = 2$, one knows $u|_{[\underline{t}, b]} \in C^{\frac{\sigma}{2}}([\underline{t}, b], C^\sigma(M))$ for $\sigma \in (0, 1)$. We refer to 5.3.1 in Chapter II of [1] for linear evolution equations generated by analytic semigroups.

6. Trajectory attractor

Since one cannot expect unique solvability for (10) under (H0)–(H2), (H4), (H5), we employ the concept of an trajectory attractor which goes back to Sell [31] and Chepyzhov and Vishik [9,10] for Navier–Stokes equations and evolution equations, respectively. We also refer to [11,22] and [26] for trajectory attractors related to energy balance models. As compared to the earlier papers on EBMs, one has to address two additional difficulties, the memory of the system and the hysteresis effect. It is well known in the case of functional differential equations that one cannot expect complete continuity of the time t -shift unless t is greater than the memory span T of the system. The hysteresis effect complicates the situation even more, since the ordinary differential equation for $w(\cdot, x)$ does not show any smoothing effect. The latter forces us to restrict our attention to the special case where h_I and h_D do not depend on w . Additionally, the explicit t -dependence does not arise in our application, and though one could allow “almost periodicity in t ”, we also assume for notational convenience that h_I and h_D do not depend on t . Thus we consider from now on (H6) h_I and h_D do not depend on t and w .

Then system (10) reduces to a functional reaction–diffusion inclusion

$$\begin{cases} c(x)\partial_t u(t, x) - \nabla \cdot [k(\cdot)|\nabla u(t, \cdot)|^{p-2} \nabla u(t, \cdot)](x) \\ \quad + h_I(x, v(t, x))[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds]^+ \\ \quad - h_D(x, v(t, x))[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t + s, x)] ds]^- \\ \quad + g(u(t, x)) \in F(t, x, u(t, x), v(t, x)), \quad t > 0, \quad x \in M \text{ a.e.}, \\ v(t, x) = \int_{-T}^0 \beta(s, x)u(t + s, x) ds, \quad t > 0, \quad x \in M, \\ u(s, x) = u_0(s, x) \quad s \in [-T, 0], \quad x \in M. \end{cases} \tag{13}$$

Clearly, the results of Section 5 apply to (13), but different from (10), one can introduce a united trajectory space which does not involve w , hence as we are going to show here, the shift semiflow on that united trajectory space is completely continuous for $t > T$. Therefore, our setting falls into the scope of dissipative dynamical systems which ensures the existence of a global attractor.

Consider $C(\mathbf{R}_+, C([-T, 0], C(M)))$ under the metric

$$d(U, V) := \sup\{\|U(t) - V(t)\|_\infty : t \in [0, 1]\} + \sum_{l=2}^\infty \frac{1}{2^l} \frac{\sup\{\|U(t) - V(t)\|_\infty : t \in [0, l]\}}{1 + \sup\{\|U(t) - V(t)\|_\infty : t \in [0, l]\}}$$

for $U, V \in C(\mathbf{R}_+, C([-T, 0], C(M)))$, then $C(\mathbf{R}_+, C([-T, 0], C(M)))$ is a Fréchet space. Mild solutions $u \in C([-T, \infty), C(M))$ of (13) can be identified with certain elements of $C(\mathbf{R}_+, C([-T, 0], C(M)))$ via $U(t) = u_t$ for $t \in \mathbf{R}_+$. In the sequel, we are going to understand U and u in this way without explicitly mentioning this relationship each time.

Since we are interested in the long-term evolution of the climate system, orbital changes (Milankovitch theory) suggest that the incoming solar radiation flux could be either quasi-periodic or more general almost periodic. To this end, let us specify (H5) to

(H7) $F(t, x, u, v) = Q(t, x) F_0(x, u, v)$ with:

$Q \in C^2(\mathbf{R} \times M, \mathbf{R})$ nonnegative and uniformly continuous, $Q(\cdot, x)$ almost periodic for $x \in M$;

$F_0 : M \times \mathbf{R} \times \mathbf{R} \rightarrow 2^{\mathbf{R}}$ is bounded and upper semi-continuous, $F_0(x, u, v)$ is a nonempty, compact intervals for every $(x, u, v) \in M \times \mathbf{R} \times \mathbf{R}$.

Clearly, (H7) is a special case of (H5). We call the closure \mathcal{Q} of $\{(s, x) \mapsto Q(t + s, x) : t \in \mathbf{R}, x \in M\}$ in the compact open topology the hull of Q . It is well known that \mathcal{Q} is compact under hypotheses (H7) for Q and that $\|\tilde{Q}\|_\infty = \|Q\|_\infty$ for $\tilde{Q} \in \mathcal{Q}$. If $\tilde{Q} \in \mathcal{Q}$ and (H7) is satisfied, we write \tilde{F} for the “induced F ”, i.e. $\tilde{F}(t, x, u, v) := \tilde{Q}(t, x) F_0(x, u, v)$.

Definition 6.1. Let (H0)–(H2), (H4), (H6), (H7) be satisfied. Then one calls $\mathcal{X} := \{U \in C(\mathbf{R}_+, C([-T, 0], C(M))) : u \text{ solves (13) with } F = \tilde{F} \text{ for some } \tilde{Q} \in \mathcal{Q}\}$ the united trajectory space associated with (13).

Lemma 6.2. *Let (H0)–(H2), (H4), (H6) and (H7) be satisfied. Then \mathcal{X} is a closed subset of $C(\mathbf{R}_+, C([-T, 0], C(M)))$.*

Proof. Assume that $(U_j) \in \mathcal{X}^{\mathbf{N}}$ converges to $U_\infty \in C(\mathbf{R}_+, C([-T, 0], C(M)))$ w.r.t. d . Let $Q_j \in \mathcal{Q}$ such that u_j is a solution of (13) with $F = Q_j F_0$. By passing to a subsequence, if necessary, we can assume that Q_j converges uniformly on compact subsets of \mathbf{R} to some $Q_\infty \in \mathcal{Q}$. Now, let $b > 0$, then it remains to establish that $u_\infty|_{[0, b]}$ is a mild solution of $\dot{u} + A_g u + \frac{1}{c(\cdot)} \partial_t w_\infty(t, x) \in Q_\infty(t, x) F_0(x, u_\infty(t, x), \int_{-T}^0 \beta(s, x) u_\infty(t + s, x) ds)$, where $\partial_t w$ stands throughout this proof for $h_I(t, x, v(t, x))$

$\left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t+s, x)] ds \right]^+ - h_D(t, x, v(t, x)) \left[\beta(0, x)u(t, x) - \int_{-T}^0 [\partial_s \beta(s, x)u(t+s, x)] ds \right]^-$. Let $(z_j) \in L^2_{\text{loc}}(\mathbf{R}_+, L^2(M))$ with $z_j(t)(x) \in Q_j(t, x)F_0(x, u_j(t, x), \int_{-T}^0 \beta(s, x)u_j(t+s, x) ds)$ for a.a. $(t, x) \in [0, b] \times M$. Clearly, (z_j) is bounded in $L^2((0, b) \times M)$, hence, by passing to a subsequence, if necessary, we can assume that $(z_j|_{[0, b]})$ converges weakly in $L^2((0, b) \times M)$ to some $(t, x) \mapsto z_\infty(t, x)$ with $z_\infty \in L^2((0, b), L^2(M))$ (obvious identification). One concludes as in the proof of Theorem 5.2 that $z_\infty(t, x) \in Q_\infty(t, x)F_0(x, u_\infty(t, x), \int_{-T}^0 \beta(s, x)u_\infty(t+s, x) ds)$ for a.e. $(t, x) \in [0, b] \times M$. On the other hand, the “solution operator” $E : L^2(M) \times L^1((0, b), L^2(M)) \rightarrow L^2((0, b), L^2(M))$, which associated with $(y, z) \in L^2(M) \times L^1((0, b), L^2(M))$ the solution of $\dot{u} + A_g u = z$, $u(0) = y$, is completely continuous, hence the solution of $\dot{u} + A_g u = z_\infty(t) - \frac{1}{c(t)} \partial_t w_\infty(t, \cdot), u(0) = u_\infty(0)$ is a limit point of the sequence $(u_j|_{[0, b]})$, consequently, equal to $u|_{[0, b]}$.

Throughout we consider \mathcal{X} as a complete metric space under $d|_{\mathcal{X} \times \mathcal{X}}$. Define the shift-semi-flow $\tilde{S} : \mathbf{R}_+ \times C(\mathbf{R}_+, C([-T, 0], C(M))) \rightarrow C(\mathbf{R}_+, C([-T, 0], C(M)))$ by $\tilde{S}(t)U$ is equal to $s \mapsto U(t+s)$. Our definition of \mathcal{X} guarantees that $\tilde{S}(t)(\mathcal{X}) \subseteq \mathcal{X}$ for $t \in \mathbf{R}_+$, and we write $S(t)$ for $\tilde{S}(t)|_{\mathcal{X}}$. Note S is a semi-flow on \mathcal{X} . One observes that S is a contraction semi-group on \mathcal{X} which implies the joint continuity of $(t, U) \mapsto S(t)U$. \square

Remark 6.3. It follows from Remark 5.3. (ii) and (iii) that there exists a ball $\mathcal{B} \subset C([-T, 0], C(M))$ such that for every $U \in \mathcal{X}$ there exists a finite time $\tau \in (0, \infty)$ with $S(t)U \in \mathcal{B}$ for $t > \tau$, i.e., S is point dissipative. Actually, τ depends only on $\|u_0\|_\infty$.

Since memory terms are involved, we cannot expect $S(t)$ to be completely continuous for $t > 0$, but we have

Lemma 6.4. *Let (H0)–(H2), (H4), (H6), (H7) be satisfied. Then $S(t)$ is completely continuous for $t > T$.*

Proof. Let $(U_j)_{j \in \mathbf{N}} \in \mathcal{X}^{\mathbf{N}}$ be d -bounded. Then (u_j) is uniformly bounded by Remark 6.3. By definition of \mathcal{X} , one finds $(Q_j) \in \mathcal{Q}^{\mathbf{N}}$ such that u_j is an L^2 -mild solution of (13) with $F(t, x, y, z) = Q_j(t, x)F_0(x, y, z)$ for $(j, t, x, y, z) \in \mathbf{N} \times \mathbf{R} \times M \times \mathbf{R} \times \mathbf{R}$. Since Q is almost periodic, one has that $\|Q_j\| = \|Q\|_\infty$ for $j \in \mathbf{N}$. We consider the case $p > 2$. Let $t > T$. Then Remark 5.4 and the embedding theorem show that there exists a $\sigma \in (0, 1 - \frac{2}{p})$ and an $\underline{t} \in [0, t - T)$ such that $u_j(\underline{t}) \in C^\sigma(M)$ for every $j \in \mathbf{N}$. Set $G := \{u_j : j \in \mathbf{N}\}$. Corollary 5.6 implies that $\{g|_{[\underline{t}, b]} : g \in G\}$ is equicontinuous from $[\underline{t}, b]$ into $C(M)$ for every $b > \underline{t}$. Also, it implies that $\{g(\tau) : g \in G\}$ is bounded in $C^\sigma(M)$ for $\tau \in [\underline{t}, \infty)$, hence relatively compact in $C(M)$. Ascoli’s theorem therefore yields G to be relatively compact in the topology of uniform convergence on compacta on $[\underline{t}, \infty)$, hence we can find a subsequence (u_{l_j}) of (u_j) which converges uniformly on compacta of $[\underline{t}, \infty]$ to some $u_\infty \in C([\underline{t}, \infty), C(M))$, which is bounded, since G is uniformly bounded. The sequence $(S(t)(U_{l_j}))$ therefore converges for each $\tilde{b} > t$ uniformly on intervals $[t, \tilde{b}]$ in $C([-T, 0], C(M))$ to $s \mapsto u_\infty(t + \tau + s)$ for $s \in [-T, 0]$ and $\tau \in [t, \tilde{b}]$, hence converges w.r.t. d . As for $p = 2$, one replaces Corollary 5.6 by the corresponding “linear result”

mentioned thereafter and follows our reasoning, or “translates” Theorem 1.8 in [41] into the language of united trajectory spaces. \square

Now, we can appeal to Theorem 3.4.8 in [20] and conclude.

Theorem 6.5. *Let (H0)–(H2), (H4), (H6), (H7) be satisfied. Then S has a global compact attractor, the so-called trajectory attractor of (13).*

Remark 6.6. The traditional approach to guarantee that the attractor of a semi-flow is connected, is to establish that the underlying space, here \mathcal{X} , is path-connected [32, Lemma 23.6 and footnote]. However, this may not be true for \mathcal{X} , if (13) is not uniquely solvable for all initial conditions. We refer to Remark 7.6.5 in [12] which indicates the problem in case of a 2×2 ode-system.

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