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ON SOME DELAYED NONLINEAR PARABOLIC EQUATIONS MODELING CO OXIDATION

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Abstract. It is well known that several features of many reaction-diffusion systems can be studied through an associated Complex Ginzburg-Landau Equation (CGLE). In particular, the study of the catalytic CO oxidation leads to the *Krischer-Eiswirth-Ertl* model, a nonlinear parabolic system of three equations, which can be controlled by a delayed feedback term. For the control of the uniform oscillations of the process we had already studied the correspondig delayed CGLE, developing first a *pseudolinearization* principle, of a very broad applicability, which led us to a range of parameters in which the stability of these uniform oscillations takes place.In this work we first present some numerical simulations which corfirm the mentioned range of parameters, and gives us also ranges of parameters for some other different behavior. Out of the setting of the CGLE, the dynamics of the process is much richer, so we also present another method for the study of the existence, uniqueness (based on monotony methods) stability (again with the *pseudolinearization* principle) and even approximation, directly for the mentioned parabolic system.

Keywords. Linearization and Pseudolinearization principles, Nonlinear Parabolic Systems, Delayed Partial Differential Equations, Complex Ginzburg-Landau Equation, Complexity.

1 Introduction

The spatiotemporal dynamics of the catalytic CO oxidation reaction on a Pt(110) surface under low-pressure conditions (see [11] and references therein) can be described by a set of parabolic partial differential equations, the

Krischer-Eiswirth-Ertl model (see [13])

$$(KEE)\left\{\begin{array}{l} u_t - D\Delta u \ = -k_2 u \ + k_1 s_{CO} p_{CO}(1-u^3) - k_3 uv, \\ v_t = -k_3 uv \ + k_4 p_{O_2}[s_{O,1\times 1} w + s_{O,1\times 2}(1-w)](1-u-v)^2, \\ w_t = k_5\left(\frac{1}{1+\exp(\frac{u_0-u}{\delta u})} - w\right), \end{array}\right. \tag{1}$$

where the variables u and v represent the CO and oxygen coverages (adsorbate concentration) and the variable w is a measure of the reconstruction of the catalytic surface. The control parameters of the systems are the temperature (assumed to be constant here) and the partial pressures of CO and O_2 in the reaction chamber.

The spatial domain is given by $\Omega = (0, L_1) \times (0, L_2)$. We define the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}, \ j = 1, 2, \tag{2}$$

on which we assume periodic boundary conditions.

The dynamics of the KEE model is very rich: in the system without diffusion fixed points, limit cycles and several types of bifurcations (also of higher codimension), transition to chaotic behavior, can be found. For the spatio-temporal system, uniform oscillations, standing and traveling waves (fronts, pulses, spirals, solitary waves), spatio-temporal chaos can be observed (for references see, e.g., [4]).

When trying to control the instabilities or the transition to chaos one can use several methods. One of the most suitable is based on implementing a delayed feedback loop, as proposed by Pyragas [15]. It is known as the time-delay autosynchronization method (*TDAS*) which consists of applying a feedback signal to the system that is proportional to the difference between the delayed value of a variable of the system and its instantaneous value. If the chosen variable is space-dependent (as, e.g., u), the feedback represents a local control. An interesting modification is when the applied feedback signal is based on a global variable (as, e.g., the spatial average of u). Then, the feedback signal is proportional to the difference between the averages of the delayed and instantaneous values of the variable. This is a global control, easy to implement in experiments. For the CO oxidation, it can be conveniently realized (in experiments and modeling) if the control signal acts on the partial pressure of CO in the following way:

$$p_{CO}(u, t, \tau) := p_{CO}^0 + \mu(\overline{u}(t-\tau) - \overline{u}(t)) \tag{3}$$

or, more in general,

$$p_{CO}(u, t, \tau) := p_{CO}^0 + \mu\left[m_1 u(t, x) + m_2 \overline{u}(t) + m_3 u(t-\tau, x) + m_4 \overline{u}(t-\tau)\right], \tag{4}$$

with

$$\overline{u}(s) = \frac{1}{|\Omega|}\int_\Omega u(s, x)dx \tag{5}$$

to include all possibilities.

Close to the onset of harmonic oscillations, the equations of the system can be reduced to the complex Ginzburg-Landau equation, which represents the normal form of a supercritical Hopf bifurcation for a distributed system with diffusive coupling. For a given reaction-diffusion system close to a Hopf bifurcation, the parameters of the corresponding complex Ginzburg-Landau equation can be calculated following the approach described in [12].

Assuming that we are close to the Hopf bifurcation, we can therefore use the CGLE instead of the KEE model. Then, the system to be studied is given by

$$\frac{\partial \mathbf{u}}{\partial t} = (1 - \mathrm{i}\omega)\mathbf{u} - (1 + \mathrm{i}\alpha)|\mathbf{u}|^2\mathbf{u} + (1 + \mathrm{i}\beta)\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{F}(\mathbf{u}, t, \tau), \tag{6}$$

where $\mathbf{u}(x, t)$ is the complex oscillation amplitude. We will consider a time-delayed feedback term F

$$\mathbf{F}(\mathbf{u}, t, \tau) = \mu e^{\mathrm{i}\xi} \left[m_1 \mathbf{u}(x, t) + m_2 \overline{\mathbf{u}}(t) + m_3 \mathbf{u}(x, t - \tau) + m_4 \overline{\mathbf{u}}(t - \tau) \right], \tag{7}$$

which has global and local contributions since

$$\overline{\mathbf{u}}(t) = \frac{1}{L} \int_L \mathbf{u}(x, t)\, \mathrm{d}x \tag{8}$$

denotes the spatial average of the complex variable $\mathbf{u}(x, t)$.

2 Simulations

We are interested in the stabilization of uniform oscillations in a parameter range where such oscillations are unstable without any feedback $\mu = 0$. To be precise, the parameters α and β fulfill the Benjamin-Feir criterion for instability with respect to phase perturbations $1 + \alpha\beta < 0$. In fact, we are in the regime of amplitude turbulence (spatio-temporal chaos). So, we will present some numerical simulations on the above problem, giving ranges for the parameters in which this stabilization takes place and we will show later that these results can be obtained rigorously by constructing a general pseudolinearization principle. It turns out that this principle has a much wider applicability.

2.1 Method

For time integration, we use an explicit Euler scheme with $\delta t = 0.002$. The Laplacian operator is discretized using a next-neighbor representation. The system size of the one-dimensional medium is $L = 128$ with a spatial resolution of $\delta x = 0.32$. We apply periodic boundary conditions and the initial conditions consist either of slightly perturbed uniform oscillations or developed spatio-temporal chaos. The overall simulation time is $\Delta t = 700$ (usually

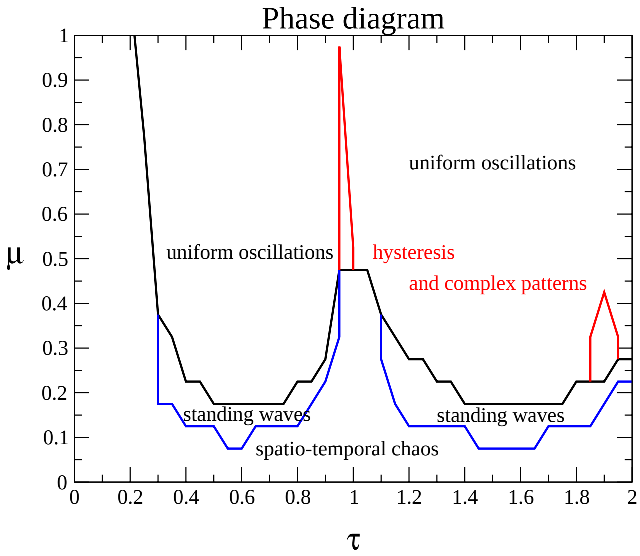

Figure 1: Ranges of parameters for stability

the systems reaches the stable asymptotic state before $\Delta t = 200$). To obtain the phase diagram, τ was changed in steps of 0.05 from 0.05 to 2.0 and μ in steps of 0.05 from 0.05 to 1.0.

2.2 Results

We investigated the cases (a): $m_1 = -1, m_2 = 0, m_3 = 1, m_4 = 0$, (b): $m_1 = -1, m_2 = -1, m_3 = 1, m_4 = 1$, (c): $m_1 = -0.7, m_2 = -0.3, m_3 = 0.7, m_4 = 0.3$, and (d): $m_1 = -1.8, m_2 = -0.2, m_3 = 1.8, m_4 = 0.2$.

Among the local and global terms within the feedback, the idea of time-delayed autosynchronization is applied, i.e., $m_1 = -m_3$ and $m_2 = -m_4$.

In the case of purely local feedback (a), we find no stabilization of uniform oscillations, but the formation of traveling waves. This has been discussed before for a slightly different setting [6].

If we apply local and global feedbacks with the same strength (b), global feedback is clearly dominant and the solutions diagram, shown in Figure 1,is very similar to the one for the strictly global case discussed in [5].

Even if the contribution of the local feedback terms is larger than the contribution of the global terms (c), we qualitatively observe a very similar picture that we do not present here through a figure.

Simulations for the case (d) with a strong local feedback component show that the region where uniform oscillations are stabilized shrinks and that the regions where stationary waves are found become larger. Nevertheless, the overall effect is that the region where spatio-temporal chaos persists becomes larger. Therefore, we can conclude that local feedback is less suited to induce regular patterns than global feedback.

3 Uniform oscillations for the GL equation

We consider the previous problem in which the domain is $\Omega = (0, L_1)\times(0, L_2)$ with periodic boundary conditions. We define the faces of the boundary and the problem as follows

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}, j = 1, 2,$$

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - (1 + i\epsilon)\Delta\mathbf{u} = (1 - i\omega)\mathbf{u} - \\ \qquad (1 + i\beta)\left|\mathbf{u}\right|^{\mathbf{2}} \mathbf{u} + \mu e^{i\chi_0}\mathbf{F}(\mathbf{u}, t, \tau) & \Omega \times (0, +\infty), \\ \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+2}}, \left.\frac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_{j+2}}, & \partial\Omega \times (0, +\infty), \\ \mathbf{u}(x,s) = \mathbf{u}_0(x, s) & \Omega \times [-\tau, 0], \end{cases} \tag{9}$$

where $\overrightarrow{n}$ is the outpointing normal unit vector, and

$$\mathbf{F}(\mathbf{u}, t, \tau) = \left[m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + m_3\mathbf{u}(t - \tau, x) + m_4\overline{\mathbf{u}}(t - \tau)\right],$$

$$\overline{\mathbf{u}}(s) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}(s, x)dx.$$

Here the parameters $\epsilon, \beta, \omega, \mu, \chi_0, m_i$ and τ are real numbers, in contrast with the solution $\mathbf{u}(x, t)=u_1(x, t) + iu_2(x, t)$. Coefficient ε measures the degree to which the diffusion matrix $\mathbf{D}$ deviates from a scalar.

We focus our attention on the so called *slowly varying complex amplitudes* defined by $\mathbf{u}(x, t) = \mathbf{v}(x, t)e^{-i\omega t}$.

In order to avoid the application of techniques for the study of the stability of periodic solutions we can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x, t) = \mathbf{v}(x, t)e^{i\theta t}$ Thus $\mathbf{z}(x, t)$ satisfies

$$\begin{cases} \frac{\partial \mathbf{z}}{\partial t} - (1 + i\epsilon)\Delta\mathbf{z} = (1 + i\theta)\mathbf{z} - (1 + i\beta)\left|\mathbf{z}\right|^2 \mathbf{z} + \\ \qquad \mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\overline{\mathbf{z}} + \right. & \text{in } \Omega \times (0, +\infty), \\ \qquad \left. e^{i(\omega+\theta)\tau}(m_3\mathbf{z}(t - \tau, x) + m_4\overline{\mathbf{z}}(t - \tau))\right] \\ \mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, \left.\frac{\partial \mathbf{z}}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial \mathbf{z}}{\partial x_j}\right|_{\Gamma_{j+2}}, & \text{on } \partial\Omega \times (0, +\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x, s)e^{i(\omega-\theta)s} & \text{on } \Omega \times [-\tau, 0]. \end{cases} \tag{10}$$

Notice that now, $\mathbf{v}_{uosc}(x, t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x, t) = \mathbf{v}_{uosc}(x, t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of (10): i.e.

$$\mathbf{0} = (1 + i\theta)\mathbf{z}_\infty - (1 + i\beta)\left|\mathbf{z}_\infty\right|^2 \mathbf{z}_\infty + \mu e^{i\chi_0}\left[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3 + m_4)\right]\mathbf{z}_\infty. \tag{11}$$

We shall assume that

$$m_1+m_2 = 0 \text{ and } m_3+m_4 = 1. \tag{12}$$

Then we get the expressions $\rho_0(\tau) = (1 + \mu \cos \chi(\tau))^{1/2}$, where $\chi(\tau) = \chi_0 + (\omega + \theta(\tau))\tau$ and with $\theta(\tau)$ given as the solution of the implicit equation

$$\theta = \beta - \mu(\sin(\chi_0 + (\omega + \theta)\,\tau) - \beta \cos(\chi_0 + (\omega + \theta)\,\tau)). \tag{13}$$

Notice that if $\mu = 0$ we deduce that $\rho_0(\tau) = 1$ and that $\theta(\tau) = \beta$ for any τ and that $\rho_0(0) = (1 + \mu \cos \chi_0)^{1/2}, \theta(0) = \beta - \mu(\sin \chi_0 - \beta \cos \chi_0)$. It is not difficult to prove the existence and uniqueness of such a function $\theta(\tau)$ and that $\theta \in C^1$. Our main result is the following:

Theorem 1 *Assume (12),* $\chi_0 \in (\pi, \frac{3\pi}{2})$,

$$3 - m_1 - 2m_3 \geq 0,\ \ m_1 + m_3 \geq 0,\ \ 3 + 2m_3 > 0, \tag{14}$$

$$\mu > \max\{\frac{3\beta - \omega + 3(\omega + \beta) \sin \chi_0 + \cos \chi_0}{5(-\beta) \sin \chi_0 \cos \chi_0 + 1},$$
$$\frac{m_3(3\beta - \omega - \varepsilon \frac{\pi^2}{L^2}) + 3(\omega + \beta) \sin \chi_0 + (m_1 + m_3) \cos \chi_0}{(3 - m_1 - 2m_3) \sin^2 \chi_0 + (m_1 + m_3) \cos^2 \chi_0 + (-\beta)(3 + 2m_3) \sin \chi_0 \cos \chi_0}\}.$$

Then there exists some $\tau_0 \in (0, 1)$ such that if we assume $\tau \in (\tau_0, 1)$ we get that

$$|\mathbf{v}(x, t) - \rho_0| \leq M e^{-\alpha t} \left\| \mathbf{u}_0(\cdot, \cdot) e^{i\omega\cdot} - \rho_0 \right\|.$$

The double bar norm above is that of the space with memory. The $\cos \chi_0 \neq 0$, and all denominators are controlled and different from zero. The proof can be divided in two parts. In the first one we shall show the applicability of an abstract result, the *pseudolinearization* principle. In a second part we shall check that the above conditions on the data of the problem allows to prove that any eigenvalue λ of the associate linearized problem has $Re(\lambda) < 0$ which implies the result.

3.1 The abstract results. The *pseudolinearization* principle

We study of the stabilization, as $t \to \infty$, of the solutions of the nonlinear abstract functional differential equation

$$\begin{cases} \frac{du}{dt}(t) + Au(t) + Bu(t) \ni F(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \tag{15}$$

on a Banach space X, where

$$u_t(\theta) = u(t + \theta), \theta \in [-\tau, 0],$$

to the associated equilibria: $w \in D(A) \subset D(B) \subset X$ such that

$$Aw + Bw \ni F(\widehat{w}(.)),$$

where $\widehat{w} \in C := C([-\tau, 0] : X)$ is the function which takes constant values equal to w. Our main goal is to extend, to a broad class of nonlinear operators A, the usual linearized stability principle saying, roughly speaking, that for the special case of A linear (single valued) and B and F are differentiable, the asymptotic stability of the zero solution of the linearized equation,

$$\begin{cases} \frac{dv}{dt}(t) + Av(t) + \mathrm{D}B(w)v(t) = \mathrm{D}F(\widehat{w})v_t(.) & \text{in } X, \\ v(s) = u_0(s) & s \in [-\tau, 0], \end{cases}$$

implies that $u(t : u_0) \to w$ as $t \to \infty$, at least if $u_0(.)$ is close enough to $\widehat{w}$.

Our main result need the following structural assumptions

(H1): $A \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with $\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X)$ such that $A + \omega I$ is a m-accretive operator$\}$ (see Brezis [10] for the case of $X = H$ a Hilbert space and the works by Benilan, Crandall, Pazy and others for the case of a general Banach space: see the monographs [3] and [17]),

(H2): the operators semigroup $T(t) : \overline{D_X(A)}^X \to X$, $t \geq 0$, generated by A, is compact

(see Vrabie [17]),

(H3): $B \in \mathcal{A}(0 : X)$, B is single valued, Fréchet differentiable, and B is dominated by A; i.e. $D_X(A) \subset D_X(B)$ and

$$|Bu| \leq k \left|A^0 u\right| + \sigma(|u|) \tag{16}$$

for any $u \in D_X(A)$ and for some $k < 1$ and some continuos function $\sigma : \mathbb{R} \to \mathbb{R}$, where, here and in what follows, $|.|$ denotes the norm in the space X (in contrast with the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $\left|A^0 u\right| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$,

(H4): $F : C \to X$ satisfies a local Lipschitz condition, i.e., for any $R > 0$ there exists $L(R) > 0$ such that

$$|F(\phi) - F(\psi)| \leq L(R) \|\phi - \psi\| \text{ for any } \phi, \psi \in C \text{ and } \|\phi\|, \|\psi\| \leq R. \tag{17}$$

(H5): there exists $\delta^F > 0$ such that $F : B^X_{\delta^F}(\widehat{w}) \to X$ is Fréchet differentiable with the Fréchet derivative $\mathrm{D}F(\widehat{w})$ given by $\mathrm{D}(F(\widehat{w}))\phi = \int_{-\tau}^0 d\eta(\theta)\phi(\theta), \phi \in C$, for $\eta : [-\tau, 0] \to B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where $B^X_{\delta^F}(\widehat{w}) = \left\{\phi \in C; \|\phi - \widehat{x}\| < \delta^F\right\}$, We further assume the main condition of our arguments:

(H6): the operator $y \to Ay + By - \mathrm{D}F(\widehat{w})\,(e^{\omega\cdot}y)$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\mathrm{Re}\,\omega = \gamma < 0$ where $e^{\omega\cdot}v \in C$ is defined by $(e^{\omega\cdot}v)(s) = e^{\omega s}\widehat{v}(s)$, with $\widehat{v}(s) = v$, for any $s \in [-\tau, 0]$ for $v \in X$.

In order to treat the case in which B is differentiable we introduce the conditions

(H7): there exists a Banach space Y and there exists $\delta^B > 0$ such that if $B_{\delta^B}(w) = \left\{z \in D(B);\ |w - z| < \delta^B\right\}$, B is Fréchet differentiable as function from $B_{\delta^B}(w)$ into Y, with the Fréchet derivative $\mathrm{D}B(w)$ locally Lipschitz continuous, and

(H8) the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^*\cdot}y\right)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{Re}\,\omega^* = \gamma^* < 0$.

We can obtain the following

Theorem 2 *Assume (H1)-(H6). Then there exists $\alpha > 0$, $\epsilon > 0$ and $M \geq 1$ such that if $u_0 \in B_\epsilon^X(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ then the solution $u(\cdot : u_0)$ of (15) exists on $[-\tau, +\infty)$ and*

$$|u(t : u_0) - w| \leq Me^{-\alpha t}\,\|u_0 - \widehat{w}\|\,, \text{ for any } t > 0. \tag{18}$$

Moreover, if we also assume (H7), that (H1)-(H5) holds on the space Y and (H8) then there exists $\alpha^* > 0$, $\epsilon^* \in (0, \epsilon]$ and $M^* \geq 1$ such that if $u_0 \in B_{\epsilon^*}^{X\cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ then

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^* e^{-\alpha^* t}(\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y), \text{ for any } t > 0. \tag{19}$$

The proof will be given elsewhere [7]

3.2 The complex Ginzburg-Landau equation

Motivated by the special form of the nonlinear term of the equation in 10 we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$. A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature: see, for instance [1]). The operator A is the generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X and the compactness of the semigroup is consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Concerning the rest of the terms of the equation in 10, we define $B\mathbf{u} = (1+i\beta)\,|\mathbf{u}|^2\,\mathbf{u}$ with $D(B) = \mathbf{L}^{12}(\Omega)$. By using the characterizarion of the semi inner-bracket $[,]$ for the spaces $L^p(\Omega)$ (see, for instance Benilan, Crandall and

Pazy [3]) it is easy to see that $\mathbf{B}$ verifies (H3). Moreover, by the results on the Frechet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [2]) we get that (H7) holds, with $\mathrm{D}B(\mathbf{y})\mathbf{v} = 3(1+i\beta)\left|\mathbf{y}\right|^2\mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. Assumption (H7) does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

The nonlocal term is defined, by

$$\begin{aligned} F(\mathbf{u}_t) &= (1+i\theta)\mathbf{u}(t) + \\ &\quad \mu e^{i\chi_0}\left[m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + e^{i(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\overline{\mathbf{u}}(t-\tau))\right], \end{aligned}$$

is locally Lipschitz continuous and its Frechet derivative is given by

$$\begin{aligned} \mathrm{D}F(\widehat{\mathbf{y}})\mathbf{v}(t) &= -(1+i\theta)\mathbf{v}(t) - \\ &\quad \mu e^{i\chi_0}\left[m_1\mathbf{v}(t) + m_2\overline{\mathbf{v}}(t) - e^{i(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\overline{\mathbf{v}}(t-\tau))\right] \end{aligned}$$

since for any $\phi \in C$, the non-local operator $\phi \to \frac{1}{|\Omega|}\int_\Omega \phi(s)dx$ is linear and we can write $\mathrm{D}F(\widehat{\mathbf{y}})\phi = \int_{-\tau}^0 d\eta(s)\phi(s)$, with

$$\begin{aligned} d\eta(s)\mathbf{v}(s) &= \delta_0(s)(1+i\theta)\mathbf{v}(s) + \\ &\quad \mu e^{i\chi_0}\left[\delta_0(s)(m_1\mathbf{v}(s) + m_2\overline{\mathbf{v}}(s)) + e^{i(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s) + m_4\overline{\mathbf{v}}(s))\right] \end{aligned}$$

for any $\mathbf{v} \in C([-\tau,\infty)\colon \mathbf{L^4}(\Omega))$ and any $s \in [-\tau,\infty)$, where $\delta_0(s), \delta_{-\tau}(s)$ denote the Dirac delta at the points $s = 0$ and $s = -\tau$ respectively. By well-known results, we have that $\eta : [-\tau, 0] \to B(X,X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analougously replacing X by Y). Assumption (H6) can be read as a condition on the stationary state $\mathbf{y}$ (a study of the eigenvalues of operator A can be found, for instance, in Temam [16]), and the analysis of the eigenvalues that can be found in [9], which also gives the estimates (**??**).

4 Stable stationary solutions

Stationary solutions, as many others, can be found for different values of the parameters (see, for instance, [13] and [5]), so the representation of the system in terms of the CGLE equation is no longer valid. Ranges of parameters leading to this framework can be found, for instance, in Krischer, Eiswirth and Ertl [13]. As it is easily shown, our method can be applied to many other alternative assumptions.We have to go back to the Krischer-Eiswirth-Ertl system.

We now prove the existence and stabilization of such solutions of the system (1) (see [13]) where the domain is given again by $\Omega = (0, L_1)\times(0, L_2)$, and the faces of its boundary are defined by (2), on which we assume periodic boundary conditions

$$(BC)\left\{ u|_{\Gamma_j} = u|_{\Gamma_{j+2}},\ \frac{\partial u}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial u}{\partial x_j}\Big|_{\Gamma_{j+2}} \right. \qquad \text{on } \partial\Omega \times (0,T). \tag{20}$$

In (1) we assume, in general, (4), (5).

Let us rewrite the system in the following *synthetic* way:

$$(KEE_s)\begin{cases} u_t - D\Delta u = f^0(u) + f^1(u,\overline{u},\overline{u}(t-\tau)) + f^2(u,v), & \text{in } \Omega\times(0,T),\\ v_t \ = f^2(u,v) + g^3(u,v,w), & \text{in } \Omega\times(0,T),\\ w_t = h^0(u) + h^1(w), & \text{in } \Omega\times(0,T), \end{cases} \tag{21}$$

(the identification is obvious). We make now some assumptions on the monotonicity of the involved functions in (KEE_s) (obviously it is equivalent to make suitable assumptions on the parameters in the formulation (1)).

$$f^0(u) \text{ is a decreasing function,} \tag{22}$$

$$\begin{cases} f^1(u,\overline{u},\overline{u}(t-\tau)) \text{ is decreasing in } u \text{ (for prescribed } \overline{u},\overline{u}(t-\tau)),\\ f^1(u,\overline{u},\overline{u}(t-\tau)) \text{ is decreasing in } \overline{u} \text{ (for prescribed } u,\overline{u}(t-\tau)),\\ f^1(u,\overline{u},\overline{u}(t-\tau)) \text{ is increasing in } \overline{u}(t-\tau) \text{ (for prescribed } u,\overline{u}), \end{cases} \tag{23}$$

$$\begin{cases} f^2(u,v) \text{ is decreasing in } u \text{ (for prescribed } v),\\ f^2(u,v) \text{ is decreasing in } v \text{ (for prescribed } u), \end{cases} \tag{24}$$

$$\begin{cases} g^3(u,v,w) \text{ is decreasing in } u \text{ (for prescribed } v,w),\\ g^3(u,v,w) \text{ is decreasing in } v \text{ (for prescribed } u,w),\\ g^3(u,v,w) \text{ is decreasing in } w \text{ (for prescribed } u,v), \end{cases} \tag{25}$$

$$h^0(u) \text{ is increasing in } u, \tag{26}$$

$$h^1(w) \text{ is decreasing in } w. \tag{27}$$

Following some ideas which seem to come from M. Müller (1926) (see this and many other references at the monograph Pao [14]), we start by introducing the notion of coupled super and subsolution associated to the system (KEE_s).

Definition 1 *Under the above conditions, the pair of vectorial functions* $(\widehat{u},\widehat{v},\widehat{w})$, $(\underline{u},\underline{v},\underline{w})$, *defined on* $(\Omega\times[-\tau,T])\times(\Omega\times(0,T))^2$, *are called super-subsolutions for the system* (KEE_s) *if* $\underline{u}\le\widehat{u}$, $\underline{v}\le\widehat{v}$ *and* $\underline{w}\le\widehat{w}$ *on* $\Omega\times(0,T)$, *they satisfy (in* $\Omega\times(0,T)$*) the set of inequalities*

$$\begin{cases} \widehat{u}_t - D\Delta\widehat{u} \ge f^0(\underline{u}) + f^1(\underline{u},\overline{(\underline{u})},\overline{(\underline{u})}(t-\tau)) + f^2(\underline{u},\underline{v}),\\ \widehat{v}_t \ \ge f^2(\underline{u},\underline{v}) + g^3(\underline{u},\underline{v},\widehat{w}),\\ \widehat{w}_t \ge h^0(\widehat{u}) + h^1(\underline{w}),\\ \underline{u}_t - D\Delta\underline{u} \le f^0(\widehat{u}) + f^1(\widehat{u},\overline{(\widehat{u})},\overline{(\widehat{u})}(t-\tau)) + f^2(\widehat{u},\widehat{v}),\\ \underline{v}_t \ \le f^2(\widehat{u},\widehat{v}) + g^3(\widehat{u},\widehat{v},\underline{w}),\\ \underline{w}_t \le h^0(\underline{u}) + h^1(\widehat{w}), \end{cases}$$

they are initially well ordered

$$\begin{cases} \underline{u}(s,x)\le u_0(s,x)\le\widehat{u}(s,x) & \textit{on } \Omega\times[-\tau,0],\\ \underline{v}(0,x)\le v_0(x)\le\widehat{v}(0,x) & \textit{on } \Omega,\\ \underline{w}(0,x)\le w_0(x)\le\widehat{w}(0,x) & \textit{on } \Omega, \end{cases}$$

and $\widehat{u}, \underline{u}$ *satisfy the boundary conditions* (BC) *on* $[-\tau, T] \times \partial\Omega$.

In the rest of this paragraph we assume that

$$\text{there exists } (\widehat{u}, \widehat{v}, \widehat{w}), (\underline{u}, \underline{v}, \underline{w}) \text{ super-subsolutions of } (KEE_s). \tag{28}$$

The iterative super and subsolution method consists of several steps (after defining the notion of super and subsolutions). The iterative construction of two vectorial sequences $(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n)$, $(\underline{u}_n, \underline{v}_n, \underline{w}_n)$ is carried out by considering the coupled system

$$\begin{cases} \widehat{u}_{nt} - D\Delta \widehat{u}_n = f^0(\underline{u}_{n-1}) + f^1(\underline{u}_{n-1}, \overline{(\underline{u}_{n-1})}, \overline{(\underline{u}_{n-1})}(t-\tau)) \\ \qquad\qquad +f^2(\underline{u}_{n-1}, \underline{v}_{n-1}), \\ \widehat{v}_{nt} \ = f^2(\underline{u}_{n-1}, \underline{v}_{n-1}) + g^3(\underline{u}_{n-1}, \underline{v}_{n-1}, \widehat{w}_{n-1}), \\ \widehat{w}_t = h^0(\widehat{u}_{n-1}) + h^1(\underline{w}_{n-1}), \\ \underline{u}_{nt} - D\Delta \underline{u}_n = f^0(\widehat{u}_{n-1}) + f^1(\widehat{u}_{n-1}, \overline{(\widehat{u}_{n-1})}, \overline{(\widehat{u}_{n-1})}(t-\tau)) \\ \qquad\qquad +f^2(\widehat{u}_{n-1}, \widehat{v}_{n-1}), \\ \underline{v}_{nt} \ = f^2(\widehat{u}_{n-1}, \widehat{v}_{n-1}) + g^3(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \underline{w}_{n-1}), \\ \underline{w}_{nt} = h^0(\underline{u}_{n-1}) + h^1(\widehat{w}_{n-1}), \end{cases}$$

$$\begin{cases} \widehat{u}_n\big|_{\Gamma_j} = \widehat{u}_n\big|_{\Gamma_{j+2}}, \ \dfrac{\partial \widehat{u}_n}{\partial x_j}\Big|_{\Gamma_j} = \dfrac{\partial \widehat{u}_n}{\partial x_j}\Big|_{\Gamma_{j+2}} \\ \underline{u}_n\big|_{\Gamma_j} = \underline{u}_n\big|_{\Gamma_{j+2}}, \ \dfrac{\partial \underline{u}_n}{\partial x_j}\Big|_{\Gamma_j} = \dfrac{\partial \underline{u}_n}{\partial x_j}\Big|_{\Gamma_{j+2}} \end{cases} \quad \text{on } \partial\Omega \times (0,T),$$

$$\begin{cases} \underline{u}_n(x,s) = \underline{u}_{n-1}(x,s) = \widehat{u}_n(x,s) = \widehat{u}_{n-1}(x,s) = u_0(s,x) & \text{on } \Omega \times [-\tau, 0], \\ \underline{v}_n(x,0) = \underline{v}_{n-1}(x,0) = \widehat{v}_n(x,0) = \widehat{v}_{n-1}(x,0) = v_0(x) & \text{on } \Omega, \\ \underline{w}_n(x,0) = \underline{w}_{n-1}(x,0) = \widehat{w}_n(x,0) = \widehat{w}_{n-1}(x,0) = w_0(x) & \text{on } \Omega. \end{cases}$$

The iteration starts, for $n = 1$ by taking as starting functions the super and subsolutions

$$\underline{u}_0(x,t) = \underline{u}(x,t),\ \widehat{u}(x,t) = \widehat{u}_0(x,t),$$
$$\underline{v}_0(x,t) = \underline{v}(x,t),\ \widehat{v}(x,t) = \widehat{v}_0(x,t),$$
$$\underline{w}_0(x,t) = \underline{w}(x,t),\ \widehat{w}(x,t) = \widehat{w}_0(x,t),$$

By well-known results, even under the presence of delay, the system associated to $n = 1$ has a unique solution (and the same for the system associated to $n > 1$ once solved the system for $n - 1$).

Thanks to the maximum principle (and the assumptions on the functions) we get that

$$\underline{u}_0(x,t) \le \underline{u}_{n-1} \le \underline{u}_n \le ...\widehat{u}_n \le \widehat{u}_{n-1} \le \widehat{u}_0,$$
$$\underline{v}_0(x,t) \le \underline{v}_{n-1} \le \underline{v}_n \le ...\widehat{v}_n \le \widehat{v}_{n-1} \le \widehat{v}_0,$$
$$\underline{w}_0(x,t) \le \underline{w}_{n-1} \le \underline{w}_n \le ...\widehat{w}_n \le \widehat{w}_{n-1} \le \widehat{w}_0.$$

The proof of convergence of the above monotone sequences towards the components of the solutions $(\widehat{U}, \widehat{V}, \widehat{W}), (\underline{U}, \underline{V}, \underline{W})$ of the original problem $\{\underline{u}_n\} \to \underline{U}, \{\widehat{u}_n\} \to \widehat{U}, \{\underline{v}_n\} \to \underline{V}, \{\widehat{v}_n\} \to \widehat{V}$, $\{\underline{w}_n\} \to \underline{W}$, $\{\widehat{w}_n\} \to \widehat{W}$ is standard (see, e.g., the treatment made in Pao [14] for a similar system of EDP and ODE equations (Section 8.2), the treatment made of global terms at the equation as $\int_\Omega u(s,x)dx$ (Section 12.9) and the treatment of delayed terms (Section 2.8)).

Notice that since $(\widehat{U},\widehat{V},\widehat{W}),(\underline{U},\underline{V},\underline{W}) \in L^\infty((\Omega\times[-\tau,T])\times(\Omega\times(0,T))^2)$ we can truncate (keeping continuity) the functions $f^0(u)$, $f^1(u,\overline{u},\eta)$, $f^2(u,v)$, $g^3(u,v,w)$, $h^0(u)$ and $h^1(w)$ by constant values for large value of its arguments. In this way, we can assume, without lost of generality that $f^0(u)$, $f^1(u,\overline{u},\eta)$, $f^2(u,v)$, $g^3(u,v,w)$, $h^0(u)$ and $h^1(w)$ are globally Lipschitz functions.

In consequence, it is easy to prove the uniqueness of solutions of (KEE_s), and so $(\widehat{U},\widehat{V},\widehat{W}) \equiv (\underline{U},\underline{V},\underline{W})$.

The stationary solutions of the system must satisfy

$$(KEE_\infty)\begin{cases} -D\Delta u_\infty = f^0(u_\infty) + f^1(u_\infty,\overline{u_\infty},\widehat{\overline{u_\infty}}) + f^2(u_\infty,v_\infty), & \text{in } \Omega,\\ 0 = f^2(u_\infty,v_\infty) + g^3(u_\infty,v_\infty,w_\infty), & \text{in } \Omega,\\ 0 = h^0(u_\infty) + h^1(w_\infty), & \text{in } \Omega, \end{cases}$$

and the boundary conditions

$$(BC_\infty)\left\{ u_\infty|_{\Gamma_j} = u_\infty|_{\Gamma_{j+2}},\ \left.\frac{\partial u_\infty}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial u_\infty}{\partial x_j}\right|_{\Gamma_{j+2}} \quad \text{on } \partial\Omega, \right. \tag{29}$$

where we used the usual notation in the theory of delayed equations $\widehat{h}(x,s) = h(x)$, for any $s \in [-\tau,0]$.

The existence of a nontrivial stationary solution, $(u_\infty,v_\infty,w_\infty)$, and the study of its stability can be carried out, again, by the similar arguments to the quoted ones at the monograph Pao [14] (see the above mentioned sections). A different question concerns the study of the delayed and global terms as tools for the stabilization near a function $(u_\infty,v_\infty,w_\infty)$ which is an unstable solution of the nondelayed system $(\tau = 0)$.

Unstable solutions (of the nondelayed system $(\tau = 0)$) arises for some special values of the parameters (see regions 3, 4, 5, 6 and 7 of the paper [13]). Notice that any uniformly spatial solution (i. e. solution of the system without diffusion) is also a solution of our system.

It is a routine matter to check the the abstract pseudo-linearization theorem can be applied to this framework by taking, for instance, $X = Y = \mathbf{L}^2(\Omega)^3$, $A\mathbf{u}$, with $\mathbf{u} = (u,v,w)$, can be formulated matricially as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \to \begin{pmatrix} -\Delta u - f^0(u) - f^2(u,v) \\ -f^2(u,v) - g^3(u,v,w) \\ -h^0(u) - h^1(w) \end{pmatrix},$$

$B\mathbf{u} \equiv \mathbf{0}$ and

$$F(\mathbf{u}_t) = \begin{pmatrix} f^1(u, \overline{u}, \overline{u}(t-\tau)) \\ 0 \\ 0 \end{pmatrix}.$$

Notice that the assumption that the operator $y \to Ay$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ comes from the fact that due to the truncation argument we have that

$$f^0(u)u+f^2(u,v)u+f^2(u,v)v+g^3(u,v,w)v+h^0(u)w+h^1(w)w \leq \omega(u^2+v^2+w^2)$$

for some real constant $\omega > 0$ large enough.

The fact that τ can be chosen large enough in order to get the right condition on the spectrum of the linearized operator follows the same lines than the study made for the Ginzburg-Landau system once that the great similitudes among the respective operators and boundary conditions.

Remark. By applying the arguments of the work [8] it seems possible to get a faster iteration algorithm.

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