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ON A FREE–BOUNDARY PROBLEM MODELING THE ACTION OF A LIMITER ON A PLASMA

J. I. Díaz and J. F. Padial

Dpto. Matemática Aplicada, Fac. de Matemáticas
U. Complutense de Madrid
Dpto. Matemática Aplicada, E.T.S. de Arquitectura
U. Politécnica de Madrid.
28040 Madrid, SPAIN

Abstract. In this paper we prove the existence of weak solutions for a 2D free–boundary problems arising in the magnetic confinement of a plasma in a Stellarator device which includes the action of a *limiter*. The model can be expressed as an *inverse thin obstacle problem* in which the *limiter* plays the role of a *thin obstacle* for the plasma. The inverse nature of the problem comes from the fact that the associated Grad–Shafranov equation involves some unknown nonlinear terms which must be determined by the current–carrying Stellarator condition.

1. **Introduction.** One of the main difficulties of the magnetically controlled plasma fusion (in axisymmetric geometric devices as Tokamaks or non axisymmetric geometric ones as Stellarators), is to determinate the conditions on the magnetic field and on the current density in order to maintain the plasma far from the camera walls. A way to prevent mechanically this is to introduce a *limiter*: a solid object which determines the boundary of the plasma by playing the role of a *thin obstacle* for it. The influence of limiters on plasma confinement has been investigated, from the experimental view point, in many situations: for the case of the TJ-II Stellarator (CIEMAT, Madrid) some evidences have been found about how they improve the confinement [6].

The magnetic confinement in a Stellarator, when the plasma is assumed to be an ideal fluid and a perfect conductor, can be modeled by using the ideal incompressible MHD systems. With the help of averaging methods and Boozer vacuum coordinate we arrive to a two–dimensional Grad–Shafranov type problem for the average poloidal flux function (see [4], [5] and their references). Here, by first time, we consider the associated theoretical aspects under the presence of a *limiter*. Using a similar approach to the already followed for the equilibrium regime [5] and modeling the action of the *limiter* by the multivalued maximal monotone graph β given by $\beta(r) = 0$ if $r < 0$, $\beta(0) = [0, +\infty)$, our models can be expressed as the following *inverse thin obstacle problem*:

Let Ω be an open bounded and regular set of $\mathbb{R}^2$ and let ω (the *limiter*) be a connected subset of Ω such that $\bar{\omega} \cap \partial\Omega$ is a nonempty connected subset of $\partial\Omega$.

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Given the parameters $\lambda > 0$, $F_v > 0$ and $\gamma < 0$, the problem is to find (u, F), with $u : \Omega \longrightarrow \mathbb{R}$, $F : \mathbb{R} \longrightarrow \mathbb{R}$, such that $F(s) = F_v$ for any $s \leq 0$ and satisfying the nonlocal problem

$$-\Delta u + \beta(u\chi_\omega(x)) \ni a(x) F(u) + \frac{1}{2}\left(F(u)^2\right)' + b(x) p'(u) \quad \text{in } \Omega, \tag{1}$$

$$u = \gamma \qquad \text{on } \partial\Omega, \tag{2}$$

where the scalar function p is such that

$$p(0) = 0, 0 \leq p'(r) \leq \lambda r_+ \text{ and } |p'(r) - p'(s)| \leq L|r-s|^\alpha \text{ for some } L > 0 \text{ and } \alpha \in]0, 1[.$$

Here χ_ω denotes the characteristic function on ω. The function p is related to the pressure of the plasma and, for instance it can be given by the usual constitutive law $p(r) = \frac{\lambda}{2}(r_+)^2$. The coefficients a, b are given functions in $L^\infty(\Omega)$ such that $a < 0$ and $b > 0$ a.e. in Ω.

A characteristic of an ideal Stellarator is that it has zero net current within each flux magnetic surface, but in practice, however, this ideal condition does not hold, and a known current arises in the interior of each magnetic surface (see [3] for a physical modeling and [5] for a mathematical treatment). Using the change of variables introduced in [5], the condition of a nonzero current inside each magnetic surface can be expressed in terms of a family of integrals, involving a given function $j : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$:

$$\int_{\{x:u>s\}} \frac{1}{2}\left(F(u)^2\right)' + b(x) p'(u)\, dx = j(s_+, \|u_+\|_{L^\infty(\Omega)}) \quad \forall\, s \in \left[\inf_\Omega u, \sup_\Omega u\right], \tag{3}$$

where j satisfies

$$j \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^+),\ j(\sigma, \sigma) = 0 \text{ for all } \sigma \geq 0, j'_s \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+) \tag{4}$$
$$\text{and } \eta := \sup\{|j'_s(s, \sigma)| : (s, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+\} < +\infty.$$

The family of integral identities (3) is known as the *current-carrying Stellarator condition*. The function u is the averaged poloidal flux.

Remark 1. Notice that there is a double free–boundary: the boundary of the plasma set $\Omega_p := \{u \geq 0\}$ and the part of it which is in contact with the limiter.

Remark 2. Since $u\chi_\omega$ must belong to the domain of the maximal monotone graph β, we get that $u(x)\chi_\omega(x) \leq 0$ a.e. $x \in \Omega$. So, if $x \in \omega$, then necessarily $u(x) \leq 0$ which implies that x is in the vacuum region or in the free boundary (and never in the plasma region since there u is strictly positive). If $x \in \Omega \setminus \omega$, then $\chi_\omega(x) = 0$ and so $\beta(u(x)\chi_\omega(x)) \in [0, +\infty[$.

The main purpose of this paper is to study the existence of solution for the *inverse thin obstacle problem* $(\mathcal{P})$ defined by (1), (2) and (3). Before stating our main result, we introduce the following useful convex cone

$$V(\Omega) = \{v \in H^1(\Omega) : v_{|\partial\Omega} \leq 0\}\ . \tag{5}$$

Theorem 1. *Suppose that* $\gamma \leq 0$ *and* $\inf_\Omega |a| > 0$. *Then there exist* $\Lambda_1, \Lambda_2 > 0$ *such that if*

$$\lambda\|b\|_{L^\infty(\Omega)} + \eta < \Lambda_1 \quad \text{and} \quad \Lambda_2 < \inf_\Omega |a| F_v$$

there is a couple (u, F), $u \in V(\Omega)$ *and* $F \in W^{1,\infty}(]\inf_\Omega u, \sup_\Omega u[)$ *solution of* $(\mathcal{P})$ *satisfying also that meas*$\{x \in \Omega : \nabla u(x) = 0\} = 0$ *and that* F *is entirely determined by* u.

2. **An equivalent formulation as a non local problem** $(\mathcal{P}_*)$**.** Notice that the obstacle term $\beta(u\chi_\omega)$ doesn't appear in the *Stellarator condition* (3). So, like in [5, Section 3], we have that for any couple (u, F) verifying the *Stellarator condition* it is possible to express F in terms of u. In this way, we reduce the original problem (1), (2) and (3) to the *non-local problem*:

$$(\mathcal{P}_*)\begin{cases} -\Delta u+\beta(u\chi_\omega(x)) \ni a\mathcal{F}_u(x)+p'(u(x))\left[b(x)-b_{*u}(|u>u(x)|)\right] \\ \qquad\qquad + j_s'(u_+(x),u_{+*}(0))\,u_{+*}'(|u>u(x)|) \qquad \text{in } \Omega \\ \qquad\qquad u-\gamma\in H_0^1(\Omega), \end{cases} \tag{6}$$

where now u is the only unknown and function $\mathcal{F}_u:\Omega\to\mathbb{R}$ is defined as follows in terms of u:

$$\mathcal{F}_u(x):=\left[F_v^2-2\int_{|u>0|}^{|u>u_+(x)|}[p(u_*)]'(r)\,b_{*u}(r)\,dr+2\int_{|u>0|}^{|u>u_+(x)|}j_s'(u_{+*}(r),u_{+*}(0))(u_{+*}'(r))^2dr\right]_+^{\frac12}$$

(we send the reader to the paper [5] for the notions of *decreasing rearrangement* u_* of u, the *relative rearrangement* b_{*u} *of* b *with respect to* u, etc., and to the book [10] for a general exposition and a large list of references). The equivalence between problems $(\mathcal{P})$ and $(\mathcal{P}_*)$ can be proved as in [5, Section 3]: We set $\hat{m}=\inf_\Omega u$ and $M=\sup_\Omega u$ (which are justified since $u\in L^\infty(\Omega)$), given $u\in W^{1,\infty}(\Omega)$, we define the function $\mathcal{F}:\mathbb{R}\to\mathbb{R}_+$ by

$$\mathcal{F}(s)=\left[F_v^2-2\int_0^{s_+}p'(r)\,b_{*u}(|u>r|)\,dr+2\int_0^{s_+}j_s'(r,u_{+*}(0))u_{+*}'(|u>r|)dr\right]_+^{\frac12}. \tag{7}$$

On the other hand, if $u\in V(\Omega)$ and $\min\{\mathcal{F}(s):s\in[\hat{m},M]\}>0$, then $\mathcal{F}\in W^{1,\infty}(]\hat{m},M[)$ and the equalities $\frac12\left(\mathcal{F}^2(s)\right)'+p'(s)b_{*u}(|u>s|)=j_s'(s_+,u_{+*}(0))u_{+*}'(|u>s|)$ and $\int_{\{x:u(x)>s\}}\frac12\left(\mathcal{F}(u(x))^2\right)'+b(x)\,p'(u(x))dx=j(s_+,u_{+*}(0))$ for all $s\in[\hat{m},M]$ are satisfied. From the above properties, we have

Theorem 2. *Let* $u\in V(\Omega)$ *such that meas*$\{x\in\Omega:\nabla u(x)=0\}=0$*. Assume* $\hat{m}=\inf_\Omega u\le 0$ *and* $\mathcal{F}_u(x)>0$ *a.e. in* Ω*. Then, if* (u,F) *is a solution of the problem* $(\mathcal{P})$ *given by (1), (2) and (3) such that* $F:[\hat{m},M]\to\mathbb{R}^+$*,* $F\in W^{1,\infty}(]\hat{m},M[)$ *and* $F(s)=F_v$ *for all* $s\le 0$*, then the function* u *is also a solution of the problem* $(\mathcal{P}_*)$ *and necessarily* $F(s)=\mathcal{F}(s)$ *for all* $s\in[\hat{m},M]$ *with* $\mathcal{F}$ *given by (7). Conversely, if* u *is a solution of the problem* $(\mathcal{P}_*)$ *and* $\mathcal{F}$ *is given by (7) then, the couple* $(u,\mathcal{F})$ *is a solution of the problem* $(\mathcal{P})$ *and* $\mathcal{F}\in W^{1,\infty}(]\hat{m},M[)$*.*

The proof the above theorem follows Theorem 2 of [5, Section 3].

Thanks to Theorem 2, we prove the existence of solution of $(\mathcal{P})$ by proving the existence of solution of problem $(\mathcal{P}_*)$.

2.1. **The approximate problem** $(\mathcal{P}_{*\epsilon})$**.** Here and in what follows, we use the notation: $I(v(x),\sigma):=\chi_{[|v>v_+(x)|,|v>0|]}(\sigma),\sigma\in\Omega_*$,

$$F_1(x,v,\hat{b}):=-\int_{\Omega_*}I(v(x),s)[p(v_*)]'(s)\hat{b}(s)\,ds$$

$$F_{\epsilon,2}(x,v):=-\int_{\Omega_*}I(v(x),s)h_\epsilon(v_{+*}'(s))j_s'(v_{+*}(s),v_{+*}(0))ds$$

$$\begin{aligned} F_\epsilon(x,v,\hat{b}) &:= \left[F_v^2 - 2F_1(x,v,\hat{b}) + 2F_{\epsilon,2}(x,v)\right]_+^{\frac{1}{2}} \\ H(v(x),\tilde{b}\,(x)) &:= p'(v(x))[b(x)-\tilde{b}\,(x)] \\ J_\epsilon(v(x)) &:= \xi_\epsilon(v'_{+*}(|v>v_+(x)|))j'_s(v_+(x),v_{+*}(0)) \end{aligned}$$

always for a.e. $x\in\Omega$ and for any function $v\in H_0^1(\Omega)$, $\hat{b}\in L^\infty(\Omega_*)$, $\hat{b}\geq 0$ and $\tilde{b}\in L^\infty(\Omega)$, $\tilde{b}\geq 0$. We used the truncation functions $h_\epsilon(s):=\frac{s^2}{1+\epsilon s^2}$, $\xi_\epsilon(s):=\frac{s}{1+\epsilon|s|}$. We shall adopt the notation $F_2:=F_{0,2}$ and $J:=J_0$ (for $\epsilon=0$) concerning to problem (6), that is, $F_2(x,v):=F_{0,2}(x,v)=\int_{|v>0|}^{|v>v_+(x)|}(v'_{+*})^2(s)j_s(v_{+*}(s),v_{+*}(0))ds$ and $J(v):=J_0(v)=v'_{+*}(|v>v_+(x)|)j_s(v_+(x),v_+*(0))$. Notice that $J_\epsilon(0)=0$ and $H(0,b_{*v})=0$ for all $v\in H_0^1(\Omega)$. Let $\beta_\epsilon(\cdot)$ be a Yosida approximation of $\beta(\cdot)$ given by $\beta_\epsilon(r):=\frac{1}{\epsilon}\left(I-(I+\epsilon\beta)^{-1}\right)(r)=\frac{1}{\epsilon}\max(0,r)$ for all $r\in\mathbb{R}$, and $\epsilon\searrow 0$. (Notice that $\beta_\epsilon(r)\geq r$ for all $r\in\mathbb{R}$).

We introduce the approximate problem $(\mathcal{P}_{*\epsilon})$: Find u^ϵ such that

$$(\mathcal{P}_{*\epsilon})\begin{cases} -\Delta u^\epsilon+\beta_\epsilon(u^\epsilon\chi_\omega)=aF_\epsilon(x,u^\epsilon,b_{*u^\epsilon})+H(u^\epsilon,b_{*u^\epsilon}(|u^\epsilon>u^\epsilon(x)|)) \\ \qquad +J_\epsilon(u^\epsilon(x)) \qquad \text{in } \Omega \\ u^\epsilon-\gamma\in H_0^1(\Omega)\cap W^{2,p}(\Omega)\,;\ \forall\ p\geq 1 \end{cases} \tag{8}$$

To simplify the boundary condition we define $w^\epsilon:=u^\epsilon-\gamma$. In order to prove the existence of w^ϵ we shall find a solution $w_m^\epsilon\in V_m$ of some auxiliary problems $(\mathcal{P}_{*\epsilon,m})$, where V_m is a finite dimensional space such that $V_m\subset V_{m+1}\subset H_0^1(\Omega)$. Later, we shall pass to the limit on m and so we shall find w^ϵ such that $w^\epsilon+\gamma$ is a solution of $(\mathcal{P}_{*\epsilon})$.

2.2. **The Galerkin method. existence of solution for a family of finite dimensional problems** $(\mathcal{P}_{*\epsilon,m})$**.** Consider $(\lambda_k,\varphi_k)_{k\geq 1}$ be the eigenvalues and eigenfunctions associated to $-\Delta$ on Ω with Dirichlet boundary conditions. Let $V_m=\text{span}\{\varphi_1,\ldots,\varphi_m\}$. On V_m, we define $[v,w]:=\sum_{k=1}^m v^k w^k$ where $v=\sum_{k=1}^m v^k\varphi_k$ and $w=\sum_{k=1}^m w^k\varphi_k$. Let $\|v\|_{V_m}:=[v,v]^{\frac{1}{2}}$ the associated norm. For $\gamma\leq 0$ fixed, we consider $T_m^\epsilon:V_m\to V_m$ defined for all $v,\varphi\in V_m$ as

$$\begin{aligned}[T_m^\epsilon v,\varphi] &:= \int_\Omega\nabla v\cdot\nabla\varphi dx+\int_\Omega\beta_\epsilon((v+\gamma)\chi_\omega)\varphi dx-\int_\Omega aF_\epsilon(x,v+\gamma,b_{*(v+\gamma)})\varphi dx \\ &\quad -\int_\Omega H(v+\gamma,b_{*(v+\gamma)}(|v+\gamma>(v+\gamma)(x)|))\varphi dx-\int_\Omega J_\epsilon(v+\gamma)\varphi dx\ . \end{aligned}\tag{9}$$

We shall prove that this operator attains zero for some $w_m^\epsilon\in V_m\setminus\{0\}$. It is clear that if w_m^ϵ satisfies $T_m^\epsilon w_m^\epsilon=0$ in V_m, then w_m^ϵ satisfies the finite dimensional problem $(\mathcal{P}_{*\epsilon,m})$ given by

$$(\mathcal{P}_{*\epsilon,m})\begin{cases} -\Delta(w_m^\epsilon+\gamma)= & P_m[-\beta_\epsilon((w_m^\epsilon+\gamma)\chi_\omega)+aF_\epsilon(x,w_m^\epsilon+\gamma,b_{*(w_m^\epsilon+\gamma)}) \\ & +H(w_m^\epsilon+\gamma,b_{*(w_m^\epsilon+\gamma)}(|w_m^\epsilon+\gamma>(w_m^\epsilon+\gamma)(x)|)) \\ & +J_\epsilon(w_m^\epsilon+\gamma)] \qquad \text{in } \Omega,\ \text{with } w_m^\epsilon\in V_m, \end{cases}$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m. To prove that T_m^ϵ has a zero in $V_m\setminus\{0\}$ we shall use Lemma 4.3 of [7]. We need to check that T_m^ϵ is a *coercive* and *continuous map.* We notice that $[T_m^\epsilon 0,\varphi]=\int_\Omega\beta_\epsilon(\gamma\chi_\omega)\varphi dx-F_v\int_\Omega a\varphi dx\neq 0$ for some φ, provided that $a\not\equiv 0$ ($v\equiv 0$ is not a solution of $(\mathcal{P}_{*\epsilon,m})$).

Theorem 3. *Assume*

$$\lambda_1 - \lambda \operatorname*{osc}_{\Omega} b > 0. \tag{10}$$

Then there exists at least $w_m^\epsilon \in V_m$ *solution of problem* $(\mathcal{P}_{*\epsilon,m})$, *i.e. satisfying*

$$[T_m^\epsilon w_m^\epsilon, \varphi] = 0 \text{ for all } \varphi \in V_m. \tag{11}$$

Proof. We prove the coercivity of T_m^ϵ estimating each term of $[T_m^\epsilon v, v]$. Minoring $\int_\Omega \beta_\epsilon\left((v+\gamma)\chi_\omega\right) v dx$ by zero and obtaining the rests of estimates as in [5, Proposition 1], we show that

$$[T_m^\epsilon v, v] \geq \int_\Omega |\nabla v|^2 dx - \left(\delta + \lambda \operatorname*{osc}_{\Omega} b\right) \int_\Omega v^2 dx - C_{\epsilon\delta} \tag{12}$$

with $C_{\epsilon\delta} = \|a\|^2_{L^\infty(\Omega)} \left[F_v^2 + \frac{2}{\epsilon}\eta|\Omega|\right] \frac{|\Omega|}{4\delta}$ and $\delta > 0$. From Poincaré's inequality and (12) we obtain that $[T_m^\epsilon v, v] \geq (\lambda_1 - \lambda \operatorname*{osc}_{\Omega} b - \delta) \int_\Omega v^2 dx - C_{\epsilon\delta}$ and thus, choosing δ small enough, we obtain the coercivity of operator T_m^ϵ. As in [5, Proposition 2], we obtain that T_m^ϵ is a continuous map ($T_m^\epsilon v$ can be expressed as $T_m^\epsilon v = \sum_{k=1}^m [T_m^\epsilon v, \varphi_k]\varphi_k$ where $\varphi \in V_m$ is an arbitrary function and so it is enough to use the continuity of the different functions appearing in the definition (9)). Then we can apply the Brouwer Fixed Point Theorem (see e.g. [7, Lemma 4.3, p. 55]) and the conclusion follows. □

2.3. **Passing to the limit $m \to \infty$: existence of solution of** $(\mathcal{P}_{*\epsilon})$ **.** Let w_m^ϵ be a solution of $(\mathcal{P}_{*\epsilon,m})$. As in Subsection 4.2 of [5] and thanks to fact that β_ϵ is a monotone function, we can obtain that

$$0 = [T_m^\epsilon w_m^\epsilon, w_m^\epsilon] \geq (\lambda_1 - \lambda \operatorname*{osc}_{\Omega} b - \delta) \int_\Omega |w_m^\epsilon|^2 dx - C_{\epsilon\delta}$$

for any $\delta > 0$ and for some positive constant $C_{\epsilon\delta}$. Thanks to the assumption (10) and choosing δ small enough, from the above inequality, we deduce that

$$\|w_m^\epsilon\|_{L^2(\Omega)} \leq C_\epsilon. \tag{13}$$

Replacing the function v by w_m^ϵ in (12) we deduce that

$$\int_\Omega |\nabla w_m^\epsilon|^2 dx \leq C_\epsilon \tag{14}$$

from (12) and (13) for some $C_\epsilon > 0$ (independent of m). Since $w_m^\epsilon \in V_m$ is a solution of $(\mathcal{P}_{*\epsilon,m})$, multiplying the equation of the problem by $-\Delta\left(w_m^\epsilon + \gamma\right)$ (notice that it is equal to $-\Delta w_m^\epsilon$) and considering the fact that

$$\int_\Omega -\Delta\left(w_m^\epsilon + \gamma\right) \beta_\epsilon\left(\left(w_m^\epsilon + \gamma\right)\chi_\omega\right) \geq 0$$

(see [2]), we obtain that

$$\begin{aligned} \|\Delta w_m^\epsilon\|_{L^2(\Omega)} \leq & \|aF_\epsilon(x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)})\|_{L^2(\Omega)} + \|J_\epsilon(w_m^\epsilon + \gamma)\|_{L^2(\Omega)} \\ & + \|H(w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)}\left(|w_m^\epsilon + \gamma > \left(w_m^\epsilon + \gamma\right)(x)|\right))\|_{L^2(\Omega)} \\ \leq & \|a\|_{L^\infty(\Omega)} \left[F_v^2 + \frac{2\eta}{\epsilon}|\Omega|\right]^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \lambda \operatorname*{osc}_{\Omega} b \|w_m^\epsilon\|_{L^2(\Omega)} + \frac{|\Omega|^{\frac{1}{2}}\eta}{\epsilon} \leq C_\epsilon. \end{aligned} \tag{15}$$

Thus, by standard regularity result, $\{w_m^\epsilon\}_{m\geq 1}$ is uniformly bounded in $W^{2,2}(\Omega)$ with respect to m. Then, there exists a subsequence of $\{w_m^\epsilon\}$, which we also denote by $\{w_m^\epsilon\}$, and a function $w^\epsilon \in W^{2,2}(\Omega)$ such that $w_m^\epsilon \underset{m\to\infty}{\rightharpoonup} w^\epsilon$ weakly in

$W^{2,2}(\Omega)$, and so $w_m^\epsilon \underset{m\to+\infty}{\longrightarrow} w^\epsilon$ strongly in $W^{1,p}(\Omega)$ $\forall p \in [1,+\infty[$ and in $\mathcal{C}(\overline{\Omega})$ (notice that the dimension is two). We define the operator $T^\epsilon : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$[T^\epsilon v, \varphi] := \int_\Omega \nabla v \cdot \nabla \varphi dx + \int_\Omega \beta_\epsilon((v+\gamma)\,\chi_\omega)\varphi dx - \int_\Omega aF_\epsilon(x, v+\gamma, b_{*(v+\gamma)})\varphi dx$$
$$- \int_\Omega H(v+\gamma, b_{*(v+\gamma)}\,(|v+\gamma > (v+\gamma)\,(x)\,|))\varphi dx - \int_\Omega J_\epsilon(v+\gamma)\varphi dx$$

if $v, \varphi \in H_0^1(\Omega)$. Then, our next step is to verify that $T^\epsilon w^\epsilon = 0$ and so, $w^\epsilon + \gamma$ will be a solution of $(\mathcal{P}_{*\epsilon})$. In order to do that, we need to prove that the limit of (11) when m goes to $+\infty$, is exactly $[T^\epsilon w^\epsilon, \varphi]$ for all $\varphi \in H_0^1(\Omega)$, and thus zero.
From [5, Section 4.3], we have (for some subsequence) that

$$F_\epsilon(x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)}) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} F_\epsilon(x, w^\epsilon + \gamma, \hat{b}^\epsilon) \text{ weakly}-* \text{ in } L^\infty(\Omega)$$

where $b_{*(w_m^\epsilon+\gamma)} \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \hat{b}^\epsilon$ weakly–$*$ in $L^\infty(\Omega_*)$. There exists a function $\tilde{b}^\epsilon \in L^\infty(\Omega)$ such that $b_{*(w_m^\epsilon+\gamma)}(|w_m^\epsilon + \gamma > (w_m^\epsilon + \gamma)(\cdot)|) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \tilde{b}^\epsilon$ weakly–$*$ in $L^\infty(\Omega)$. Moreover

$$H(x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)}(|w_m^\epsilon + \gamma > (w_m^\epsilon + \gamma)(x)|)) \underset{m\to\infty}{\rightharpoonup} H(x, w^\epsilon + \gamma, \tilde{b}^\epsilon(x)) \text{ weakly-}* \text{ in } L^\infty(\Omega).$$

By the regularity of β_ϵ, we have that $\beta_\epsilon((w_m^\epsilon + \gamma)\,\chi_\omega) \underset{m\to+\infty}{\longrightarrow} \beta_\epsilon((w^\epsilon + \gamma)\,\chi_\omega)$ strongly in $W^{1,p}(\Omega)$ $\forall p \in [1,+\infty[$ and in $\mathcal{C}(\overline{\Omega})$. Finally, we have

$$J_\epsilon(w_m^\epsilon + \gamma) \to J_\epsilon(w^\epsilon + \gamma) \text{ strongly in } L^1(\Omega).$$

Then, if we set $u^\epsilon := (w^\epsilon + \gamma)$, u^ϵ verifies the weak formulation of the problem

$$(\tilde{\mathcal{P}}_{*\epsilon}) \begin{cases} -\Delta u^\epsilon + \beta_\epsilon(u^\epsilon \chi_\omega) = aF_\epsilon(x, u^\epsilon, \hat{b}^\epsilon) + p'(u^\epsilon)[b - \tilde{b}_\epsilon] + J_\epsilon(u^\epsilon) & \text{in } \Omega \\ u^\epsilon - \gamma \in H_0^1(\Omega) \cap W^{2,2}(\Omega) \qquad (\ \subset \mathcal{C}(\overline{\Omega})\ ) & \end{cases}$$

for any $\epsilon > 0$. To obtain a solution of $(\mathcal{P}_{*\epsilon})$, we need to identify $F_\epsilon(x, u^\epsilon, \hat{b}^\epsilon)$ as $F_\epsilon(x, u^\epsilon, b_{*u^\epsilon})$ and $\tilde{b}^\epsilon$ as $b_{*u^\epsilon}(|u^\epsilon > u^\epsilon(x)|)$.

Proposition 1. *Let w^ϵ be a solution of problem $(\tilde{\mathcal{P}}_{*\epsilon})$. If meas$\{x \in \Omega : \nabla u^\epsilon(x) = 0\} = 0$ then $\hat{b}^\epsilon = b_{*u^\epsilon}$ in Ω_* and $\tilde{b}^\epsilon\,(x) = b_{*u^\epsilon}(|u^\epsilon > u^\epsilon(x)|)$ in Ω. In particular, u^ϵ is a solution of $(\mathcal{P}_{*\epsilon})$.*

Proof. Use the analyticity of w_m^ϵ and Theorem 1 of [9]. □

Let us start by giving a condition on the data in order to have meas$\{x \in \Omega : \nabla u^\epsilon(x) = 0\} = 0$.

Lemma 1. *Let $\{u^\epsilon\}$ verifying $(\tilde{\mathcal{P}}_{*\epsilon})$ and such that $u^\epsilon - \gamma \in W_0^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then $\Delta u^\epsilon \in L^\infty\,(\Omega)$ and $u^\epsilon \in V\,(\Omega)$.*

Proof. Let $\epsilon > 0$ and let u^ϵ be any solution of $(\mathcal{P}_{*\epsilon})$. The function $F_\epsilon(x, u^\epsilon, \hat{b}^\epsilon)$ is bounded in Ω (the integral F_1 is positive from $b > 0$ and $p' \geq 0$, and $F_{\epsilon,2}$ is bounded from $|\xi_\epsilon| \leq 1/\epsilon$ and $|j_s'| \leq \eta$ –see (4)). In the same way, $J_\epsilon(u^\epsilon)$ is bounded in Ω and $H(u^\epsilon, \tilde{b}^\epsilon)$ is majored by $\lambda \left\| u_+^\epsilon \right\|_{L^\infty(\Omega)} \text{osc}_\Omega\, b$. The boundedness of $\beta_\epsilon\,(u^\epsilon \chi_\omega)$ comes from the regularity of β_ϵ and the fact that $u^\epsilon \in \mathcal{C}(\overline{\Omega})$. So Δu^ϵ is bounded in Ω. □

The right hand side of equation of problem $(\tilde{\mathcal{P}}_{*\epsilon})$ can be estimated as follows:

$$0 \le F_1\left(x, v, \hat{b}\right) = \int_{|v>0|}^{|v>v_+(x)|} [p(v_*)]'(s)\,\hat{b}(s)\,ds \le \lambda \|\hat{b}\|_{L^\infty(\Omega_*)} \int_{|v>0|}^{|v>v_+(x)|} v_{+*}(s)\,ds$$
$$\le \lambda|\Omega|\|\hat{b}\|_{L^\infty(\Omega_*)}\|v_+\|_{L^\infty(\Omega)} \le \lambda|\Omega|\|b\|_{L^\infty(\Omega_*)}\|v_+\|_{L^\infty(\Omega)} \quad (16)$$

$$|F_{\epsilon,2}(x,v)| = \left|\int_{|v>0|}^{|v>v_+(x)|} h_\epsilon\left(v'_{+*}(s)\right) j'_s\left(v_{+*}(s), v_{+*}(0)\right) ds\right| \le \eta|\Omega| \left\|v'_{+*}(s)\right\|^2_{L^\infty(\Omega_*)} \quad (17)$$

$$0 \le F_\epsilon(x, v, \hat{b}) =\le F_v + (2\eta|\Omega|)^{\frac{1}{2}} \left\|v'_{+*}(s)\right\|_{L^\infty(\Omega_*)} \quad (18)$$

$$|H(v,\tilde{b})| \le \lambda \|v_+\|_{L^\infty(\Omega)} ||b - \tilde{b}||_{L^\infty(\Omega)} \le \lambda \operatorname*{osc}_\Omega b \|v_+\|_{L^\infty(\Omega)} \quad (19)$$

$$|J_\epsilon(v(x))| \le \eta \left\|v'_{+*}(s)\right\|_{L^\infty(\Omega_*)} \ . \quad (20)$$

Lemma 2. *Let* $\{u^\epsilon\}$ *solution of* $(\tilde{\mathcal{P}}_{*\epsilon})$ *such that* $u^\epsilon - \gamma \in W_0^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. *If*

$$\nu := \frac{1}{4\pi}\left(\eta^{\frac{1}{2}}(2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}||a||_{L^\infty(\Omega)} + \eta^{\frac{1}{2}}) + 4\pi|\Omega|\lambda \operatorname*{osc}_\Omega b\right) < 1, \quad (21)$$

then, we have the following estimates uniformly in ϵ*:*

$$||u^{\epsilon\prime}_{+*}||_{L^\infty(\Omega_*)} \le \frac{||a||_{L^\infty(\Omega)}F_v + \lambda \operatorname*{osc}_\Omega b \left\|u^\epsilon_+\right\|_{L^\infty(\Omega)}}{4\pi\left(1 - \nu + |\Omega|\lambda \operatorname{osc}_\Omega b\right)}, \quad (22)$$

$$\|u^\epsilon_+\|_{L^\infty(\Omega)} \le \frac{||a||_{L^\infty(\Omega)}F_v}{4\pi(1-\nu)} := S, \quad (23)$$

$$\textit{and thus}\ \ ||u^{\epsilon\prime}_{+*}||_{L^\infty(\Omega_*)} \le S + \lambda \operatorname*{osc}_\Omega b ||a||_{L^\infty(\Omega)} F_v \ . \quad (24)$$

Proof. To prove this lemma, we shall follow the same steps and arguments than used in the proof of Lemma 4 of [5], but adapting them to our equation (1). In order to simplify the notation we denote by v to u^ϵ in the proof of this lemma. By Lemma 1, $\Delta v \in L^\infty(\Omega)$ and $v \in V(\Omega)$. Then

$$\int_\Omega \Delta v(v_+ - t)_+ dx = \int_{\{v_+>t\}} \Delta v(v_+ - t) dx \qquad \text{for all } t > 0. \quad (25)$$

Since $(v_+ - t)_+ \in H_0^1(\Omega)$, integrating by parts, we have

$$\int_\Omega \Delta v(v_+ - t)_+ dx = -\int_{\{v_+>t\}} |\nabla v_+|^2 dx \ . \quad (26)$$

By classical arguments (see, for instance, [8]), we have

$$\frac{d}{dt}\int_\Omega \Delta v(v_+ - t)_+ dx = -\int_{\{v_+>t\}} \Delta v dx \ . \quad (27)$$

Combining (25), (26) and (27), one has

$$-\frac{d}{dt}\int_{\{v_+>t\}} |\nabla v_+|^2 dx = -\int_{\{v_+>t\}} \Delta v dx. \quad (28)$$

By the assumption that v is a solution of $(\tilde{\mathcal{P}}_{*\epsilon})$, we have that

$$-\int_{\{v_+>t\}} \Delta v dx = -\int_{\{v_+>t\}} \beta_\epsilon (v\chi_\omega)\, dx + \int_{\{v_+>t\}} a\Big[F_v^2 - 2F_1(x,v,\hat{b}^\epsilon) + 2F_{\epsilon,2}(x,v)\Big]_+^{\frac{1}{2}} dx + \int_{\{v_+>t\}} H(v,\tilde{b}^\epsilon) + J_\epsilon(v) dx.$$

Since $\beta_\epsilon (v\chi_\omega) \geq 0$, the last equality gives

$$-\int_{\{v_+>t\}} \Delta v dx \leq \int_{\{v_+>t\}} a\left[F_v^2 - 2F_1(x,v,\hat{b}^\epsilon) + 2F_{\epsilon,2}(x,v)\right]_+^{\frac{1}{2}} dx + \int_{\{v_+>t\}} H(v,\tilde{b}^\epsilon) + J_\epsilon(v) dx$$

and by the boundedness (18) (19) and (20), we get that

$$-\int_{\{v_+>t\}} \Delta v dx \leq ||a||_{L^\infty(\Omega)} \left(F_v + (2\eta|\Omega|)^{\frac{1}{2}} \left\|v'_{+*}(s)\right\|_{L^\infty(\Omega_*)}\right)|v_+>t| + \left(\lambda \operatorname*{osc}_\Omega b \left\|v_+\right\|_{L^\infty(\Omega)} + \eta\left\|v'_{+*}(s)\right\|_{L^\infty(\Omega_*)}\right)|v_+>t|$$

and so

$$-\int_{\{v_+>t\}} \Delta v dx \leq K|v_+>t| \tag{29}$$

with $K = ||a||_{L^\infty(\Omega)}F_v + \lambda \operatorname{osc}_\Omega b \left\|v_+\right\|_{L^\infty(\Omega)} + (||a||_{L^\infty(\Omega)} (2\eta|\Omega|)^{\frac{1}{2}} + \eta) \left\|v'_{+*}(s)\right\|_{L^\infty(\Omega_*)}$. Arguing as in Talenti [11] and using the De Giorgi isoperimetric inequality, we get from (28) and (29) that

$$4\pi|v_+>t| \leq \left(-\frac{d}{dt}|v_+>t|\right)\left(-\frac{d}{dt}\int_{\{v_+>t\}} |\nabla v_+|^2 dx\right) \leq \left(-\frac{d}{dt}|v_+>t|\right)|v_+>t|K \tag{30}$$

for a.e. $t \in]\inf_\Omega v, \sup_\Omega v[$. Thus $4\pi \leq \left(-\frac{d}{dt}|v_+>t|\right)K$. Now, by standard arguments (see [11, 8]) we obtain

$$-\frac{d}{ds}v_{+*}(s) \leq \frac{K}{4\pi} \qquad \text{a.e. in } \Omega_* . \tag{31}$$

By (31) and definition of K,

$$||v'_{+*}||_{L^\infty(\Omega_*)} \leq \frac{1}{4\pi}||a||_{L^\infty(\Omega)}F_v + \frac{1}{4\pi}\lambda \operatorname*{osc}_\Omega b \left\|v_+\right\|_{L^\infty(\Omega)} + \frac{1}{4\pi}(||a||_{L^\infty(\Omega)} (2\eta|\Omega|)^{\frac{1}{2}} + \eta) \left\|v'_{+*}(s)\right\|_{L^\infty(\Omega_*)},$$

and so

$$||v'_{+*}||_{L^\infty(\Omega_*)} \leq \frac{||a||_{L^\infty(\Omega)}F_v + \lambda \operatorname{osc}_\Omega b \left\|v_+\right\|_{L^\infty(\Omega)}}{4\pi\left(1 - \frac{1}{4\pi}\eta^{\frac{1}{2}}(2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}||a||_{L^\infty(\Omega)} + \eta^{\frac{1}{2}})\right)}.$$

The estimate (22) holds with $\nu = \frac{1}{4\pi}\left(\eta^{\frac{1}{2}}(2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}||a||_{L^\infty(\Omega)} + \eta^{\frac{1}{2}}) + 4\pi|\Omega|\lambda \operatorname{osc}_\Omega b\right)$. Setting $\kappa := \eta^{\frac{1}{2}}(2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}||a||_{L^\infty(\Omega)} + \eta^{\frac{1}{2}})$, $C_1 := \frac{||a||_{L^\infty(\Omega)}F_v}{4\pi-\kappa}$ and $C_2 := \frac{\lambda \operatorname{osc}_\Omega b}{4\pi-\kappa}$ (positive constants independent of ϵ), the last inequality can be written as

$$||v'_{+*}||_{L^\infty(\Omega_*)} \leq C_1 + C_2 \left\|v_+\right\|_{L^\infty(\Omega)} . \tag{32}$$

Since we know already that $v_{+*} \in H^1_{loc}(\Omega_*)$, (31) infers that $v_{+*} \in W^{1,\infty}(\Omega_*)$ and an integration leads to $\|v_+\|_{L^\infty(\Omega)} = v_{+*}(0) \leq \frac{|\Omega|}{4\pi}K$ (since $v_{+*}(|\Omega|) = 0$). Thus

$$\|v_+\|_{L^\infty(\Omega)} \leq \frac{|\Omega|}{4\pi}\left(||a||_{L^\infty(\Omega)}F_v + \lambda \operatorname*{osc}_{\Omega} b \, \|v_+\|_{L^\infty(\Omega)} + \kappa\left(C_1 + C_2\|v_+\|_{L^\infty(\Omega)}\right)\right)$$

from (32). Collecting terms,

$$\|v_+\|_{L^\infty(\Omega)} \leq \frac{||a||_{L^\infty(\Omega)}F_v}{4\pi\left(1 - \frac{1}{4\pi}\left(\eta^{\frac{1}{2}}(2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}||a||_{L^\infty(\Omega)} + \eta^{\frac{1}{2}}) + 4\pi|\Omega|\lambda \operatorname*{osc}_{\Omega} b\right)\right)} := S \ .$$

The assertion (23) holds with $\nu = \frac{1}{4\pi}\left(\eta^{\frac{1}{2}}(2^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}||a||_{L^\infty(\Omega)} + \eta^{\frac{1}{2}}) + 4\pi|\Omega|\lambda \operatorname{osc}_\Omega b\right)$. Finally, (24) comes from (22) and (23). □

Corollary 1. *Let $\{u^\epsilon\}$ solution of $(\tilde{\mathcal{P}}_{*\epsilon})$ as in Lemma 2, then u^ϵ, ∇u^ϵ and Δu^ϵ are uniformly bounded in $L^2(\Omega)$ with respect to ϵ.*

Proof. As we make for obtain (13), (14) and (15); but using the boundeness given in (16), (17), (18), (19), and (20) and Lemma 2. □

In order to get a solution of problem $(\mathcal{P}_{*\epsilon})$, we need to verify the assumption $\text{meas}\{x \in \Omega : \nabla u^\epsilon(x) = 0\} = 0$ (see Proposition 1).

Lemma 3. *If $\lambda\|b\|_{L^\infty(\Omega)}$ and η are small enough, and $\inf_\Omega |a|$ and F_v large enough, that is*

$$\begin{aligned}&\inf|a|\left[F_v^2 - 2\lambda||b||_{L^\infty(\Omega)}|\Omega|S + 2\eta|\Omega|\left(S + \lambda \operatorname*{osc}_{\Omega} b||a||_{L^\infty(\Omega)}F_v\right)^2\right]_+^{\frac{1}{2}} \\ >&\lambda \operatorname*{osc}_{\Omega} bS + \eta\left(S + \lambda \operatorname*{osc}_{\Omega} b||a||_{L^\infty(\Omega)}F_v\right)\end{aligned} \tag{33}$$

then $\text{meas}\{x \in \Omega : \nabla u^\epsilon(x) = 0\} = 0$. *In particular, u^ϵ satisfies problem $(\mathcal{P}_{*\epsilon})$.*

Proof. We argue by contradiction. Suppose that $\text{meas}\{x \in \Omega : \nabla u^\epsilon(x) = 0\} \neq 0$. Then, from the equation $(\tilde{\mathcal{P}}_{*\epsilon})$

$$\beta_\epsilon(u^\epsilon\chi_\omega) = a[F_v^2 - 2F_1(u^\epsilon, \hat{b}^\epsilon) + 2F_{\epsilon,2}(u^\epsilon)]_+^{\frac{1}{2}} + H(u^\epsilon, \tilde{b}^\epsilon) + J_\epsilon(u^\epsilon) \tag{34}$$

a.e. on $\{x \in \Omega : \nabla u^\epsilon(x) = 0\}$. Since $\beta_\epsilon(u^\epsilon\chi_\omega) \geq 0$ for any $x \in \{x \in \Omega : \nabla u^\epsilon(x) = 0\}$

$$0 \leq a[F_v^2 - 2F_1(u^\epsilon, \hat{b}^\epsilon) + 2F_{\epsilon,2}(u^\epsilon)]_+^{\frac{1}{2}} + H(u^\epsilon, \tilde{b}^\epsilon) + J_\epsilon(u^\epsilon).$$

By the assumption that $a < 0$, we get that $\inf|a|[F_v^2 - 2F_1(u^\epsilon, \hat{b}^\epsilon) + 2F_{\epsilon,2}(u^\epsilon)]_+^{\frac{1}{2}} \leq H(u^\epsilon, \tilde{b}^\epsilon) + J_\epsilon(u^\epsilon)$. Using the estimates on $F_1(u^\epsilon, \hat{b}^\epsilon)$, $F_{\epsilon,2}(u^\epsilon)$, $H(u^\epsilon, \tilde{b}^\epsilon)$ and $J_\epsilon(u^\epsilon)$ given in (16), (17), (19) and (20) respectively, we get that

$$\begin{aligned}&\inf|a|\left[F_v^2 - 2\lambda||b||_{L^\infty(\Omega)}|\Omega|S + 2\eta|\Omega|\left(S + \lambda \operatorname*{osc}_{\Omega} b||a||_{L^\infty(\Omega)}F_v\right)^2\right]_+^{\frac{1}{2}} \\ \leq&\lambda \operatorname*{osc}_{\Omega} bS + \eta\left(S + \lambda \operatorname*{osc}_{\Omega} b||a||_{L^\infty(\Omega)}F_v\right)\end{aligned}$$

from the estimates given in Lemma 2. This contradicts the assumption (33). On the other hand, from Proposition 1, u^ϵ satisfies problem $(\mathcal{P}_{*\epsilon})$. □

3. Existence of solution for problem $(\mathcal{P}_*)$ and problem $(\mathcal{P})$.

Theorem 4. *Assume* $\inf_\Omega |a| > 0$, $\gamma \in \mathbb{R}^-$ *and that* $\lambda\|b\|_{L^\infty(\Omega)} + \eta < \Lambda_1$ *and* $\inf_\Omega |a|F_v > \Lambda_2$ *for some suitable positive constants* Λ_1 *and* Λ_2. *Then there exists* u *solution of* $(\mathcal{P}_*)$.

Proof. Our aim is to let $\epsilon \to 0$. By the uniform estimates on $||u^\epsilon||_{L^2(\Omega)}, ||\nabla u^\epsilon||_{L^2(\Omega)}$ and $\|\Delta u^\epsilon\|_{L^2(\Omega)}$ given in Corollary 1 there exists some subsequence of (u^ϵ) (which we will again denote by u^ϵ) and a function $\alpha \in L^2(\Omega)$ such that $\Delta u^\epsilon \underset{\epsilon\to 0}{\rightharpoonup} \alpha$ weakly in $L^2(\Omega)$. By standard regularity, u^ϵ belongs to a bounded set of $W^{2,2}(\Omega)$. Then, we have (for some subsequence) that

$$u^\epsilon \underset{\epsilon\to 0}{\rightharpoonup} u \quad \text{weakly in } W^{2,2}(\Omega) \qquad \text{and} \qquad u^\epsilon \underset{\epsilon\to 0}{\longrightarrow} u \quad \text{strongly in } \mathcal{C}(\bar{\Omega})$$

(notice that the dimension of space is two). In particular, $\alpha = \Delta u$, $\Delta u \in L^2(\Omega)$, $u \in V(\Omega)$. The estimates (22), (23) and (24) of Lemma 2 remain true replacing u^ϵ by u. Moreover $\beta_\epsilon (u^\epsilon \chi_\omega) \rightharpoonup B$ weakly in $L^2(\Omega)$ and as β is maximal monotone we get that $B(x) \in \beta(u\chi_\omega)$ a.e. $x \in \Omega$ (see [1]). Arguing like in Section 2.3, we prove the convergence of equation (8) of the problem $(\mathcal{P}_{*\epsilon})$ term by term to the equation of problem $(\mathcal{P}_*)$. From Lemma 3 we obtain that meas$\{x \in \Omega : \nabla u(x) = 0\} = 0$. and by Proposition 1 we can identify all the terms which appear after to take the limit in $(\mathcal{P}_{*\epsilon})$. In this way, we get the conclusion that u is a solution of $(\mathcal{P}_*)$. □

Proof. By Theorem 4 there exists u solution of $(\mathcal{P}_*)$ such that $u \in V(\Omega)$. Moreover, the assumptions of Theorem 2 are fulfilled and so the couple $(u, \mathcal{F})$ is a solution of $(\mathcal{P})$. □

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E-mail address: ji_diaz@mat.ucm.es
E-mail address: jf.padial@upm.es