FINITE EXTINCTION TIME VIA DELAYED FEEDBACK ACTIONS

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Abstract. We study the "finite extinction phenomenon" for solutions of parabolic reaction-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \Delta u + b(t) f(u(t-\tau, \mathbf{x})) = 0, \quad (t, \mathbf{x}) \in (0, +\infty) \times \Omega,$$

with a delay term $\tau > 0$. Here Ω is an open and bounded set in $\mathbb{R}^N$, $b \geq 0$, f is a continuous function, $u(t,x)$ satisfies a homogeneous Neumann or Dirichlet boundary condition on $(0, +\infty) \times \partial\Omega$ and some functional initial condition $u(s,x) = u_0(s,x)$ on $(-\tau, 0) \times \Omega$ for a given function $u_0 \in C([-\tau, 0] : L^p(\Omega))$, for some $p \in [1, +\infty]$. The "reaction term" $b(t) f(u(t-\tau, x))$ can be understood as a delayed feedback control.

Keywords. Finite extinction time, delayed feedback controls, linear heat equation, quenching in time-delay equations.

AMS (MOS) subject classification: 35R10, 35R35, 35K20.

1 Introduction

In the last years the "finite extinction phenomenon" (there exists $t_e \geq 0$ such that $u(t,x) \equiv 0 \quad \forall t \geq t_e$, and *a.e.* $x \in \Omega$) has been proved for solutions of suitable parabolic reaction-diffusion equations (usually involving some non-Lipschitz nonlinear terms): see, e.g., the presentation made in Chapter 2 of Antontsev, Díaz and Shmarev[1] and its references.

The main goal of this work is to show how the finite extinction phenomenon may be the result of the mere presence of a suitable time-delayed reaction term. More precisely, given an open and bounded set Ω in $\mathbb{R}^N$, we consider the Neumann problem

$$(P_N) \begin{cases} \frac{\partial u}{\partial t} - \Delta u + b(t) f(u(t-\tau, \mathbf{x})) = 0 & (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t, \mathbf{x}) = 0 & (0, +\infty) \times \partial\Omega, \\ u(s, \mathbf{x}) = u_0(s, \mathbf{x}) & (-\tau, 0) \times \Omega, \end{cases}$$

and the Dirichlet problem

$$(P_D) \begin{cases} \frac{\partial u}{\partial t} - \Delta u + b(t) f(u(t-\tau, \mathbf{x})) = 0 & (0, +\infty) \times \Omega, \\ u(t, \mathbf{x}) = 0 & (0, +\infty) \times \partial\Omega, \\ u(s, \mathbf{x}) = u_0(s, \mathbf{x}) & (-\tau, 0) \times \Omega, \end{cases}$$

where we assume that $u_0 \in C([-\tau, 0], L^p(\Omega))$ for some $p \in [1, +\infty]$ and the following structural conditions

$$f \text{ is a continuous function, } f : \mathbb{R} \to \mathbb{R}, \; f(0) = 0, \tag{1}$$

$$b \in L^1_{loc}(0, +\infty), \quad b \geq 0. \tag{2}$$

We recall that if $b(t) \equiv 0$ the "finite extinction phenomenon" cannot hold because of such well-known properties for linear parabolic equations as the unique continuation property or the strong maximum principle. We also mention here that, in the case of zero delay $\tau = 0$, extinction in finite time is typical of equations containing a strong absorption term. For instance, in the case of reaction-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \Delta u + \lambda |u|^{m-1} u = 0 \tag{3}$$

for some $\lambda, m > 0$ it is well-known (see e.g. Antontsev, Díaz and Shmarev[1] and its references) that the finite extinction phenomenon takes place if and only if

$$m \in (0, 1). \tag{4}$$

We point out that a sistematic study about under which non local terms $G(t, u_t)$ (with the usual notation in fuctional equations $u_t(s, \cdot) := u(t+s, \cdot)$ for $s \in [-\tau, 0]$) the solutions of equations of the type

$$\frac{\partial u}{\partial t} - \Delta u + G(t, u_t) + \lambda |u|^{m-1} u = 0,$$

was made in Redheffer and Redlinger[9] but always under condition (4). Our point of view is different since we are interested in the pure memory effects, and no condition of the type (4) will be required here.

Most of our results deal with the linear case $f(s) = \lambda s$ for some $\lambda > 0$. We show that if $b(t)$ becomes extinct after a "small" time $t_b = 2\tau$ (typical of switched controls), $b(t)$ being inactive (i.e. zero) on $[0, \tau]$, then the solution $u(t,x)$ becomes extinct after the finite time t_b.

We also studied there the way in which the solutions behaves near the extinction time and proved that, at least in some cases, extinction takes place globally in Ω in contrast with some well-known results in the literature concerning extinction processes for reaction-diffusion equations as (3) under (4).

For instance, we prove that if we consider the linear heat equation with memory

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \lambda b(t) u(t-\tau, \mathbf{x}) = 0 & (0, +\infty) \times \Omega, \\ u(t, \mathbf{x}) = 0 & (0, +\infty) \times \partial\Omega, \\ u(s, \mathbf{x}) = u_0(s, \mathbf{x}) & (-\tau, 0) \times \Omega, \end{cases}$$

with $\lambda > 0$ and $u_0 \in C([-\tau, 0] : L^p(\Omega))$, then the solution vanishes in finite time once we assume that $b(t)$ becomes extinct after some finite time $t_b = 2\tau$, being also zero on $[0, \tau]$, and satisfies

$$1 = \lambda \int_0^{t_b} b(s) ds. \tag{5}$$

Moreover, if $u_0(s, \mathbf{x}) = \mu(s)\varphi_n(\mathbf{x})$ *a.e.* $\mathbf{x} \in \Omega$ and for any $s \in [-\tau, 0], \mu \in C([-\tau, 0])$, where φ_n is n-th eigenfunction, $n \geq 1$, of the $-\Delta$ operator with homogeneous Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta\varphi_n = \lambda_n \varphi_n & in \quad \Omega, \\ \varphi_n = 0 & on \quad \partial\Omega. \end{cases}.$$

and if we replace (5) by the condition

$$1 = \lambda \int_\tau^{2\tau} b(s) e^{\lambda_n s} ds, \tag{6}$$

then there exists a function $W(t)$ with $W(t) \equiv 0 \quad \forall t \in [2\tau, +\infty)$ such that the solution u of problem (P_D) satisfies $u(t, \mathbf{x}) = W(t)\varphi_n(\mathbf{x})$ a.e. $\mathbf{x} \in \Omega$ and for any $t \in [0, 2\tau]$.

We finish this section by pointing out that problems (P_N) and (P_D) can be understood in the framework of Control Theory (see **Remark 8**).

2 Existence and uniqueness of solutions

Concerning existence and uniqueness we have

Theorem 1. *Let $u_0 \in C([-\tau, 0] : L^p(\Omega))$ for some $p \in [1, +\infty]$. Assume (2) and that one of the following conditions holds:*

$$p = +\infty \text{ and } f \text{ is merely continuous}, \tag{7}$$

or

$$p < +\infty \text{ and } f \text{ is globally Lipschitz continuous}. \tag{8}$$

Then there exists a unique mild solution u which belongs to $C([-\tau, \infty), L^p(\Omega))$ for both (P_N) and (P_D).

Remark 1. The existence part of the above theorem (for instance for the case of (P_D)) can be obtained as in Ha[6] (see the abstract Theorem 2.4.1) by using the well-known results that the operator $A : D(A) \to L^p(\Omega)$ is m-accretive in $L^p(\Omega)$, where $D(A) = \{w \in W_0^{1,p}(\Omega) : \Delta w \in L^p(\Omega)\}$ and $\overline{D(A)} = L^p(\Omega)$ (see, e.g. Bénilan, Crandall and Pazy[3]). The modifications for the case of the Neumann problem are obviuos and we shall not enter into the details here. We recall that, given the abstract problem

$$(AP) \begin{cases} \frac{du}{dt} + Au \ni G(t, u_t, u) & t \in (0, T), \\ u(s) = u_0(s) & s \in (-\tau, 0), \end{cases}$$

where $T > 0$, $A : D(A) \to \mathcal{P}(X)$ is an m-accretive operator on the Banach space X and $G : [0, T) \times C([-\tau, 0] : X) \times X \to X$ (where $u_t(s, \cdot) := u(t + s, \cdot)$ for $s \in [-\tau, 0]$), a function $u : [-\tau, \infty) : X$ is called an *integral solution* of (AP) if $u \in C([-\tau, T] : X)$, $u(s) = u_0(s)$ for any $s \in (-\tau, 0)$ and it satisfies

$$\|u(t) - v\| \leq \|u(s) - v\| + \int_s^t [G(h, u_h, u(h)) - w, u(h) - v]_+ dh$$

for any $[v, w] \in A$ and $s, t \in [0, T]$, where $[.,.]_+$ denotes the semi-inner product on the space X. See Ha[6] for details of the relation of the integral solution with the notions of *classical, strong* and *mild* (also called as *limit)* solutions of (AP).

Notice that under conditions (7) or (8) we can ensure that the function $G : [0, +\infty) \times C([-\tau, 0] : L^p(\Omega)) \times L^p(\Omega) \to L^p(\Omega)$ defined by

$$G(t, \psi, \varphi) = -b(t) f(\psi(\tau, \cdot))$$

for any $(\psi, \varphi) \in C([-\tau, 0] : L^p(\Omega)) \times L^p(\Omega)$ and so

$$G(t, u_t, u) = -b(t) f(u_t(\tau, \cdot)) = -b(t) f(u(t - \tau, \cdot))$$

are well defined. For an existence result in the class of Hölder continuous funcions $C^\alpha(\overline{\Omega})$, instead of the above $L^p(\Omega)$ spaces, see Redheffer and Redlinger[9]. We also mention that once we assume that $A : D(A) \to X$ is a linear m-accretive operator on the Banach space X we know that A generates a semigroup of contractions $S(t) : \overline{D(A)} \to X$ and the nonhomogeneous Cauchy problem

$$(NHCP) \begin{cases} \frac{du}{dt} + Au \ni g(t) & t \in (0, T), \\ u(s) = u_0 & s \in (-\tau, 0), \end{cases}$$

can be solved by the variation of constants formula

$$u(t) = S(t)u_0 + \int_0^t S(t - s) g(s) ds. \tag{9}$$

Remark 2. The above Theorem 1 remains true for other boundary conditions leading to accretive operators. It also can be obtained under more general conditions: cuasilinear operators, other functional expressions of the type $G(t, u_t, u)$, etc., see Vrabie[10], and Casal, Diaz and Vegas[4].

Remark 3. Since the function $b(t)$ which will be used in the next section has the property that $b(t) \equiv 0$ for $t \in [0, \tau]$, one has the following interesting fact: The solutions of the problems (P_N) and (P_D) do not depend on the whole initial history, $u_0(s,x)$, $s \in [-\tau, 0)$, but only on $u_0(0,x)$. In particular, the problems

$$(P_N^*)\begin{cases} \frac{\partial u}{\partial t} - \Delta u + b(t)f(u(t-\tau,\mathbf{x})) = 0 & (0,+\infty)\times\Omega, \\ \frac{\partial u}{\partial n}(t,\mathbf{x}) = 0 & (0,+\infty)\times\partial\Omega, \\ u(0,\mathbf{x}) = U_0(\mathbf{x}) & \Omega, \end{cases}$$

$$(P_D^*)\begin{cases} \frac{\partial u}{\partial t} - \Delta u + b(t)f(u(t-\tau,\mathbf{x})) = 0 & (0,+\infty)\times\Omega, \\ u(t,\mathbf{x}) = 0 & (0,+\infty)\times\partial\Omega, \\ u(0,\mathbf{x}) = U_0(\mathbf{x}) & \Omega, \end{cases}$$

are well posed under the condition $U_0(x) \in L^p(\Omega)$, for some $p \in [1, +\infty]$. Indeed, it is enough to construct the constant backwards extension, $u_0(s,x) = U_0(x)$ for any $s \in [-\tau, 0]$, and then apply Theorem 1.

Remark 4. It is well known (see, e.g. Pao[8] Chapter 1, Theorem 8.1) that if f is nonincreasing the following general comparison principle holds: given $T > 0$, if $\underline{u}$, $\overline{u} \in C([-\tau, T] : L^p(\Omega))$ are sub- and supersolutions of (P_D), i.e. such that

$$\begin{array}{ll} \frac{\partial \overline{u}}{\partial t} - \Delta\overline{u} + b(t)f(\overline{u}(t-\tau,\mathbf{x})) + g(\overline{u}(t,\mathbf{x})) \geq 0, & (0,T)\times\Omega, \\ \overline{u}(t,\mathbf{x}) \geq 0, & (0,T)\times\partial\Omega, \\ \overline{u}(s,\mathbf{x}) \geq u_0(s,\mathbf{x}), & (-\tau,0)\times\Omega, \end{array}$$

(replacing $\geq$ by $\leq$ for the case of $\underline{u}$) then $\underline{u} \leq u \leq \overline{u}$ on $[-\tau, T]\times\Omega$. The same result for the Neumann problem when we replace the boundary inequality by $\frac{\partial \overline{u}}{\partial n}(t,x) \geq 0$ on $(0,T)\times\partial\Omega$.

3 On the finite extinction phenomenon

The main result of this section shows that if $b(t)$ becomes extinct after a finite time $t_b > 0$ then the same happens for the solution of "small enough" bounded initial data.

Theorem 2. *Assume the conditions of Theorem 1 as well as the following additional conditions:*

$$f(r) = \lambda r, \text{for some } \lambda > 0, \tag{10}$$

$$b(t) \equiv 0 \text{ for almost all } t \in [0,\tau]\cup[2\tau,+\infty), \tag{11}$$

$$\int_\tau^{2\tau} b(s)ds = \frac{1}{\lambda}. \tag{12}$$

Then, for any $u_0 \in C([-\tau, 0] : L^p(\Omega))$, for some $p \in [1, +\infty]$, the corresponding solution u of (P_D) or (P_N) satisfies

$$u(t,\mathbf{x}) \equiv 0 \quad \forall t \geq 2\tau, \quad a.e.\ \mathbf{x}\in\Omega.$$

Before presenting the proof of Theorem 2 we must analyze the situation for the simpler case of the ordinary differential equation with memory (or ordinary delay-differential equation)

$$\begin{cases} U'(t) + b(t)f(U(t-\tau)) = 0, \\ U(s) = U_0(s) \quad for \quad s\in(-\tau,0]. \end{cases} \tag{13}$$

We begin by considering the case of constant initial data:

Lemma 1. *Let f be a increasing function and $b(t)$* such that

$$b(t) \equiv 0 \text{ for a.a. } t \in [t_b, +\infty) \text{ for some } t_b \in (0,\tau]. \tag{14}$$

Assume that

$$U_0(s) = K \quad \text{for all } s \in (-\tau, 0],$$

with

$$K = f(K)\int_0^{t_b} b(s)ds \tag{15}$$

Then, the unique solution U of *(13) verifies that*

$$U(t) \equiv 0 \quad \forall t \geq t_b.$$

*Proof of the **Lemma 1**.* The existence and uniqueness of a strong solution to (*13*) can be found in classical books as, for instance, Hale[7]. Integrating on $(0,t)$ for $t \in (0,\tau]$ we get

$$\int_0^t U'(t)dt = U(t_0) - K = -f(K)\int_0^t b(s)ds.$$

But from (15) we get that $U(t_b) = 0$ and as $U'(t) = 0$ for a.e. $t \in [t_b, +\infty)$ we obtain the result.■

Remark 5. If we assume, for instance,

$$f(u) = \lambda\,|u|^{m-1}\,u, \quad \text{for some } m > 0,$$

then we see that (15) leads to different type of conditions according the values of m. Thus, (15) can be equivalently written as

$$K^{-(m-1)} = \lambda\int_0^{t_b} b(s)ds,$$

if $m > 1$,

$$K^{1-m} = \lambda\int_0^{t_b} b(s)ds,$$

if $m \in (0,1)$ and (which is remarkable) it applies for any K if we asume that

$$1 = \lambda\int_0^{t_b} b(s)ds, \tag{16}$$

if $m = 1$.

Remark 6. If $U_0(s)$ is not constant but satisfies the condition

$$U_0(0) = -\int_0^\tau b(s)U_0(s-\tau)ds = \int_{-\tau}^0 b(s+\tau)U_0(s)ds$$

then the conclusion of Theorem 2 holds in the linear case $f(u) = \lambda u$. Notice that the class of such initial data is a linear subspace of $C([-\tau, 0])$ which contains the constant functions.

Proof of Theorem 2. From the equation we see that

$$u_t - \Delta u = -\lambda b(t)u(t-\tau).$$

So, for $t \in [0,\tau]$, since $b(t) = 0$, we can use the semigroup notation and write that $u(t) = S(t)u_0$ in $X = L^p(\Omega)$, where $S(t)$ is the semigroup generated by the operator $-\Delta u$ with the corresponding boundary conditions and with $u_0 = u(.,0)$. On the other hand , if $t \in [\tau.2\tau]$, we can adapt the main idea of the **Lemma 1**. Indeed, by using (9) we obtain

$$\begin{aligned} u(t) &= u(\tau) - \lambda \int_\tau^t S(t-s)b(s)u(s-\tau)ds = \\ &= u(\tau) - \lambda \int_\tau^t S(t-s)b(s)S(s-\tau)u_0 ds = \\ &= u(\tau) - \lambda \left(\int_\tau^t b(s)\right) S(t-\tau)u_0, \end{aligned}$$

where we have used a commutation formula which holds because $S(t)$ is linear. Then, if $t = 2\tau$, from assumption (12) we get that

$$u(2\tau) = u(\tau) - S(\tau)u_0 = 0 \text{ in } X.$$

Finally, since $b(t) = 0$ for $t \in [2\tau, +\infty)$ we conclude that if $t \in [2\tau, +\infty)$ then $u(t) = S(t)0 = 0$.■

Remark 7. The above proof can be applied to many other linear semigroups. In particular it holds for the simpler case of the (13) giving a different answer from that of **Lemma 1**. This proof also shows the way to a great variety of possible generalizations to other retarded equations associated to different linear problems as, e.g. the ones associated to higher order elliptic operators, the Stokes problem, etc. (see, e.g. Vrabie[10] and Bénilan, Crandall and Pazy[3]). Some of these extensions are in the work by Casal, Díaz and Vegas[4] in the framework of a more abstract setting, which can be used in a broader class of models and applications.

Remark 8. If we consider the zero controllability problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = v, & (0,+\infty)\times\Omega, \\ u(t,\mathbf{x}) = 0, & (0,+\infty)\times\partial\Omega, \\ u(0,\mathbf{x}) = U_0(\mathbf{x}), & \Omega, \end{cases}$$

where we want to find a control v such that $u(2\tau,x) = 0$ for a.e. $x \in \Omega$, we can use Theorem 2 to construct

$$v(t,\mathbf{x}) = -b(t)u(t-\tau,\mathbf{x}),$$

and consider the constant backwards extension of U_0 mentioned in **Remark 3**. Since v becomes extinct after a finite time $t_b > 0$ (a typical characteristic of switched controls), the problems (P_N) and (P_D) correspond to switch-type delayed feedback control problems leading to global zero controlabillity at instant t_b for the initial state $u_0(s,x)$.

Our next result investigates the way in which the solution of problem (P_D) reaches identically zero state. We recall that in the case of semilinear equations with a strong absorption term, (3) under (4), it is known that a "dead core" appears giving rise to a free (or moving) boundary defined as the boundary of the support of $u(t,.)$ (see Chapter 3 of Antontsev, Díaz and Shmarev[1] and its references). In fact, under symmetry assumptions on the initial data such a dead core ends being the complete domain Ω except a single point (see Friedman and Herrero[5]). Our next result shows that, for many cases for which the finite-time extinction arises just by addition of a delay term, the decay to zero is spatially uniform on the whole domain Ω.

Theorem 3. *Let $u_0 \in C([-\tau,0] : L^\infty(\Omega))$ be such that*

$$u_0(s,\mathbf{x}) = \mu(s)\varphi_n(\mathbf{x}), \text{ a.e. } \mathbf{x} \in \Omega \tag{17}$$

and for any $s \in [-\tau,0], \mu \in C([-\tau,0])$, where φ_n is n-th eigenfunction, $n \geq 1$, of the $-\Delta$ operator with homogeneous Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta\varphi_n = \lambda_n\varphi_n & in \quad \Omega, \\ \varphi_n = 0 & on \quad \partial\Omega. \end{cases}.$$

Let $f(r) = \lambda r$ for some $\lambda > 0$ and assume b such that

$$b(t) \equiv 0 \text{ for a.e. } t \in [0,\tau] \cup [2\tau, +\infty), \tag{18}$$

with

$$1 = \lambda \int_\tau^{2\tau} b(s)e^{\lambda_n s}ds, \tag{19}$$

Then, there exists a function $W(t)$ with $W(t) \equiv 0 \quad \forall t \in [2\tau, +\infty)$ such that the solution u of problem (P_D) satisfies that

$$u(t,\mathbf{x}) = W(t)\varphi_n(\mathbf{x}) \text{ a.e. } \mathbf{x} \in \Omega \text{ and for any } s \in [0,2\tau].$$

Proof. Consider the function

$$\underline{u}(t,\mathbf{x}) = \varphi_n(\mathbf{x})\,W(t).$$

It is a routine matter to check that

$$\underline{u}_t - \Delta\underline{u} + \lambda b(t)\underline{u}(t-\tau,\mathbf{x})) =$$

$$\varphi_n(\mathbf{x})\left(W'(t) + \lambda_n W(t) + \lambda b(t)W(t-\tau)\right)$$

So, by taking $W(t)$ as solution of the ODE with delay

$$\begin{cases} W'(t) + \lambda_n W(t) + \lambda b(t)W(t-\tau) = 0, \\ W(s) = \mu(s), \quad for \quad s \in (-\tau, 0), \end{cases}.$$

we find that $\underline{u}$ is a solution of (P_D) which must coincide with u by uniqueness of solutions.. It remains to prove that $W(t) \equiv 0 \quad \forall t \in [2\tau, +\infty)$. But this is easy: as in the Proof of Theorem 2, for $t \in [0,\tau]$, since $b(t) = 0$, we have that $u(t) = \mu(0)$. On the other hand, if $t \in [\tau, 2\tau]$, we can adapt again the main idea of the **Lemma 1**. Indeed, we must have

$$\begin{cases} W'(t) + \lambda_n W(t) = -\lambda b(t)\mu(0), \\ W(s) = \mu(0), \end{cases}.$$

and so

$$W(t) = \mu(0)e^{-\lambda_n t}(1 - \lambda \int_\tau^t b(s)e^{\lambda_n s}ds).$$

Using assumption (19) and that $b(t) = 0$ for $t \in [2\tau, +\infty)$ we conclude that $W(t) = 0$ for any $t \in [2\tau, +\infty)$.■

4 Acknowledgements

Research partially supported by project MTM2005-03463 of the DGISGPI (Spain).

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