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MATHEMATICAL MODELS OF HYDROLOGICAL CYCLE:CHANNEL LEVEL AND SURFACE GROUND WATER FLOWS

S. N. Antontsev [1,2], J.I. Díaz [2]

1: Departamento de Matemática
Universidade da Beira Interior
6201-001 Covilhã, Portugal
e-mail: anton@ubi.pt, antontsevsn@mail.ru
2: Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain
e-mail: diaz.racefyn@unsde.es

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Abstract. *This paper is devoted to a mathematical analysis of some general models of mass transport and other coupled physical processes developed in simultaneous flows of surface, soil and ground beginabstract waters. Such models are widely used for forecasting (numerical simulation) of a hydrological cycle for concrete territories. The mathematical models that proved a more realistic approach are obtained by combining several of mathematical models for local processes. The water - exchange models take into account the following factors: water flows in confined and unconfined aquifers, vertical moisture migration with allowing earth surface evaporation, open-channel flow simulated by one-dimensional hydraulic equations,transport of contamination, etc. These models may have different levels of sophistication. We illustrate the type of mathematical singularities which may appear by considering a simple model on the coupling of a surface flow of surface and ground waters with the flow of a line channel or river.*

1 INTRODUCTION

It is well known that one of the central issues of the Mathematical Environment is the study of general mathematical models for the hydrological cycle (MMHC) obtained trough the consideration of mass transport balances with other connected physical processes arising in the coupling of the different type of water flows: surface, soil and ground waters. The water exchange models (MMHC) take into account many different factors as, for instance: water flows in confined and unconfined aquifers, vertical moisture migration with allowing earth surface evaporation, open-channel flow simulated by one-dimensional hydraulic equations, transport of contamination, etc. These models may have different levels of sophistication leading, in any case, to different systems of nonlinear partial differential equations. Some illustration of the hydrological cycle and a possible grid used in numerical simulation are presented in the Figure 1. Some example of the vertical and horizontal sections of the spatial modelling area are presented in the Figures 2 and 3, 4.

The mathematical treatment of such type of models (of a great diversity) started in the second half of the past century (see, for instance, [1, 4, 8, 10], [13]-[24], [26]-[28], [31]-[33] and their references). The study of these mathematical models and their numerical approximation has lead to important theoretical advances in the study of nonlinear partial differential equations (very often of mixed and degenerate type). The mathematical results can be of a very different nature: questions concerning the mathematical well-possedness of the models (such as the existence and uniqueness of some suitable notion of solution), the study of the qualitative properties of such solutions (as, for instance, the asymptotic behavior with respect to time and spatial variables), the stability and continuous dependence with respect to initial data and physical parameters, etc.

In the paper we shall illustrate the type of mathematical singularities which may appear by considering a simple model on the coupling of a surface flow of surface and ground waters with the flow of a line channel or river. After some comments on the modelling, we present some results on the mathematical treatment of this simple model. This type of considerations seem of relevance in the study of propagation of desert.

2 A collection of mathematical models for the hydrological cycle

2.1 Basic equations

To fix ideas, we start by making some comments on the spatial domain in study. We consider a bounded multiply connected region $\Omega \subset \mathbf{R}^2$ of exterior boundary $\Gamma = \partial\Omega = \sum_{i=1}^{n} \Gamma_i$ (see Figure 4). We assume that at the interior of Ω there is a system of channels or rivers described by a set of curves $\Pi = \sum_{i=1}^{l} \Pi_i$. The interior of Ω may also contain some basins or lakes of boundaries given by some closed curves $\Gamma_0 = \sum_{j=1}^{m} \Gamma_{0j}$. We can assume also that some curves Π_i may have some points of intersection with Π (which we denote as $N = \sum_{i,j=1} N_{ij}$), with the boundary Γ (which we denote as $P = \sum_{i=1} P_i$), or with the lakes boundaries Γ_0 (which we denote as $P_0 = \sum_{i=1} P_{0i}$).

The different mathematical models on the hydrological cycle are based on the consideration of some of the following local subsystems: *1. Vertical filtration in a porous ground.* By applying the Darcy law, it is well known that [1, 29, 33] if we denote by ϑ to the volumetric moisture content and by ψ to the pressure of the soil moisture then we arrive to the so called Richards equation

$$\frac{\partial\vartheta(\psi)}{\partial t} = \frac{\partial}{\partial x_3}\left[\ K\left(\psi\right)\left(\frac{\partial\psi}{\partial x_3}+1\right)\right] + f\left(H,\ \vartheta, x_3, x, t\right), \tag{1}$$

where K is the hydraulic conductivity, $f\left(H,\ \vartheta, x_3, x, t\right)$ is a source / absorption term and x_3 is the vertical coordinate direct upward. The nonlinear parabolic equation takes place on the set

$$H(x,t) < x_3 < H_e(x),\ x = (x_1, x_2) \in \Omega \in \mathbf{R}^2,$$

where$H(x,t)$ is the level of the ground water (elevation of the ground free surface) and $H_e(x_1, x_2)$ is the given surface of the earth. A typical constitutive law used to transform the equation in a self-contained equation for ψ is the one given in the following terms

$$\vartheta = \vartheta_s / \left[\ 1 + \left(\ -\frac{\psi}{a}\right)^m\right],\ \ \psi < 0,\ K = K_s\left[\ (\ \vartheta - \vartheta_r) \,/\, (\ \vartheta_s - \vartheta_r)\right]^n,$$

2. Horizontal plane filtration equations for the levels of ground waters. By using the so called Boussinesq and Shchelkachev equations, it is well known ([26, 28, 30, 33]) that if we denote by $H(x,t)$ and $H_1(x,t)$, respectively, to the elevation of the groundwater free surface in the upper layer and the piezometric head in the lower layer, then we arrive to the coupled system of

$$\mu\frac{\partial H}{\partial t} = \operatorname{div}\left(M\nabla H\right) - \frac{k'}{T'}(H - H_1) + f_\Omega,\ x = (x_1, x_2) \in \Omega, t \in (0,T), \tag{2}$$

$$\mu_1\frac{\partial H_1}{\partial t} = \operatorname{div}\left(k_1 T_1 \nabla H_1\right) + \frac{k'}{T'}(H - H_1),\ x \in \Omega, t \in (0,T), \tag{3}$$

where

$$M = k(x)(H - H_1)\,,\ f_\Omega = f_\Omega(H, \vartheta, x, t),$$

μ is the yield coefficient (the deficiency of saturation), μ_1 the storage coefficient, $k,\ k_1$ and k' are the hydraulic conductivity (percolation) coefficients for the corresponding layers (see Figure 3), f_Ω is a source function (see [4, 33]). Here the div operator must be understood only in the spatial variable. The last term in (3) characterize the rate of vertical flow from the upper layer to the one through the semipermeable intermediate layer.

3. Water level flow in open channels. By applying a diffusion wave approximation to the Saint Venant equations, it is well known ([4, 15, 23, 26, 33] and its references) that if we denote by $u(s,t)$ to the water level in the channel stream, by ω to the cross sectional area ($\omega_u = B$ is the width) and to s to the channel curvilinear length variable measured along its axial cross-section, then we arrive to the nonlinear parabolic equation

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial s}\left(\psi\phi\left(\frac{\partial u}{\partial s}\right)\right) - Q + f_\Pi, \; s \in \Pi, t \in (0,T), \tag{4}$$

under the constitutive laws

$$\omega = \omega(s,u), \quad \phi\left(\frac{\partial u}{\partial s}\right) = \left|\frac{\partial u}{\partial s}\right|^{\frac{1}{2}} \operatorname{sign}\left(\frac{\partial u}{\partial s}\right),$$

and

$$Q = \alpha \; u|_\Pi + \alpha_0 \left[M\frac{\partial H}{\partial n}\bigg|_{\Pi_+} - M\frac{\partial H}{\partial n}\bigg|_{\Pi_-}\right],$$

where $\psi(s,u) = C\omega R^{\frac{2}{3}}$ is the discharge modulus, C is the coefficient Chezy, R is the hydraulic radius, f_Π is a source function,

$$[MH_n] = \left(MH_n|_{\Pi_+} + MH_n|_{\Pi_-}\right)$$

is the total filtration inflow of ground water from the right Π_+ and left Π_-banks of the channel, and $H_n = \partial H/\partial n$ is the outer normal derivative to the boundary of Π. We point out that in many other references the Saint Venant equations are applied to other constitutive laws leading to first order hyperbolic equations (see, e.g., [24]).

4. Water level balance in reservoirs. It is well known ([4]) that denoting by $z(t)$ to the level on the boundaries of reservoirs we arrive to the equation

$$\lambda\frac{dz}{dt} = -\oint_{\Gamma_0} M\frac{\partial H}{\partial n} ds - (\psi\phi)_{\Gamma_0}, t \in (0,T). \tag{5}$$

To obtain other models describing the quality of ground and surface water flows we need to coupled the above equation with some other equations expressing the mass transfer between the different chemical components (see, e.g., [25]):

a. Solute transport equation. For instance, in the case of a confined aquifer we must add the diffusion equation

$$\frac{\partial(mC)}{\partial t} = \operatorname{div}(D\nabla C - vC) + \Phi(C,N) + f, \tag{6}$$

where

$$v = -M\nabla H, \quad D = D_0 + \lambda|v| \text{ and } \; m = m_0 + \mu(H - H_p).$$

b. Dynamics of a reactive solid medium. In some cases, there is a chemical reaction modifying the skeleton of the porous medium and we must add then a kinetic equation of the form

$$\frac{\partial N}{\partial t} = \Phi(C,N). \tag{7}$$

c. Solute transport phenomena in open-channels. In many cases, as for instance in rivers we must add a transport equation of the form

$$\frac{\partial(\varpi S)}{\partial t} = \frac{\partial}{\partial s}(D_1\frac{\partial S}{\partial s} - v_1 S) - (q_1 C) + f, \tag{8}$$

under some constitutive law of the type

$$D_1 = D_0^1 + \lambda_1|v_1|,\ v_1 = -\Psi(s,u)|u_s|^{1/2} sign(u_s).$$

2.2 Elaboration of a coupled model: the case of simultaneous surface ground water and open channel flows

Usually the mathematical models for the hydrological cycle take into account several simultaneous processes and the modelling is carried out by coupling some of the above mentioned equations involved in the phenomena under consideration completed with the corresponding initial and boundary conditions.

To illustrate this we consider now, for instance, the interplay process between surface (lake channel) and ground waters (on which, for the sake of simplicity in the formulation, we neglect the unsaturate zone and assume that there is only one nonpressure layer). A simplified representation can be found in the Figures 3 and 4. Then the mathematical model equations collects of the equations (2)-(5) which here we reduce to the following ones:

$$\mu\frac{\partial H}{\partial t} = \mathrm{div}\,(M\nabla H) + f_\Omega,\ x = (x_1, x_2) \in \Omega, t \in (0,T), \tag{9}$$

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial s}\left(\psi\phi\left(\frac{\partial u}{\partial s}\right)\right) - Q + f,\ s \in \Pi, t \in (0,T), \tag{10}$$

$$\lambda\frac{\partial z}{\partial t} = -\oint_{\Gamma_0} M\frac{\partial H}{\partial n}ds - (\psi\phi)_{\Gamma_0}, x \in \Gamma_0,\ t \in (0,T), \tag{11}$$

for the unknown $\mathbf{W}(x,t) = (H(x,t),\ u(s,t), z(t))$.

Obviously, the above system of parabolic partial differential equations must be completed by adding a set of initial and boundary conditions:

$$\mathbf{W}(x,0) = \mathbf{W}_0(x),\ x \in \Omega, \tag{12}$$

$$\sigma_1 M\frac{\partial H}{\partial n} + \sigma_2 H = g, \quad (x,t) \in \Gamma_T = \Gamma\times(0,T), \tag{13}$$

$$\kappa_1\psi(s,u)\phi(\frac{\partial u}{\partial s}) + \kappa_2 u = g, \quad (x,t) \in P_T = P \times (0,T), \tag{14}$$

$$M\frac{\partial H}{\partial n}\bigg|_{\Pi^\pm} = \alpha\,(u - H_\pm) + \alpha_0\,(H_+ - H_-)\,,\ (x,t) \in \Pi_T = \Pi \times (0,T), \tag{15}$$

$$u_i = u_j,\ \sum_{i=1} \psi(s,u_i)\phi(\frac{\partial u_i}{\partial s}) = 0\ \ (s,t) \in N_T = N \times [0,T], \tag{16}$$

$$H(x,t) = z(x,t),\ x \in \Gamma_0,\ t \in (0,T). \tag{17}$$

The complete mathematical model is then described by the equations (9)-(11), (12)-(17) and, as we shall develop in the following Section, its mathematical treatment is far to be obvious. We shall require the analysis of a combined-type of nonlinear partial differential equations which are defined on different sets of space variables: equation (9) is defined in the two dimensional domain Ω, equation (10) on the curve Π and although equation (11) is a time ordinary differential equation its right hand side is given by a nonlocal operator depending on H and $\frac{\partial H}{\partial n}$. Moreover, the parabolic equations may degenerate changing type or order at certain values of the solution that is sought (case of (9)) or/and its derivatives (case of (10)). Finally, notice that all these equations contain numerous physical parameters which may leads to completely different behaviour of its solutions. We point out that the interaction between the different physical processes is given by the coupling source functions included into differential equations, as well as by the common boundary conditions.

Due to the presence of nonlinear terms, the solutions of such equations may exhibit many different behaviors that cannot occur in the (more or less well-known) case of linear models. The list of peculiar effects of this kind includes properties as the finite time of localization (or extinction in finite time), finite speed of propagation of disturbances from the initial data, waiting time effect, etc.

A previous study to the questions of the mathematical well-possedness of the above system (existence, uniqueness and some qualitative properties of solutions) was carried out in [7]. In the following section we shall recall a part of those results, adding some new ones and developing other qualitative properties which explain the mathematical peculiarities of such model.

3 Mathematical treatment of a simplified model coupling the channel level and surface ground water flows

In this section we shall give an idea of the mathematical analysis of the model coupling the channel level and surface ground water flows mentioned before (see Figures 5,6). As a matter of fact, for the sake of the exposition, we shall limit ourselves to the consideration of a simplified case in which the ground is assumed to be homogeneous and isotropic, the impermeable base is assumed to be horizontal

$$M = H, \tag{18}$$

the flow cross-section of the channel is assumed by uniform and with area given by

$$\omega(s,u) = u, \tag{19}$$

and we assume the constitutive relation $\psi(s,u) = |u|^{\alpha}$, where the parameter α is defined by the geometry of channel.

We shall assume also coefficients $\sigma_1 = \kappa_1 = 0$ and the coincidence among the levels of the ground waters on the left and right banks and the level of the channel water. Last condition corresponds, formally, to assume that (15) holds for $\alpha = \infty$ and $\alpha_0/\alpha = 0$.

3.1 Statement of the mathematical problem

3.1.1 System of Equations

Under the above simplifications, the stated equations the model reduce to the system

$$\frac{\partial H}{\partial t} = \nabla\left(H \nabla H\right) + f_{\Omega}, \qquad (x,t) \in \Omega_T^{\Pi} = \Omega^{\Pi} \times (0,T), \;\; \Omega^{\Pi} = \Omega / \Pi, \tag{20}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(|\, u \,|^{\alpha} \left| \frac{\partial u}{\partial s} \right|^{-1/2} \frac{\partial u}{\partial s} \right) + \left[H \frac{\partial H}{\partial n} \right]_{\Pi} + f_{\Pi}, \;\; (s,t) \in \Pi_T, \tag{21}$$

3.1.2 Initial and boundary conditions

Under the above simplifications, the set of initial and boundary conditions takes the form

$$H(x,0) = H_0(x), \qquad u(s,0) = u_0(s), \tag{22}$$

$$H_{+} = H_{-} = u, \qquad (s,t) \in \Pi_T = \Pi \times (0,T), \tag{23}$$

$$H = g, \qquad (x,t) \in \Gamma_T, \;\; u = g, \;\; (x,t) \in P_T \cup N_T. \tag{24}$$

An illustration of the cross-section and planar view of the modelling domain is presented in the Figures 5, 6. We shall assume (for simplicity) that there exist some functions $H_0(x,t)$, $u_0(x,t)$ defined on $\Omega \times (0,T)$ and such that

$$H_0|_{\Gamma_T} = g, \quad u_0|_{\Pi_T} = g; \;\; H_0(x,0) = H_0(x), \, x \in \Omega \text{ and } u_0(s,0) = u_0(s), s \in \Pi, \tag{25}$$

$$|H_0|\,, |u_0|\,, \| |\nabla H_0| + |H_{0t}| \|_{2,\Omega_T}\,, \|u_{0s}\|_{3/2,\Pi_T}\,, \;\; \|u_{0t}\|_{2,\Pi_T} \le C < \infty. \tag{26}$$

We assume also

$$\int_0^T \left(\max_x |f_{\Omega}(x,t)| + \max_s |f_{\Pi}(s,t)| \right) dt \le C < \infty \tag{27}$$

The above conditions can be weakened in some of the results which follow but we are not trying to state the more general statements of our results.

Remark 1 *Notice that due to the boundary coupling given by* (23) *the partial differential for* u *can be understood as a boundary condition (on* Π_T*) for* H*. So that, it is a dynamic boundary condition involving a diffusion term. This type of "boundary conditions" also arises in the study of some systems which appears when coupling the surface Earth temperature with the ones of a deep ocean (see* [20], [21]*).*

Remark 2 *As mentioned before, the equation* (21) *become degenerate on the set of points where* $u = 0$ *and singular on the set of points where* $\frac{\partial u}{\partial s} = 0$*. This type of equations arises, mainly, in the study of suitable non-Newtonian flows (see, e. g.* [3]*). Here, by the contrary, no assumption about the Newtonian type of the fluid is made (for other contexts leading to doubly nonlinear parabolic equations quite similar to equation* (21) *see the paper* [22]*).*

3.2 Existence and uniqueness theorems

Definition 1 *A non negative pair of bounded functions* $(H, u) = \mathbf{W}$ *such that*

$$0 \leq H(x,t), u(s,t) \leq C < \infty \tag{28}$$

and

$$\int_0^T \left(\int_{\Omega^\pm} H| \nabla H |^2 dx + \int_\Pi \left(u^\alpha | u_s |^{\frac{3}{2}} \right) ds \right) dt \leq C < \infty \tag{29}$$

is called a weak solution of the model $(20) - (24)$ *if for every test function* η *such that*

$$\eta \in W_2^{1,1}(\Omega_T) \cap W_{3/2}^{1,1}(\Pi_T), \ \eta = 0, \ (x,t) \in \Gamma_T = \Gamma \times (0,T)$$

and for every $t \in [0,T]$ *the following identity holds*

$$\int_0^t \int_\Omega (-H\eta_t + H\nabla H \cdot \nabla \eta) \, dxdt + \int_\Omega H(x,\tau)\eta(x,\tau)dx|_{\tau=0}^{\tau=t} \tag{30}$$

$$+ \int_0^t \int_\Pi (-u\eta_t + \psi\varphi(u_s)\eta_s) \, dsdt + \int_\Pi u(s,\tau)\eta(s,\tau)ds|_{\tau=0}^{\tau=t}$$

$$= \int_0^t \int_\Omega f_\Omega \eta dxdt + \int_0^t \int_\Pi f_\Pi \eta dsdt.$$

Theorem 1 *Let us assume that* (26), (27) *hold and*

$$0 \leq f_\Omega, f_\Pi, H_0, u_0 \leq C_0 < \infty, 0 < \alpha < \infty. \tag{31}$$

Then the above model has at least one weak solution $\mathbf{W}(x,t) = (H,u)$ *. If we assume that* $\mathbf{W}_t \in L^1$ *then the weak solution is unique. Moreover, if we assume additionally that*

$$0 < \delta \leq u_0, H_0, g \leq C_0 < \infty, \tag{32}$$

$$\int_0^T \left(\max_x |\partial f_\Omega / \partial t| + \max_s |\partial f_\Pi / \partial t| \right) dt \leq C < \infty \tag{33}$$

$$H_{0xt}, H_{0tt} \in L^2(\Omega_T^\pm),\ u_{0s} \in L^\infty(0,T; L^{\frac{3}{2}}(\Pi)), u_{0st} \in L^2(\Pi_T), \tag{34}$$

then there exist a small $T_0 > 0$ *such that this weak solution is unique and the following estimates are valid*

$$\sup_{0 \leq t \leq T_0} \left(\int_{\Omega^\pm} |\nabla H|^2 dx + \int_\Pi |\underset{s}{u}|^{\frac{3}{2}} ds \right) \leq C < \infty \tag{35}$$

$$\int_0^{T_0} \int_{\Omega^\pm} \left({H_t}^2 + |H_{xx}|^2 \right) dxdt + \int_0^T \int_\Pi \left({u_t}^2 + |u_s|^{-1/2} |u_{ss}|^2 \right) dsdt \leq C < \infty. \tag{36}$$

Proof 1 *First we assume that* (32) *holds. Then it follows from the definition of weak solution that* $0 < \delta \leq H, u$*. The weak solution can be constructed, for instance, as the limit of a sequence of Galerkin's approximations. From the assumptions on the domain* Ω *we know that there exists a complete system of the functions* $\Phi_k \in W_2^1(\Omega)$*, with* $\Phi_k(\vec{x})_\Gamma = 0$*, which are dense in* $W_2^1(\Omega)$*. Respectively. we can assume that the set of curves* Π *admits also a complete system of functions* $\Psi_k \in W_{\frac{3}{2}}^1(\Pi)$*, with* $\Psi_k(s)_{\Gamma \cup \Pi} = 0$*, which is dense in* $W_{\frac{3}{2}}^1(\Pi)$*. Moreover, without loss of the generality we can assume that the functions* Φ_k *and* Ψ_k *are orthogonal in* $L^2(\Omega)$ *and* $L^2(\Pi)$ *respectively Then, we can construct a sequence of approximate solutions of the form*

$$\mathbf{W}^N = (H^N, u^N) = \left(\sum_{k=1}^N H_k(t)\Phi_k(\vec{x}) + H_0, \sum_{k=1}^N u_k(t)\Psi_k(s) + u_0 \right), \tag{37}$$

where the functions H_0 *and* u_0 *satisfy the corresponding boundary conditions.*

We substitute last expression into the corresponding partial differential equations, multiply by $\Phi_j^\pm(x)$ *and* $\Psi_j(s)$*, respectively, and integrate over* $\Omega_\pm$ *and* Π*.*

This leads us to a suitable Cauchy problem

$$\frac{d\mathbf{Y}^N}{dt} = \mathbf{F}(t, \mathbf{Y}^N), \quad \mathbf{Y}^N(0) = \mathbf{Y}_0^N, \quad (\mathbf{Y}^N = (H_1, .., H_N, u_1, .., u_N)), \tag{38}$$

for some given smooth (with respect to $\mathbf{Y}$*) vectorial function* $\mathbf{F}(t, \mathbf{Y})$*. Multiplying equations* (38) *by the vector* $\mathbf{Y}^N$ *and summing, we arrive at the estimate* (35)*. To prove the estimates* (36) *we differentiate* (38) *with respect to* t *and multiply by* $d\mathbf{Y}^N/dt$*. Obtained estimates* *permit us to pass to the limit when* $N \to \infty$ *and* $\delta > 0$ *and next to pass to the limit when* $\delta \to 0$.

To prove the uniqueness of weak solutions under the additional information that $\mathbf{W}_t \in L^1$ *we apply the following continuous dependence formula*

$$\begin{aligned} &\int_{\Omega^\pm} [H_1(t,x) - H_2(t,x)]_+ dx + \int_\Pi [u_1(t,s) - u_2(t,s)]_+ ds \\ &\leq \int_{\Omega^\pm} [H_1(0,x) - H_2(0,x)]_+ dx + \int_\Pi [u_1(0,s) - u_2(0,s)]_+ \\ &+ \int_0^t \int_{\Omega^\pm} [f_{1,\Omega}(\tau,x) - f_{2,\Omega}(\tau,x)]_+ dx + \int_\Pi [f_{1,\Pi}(\tau,s) - f_{2,\Pi}(\tau,s)]_+ ds \end{aligned}$$

which holds when we work for two couples of solutions $\mathbf{W}_i(x,t) = (H_i, u_i)$ *associated to two set of data* $H_i(0,x)$, $u_i(0,s)$, $f_{i,\Omega}(t,x)$ *and* $f_{i,\Pi}(t,s)$ *(but satisfying the same Dirichlet boundary conditions* $H_i = g$, $(x,t) \in \Gamma_T$, $u_i = g$, $(s,t) \in \partial\Pi_T$*) for* $i = 1,2$. *This formula is obtained by multiplying the difference of the associate equations* (20) *by a regularized approximation of the function* $sign_+(H_1(t,x) - H_2(t,x))[= 1$ *if* $H_1(t,x) - H_2(t,x) > 0$ *and* $= 0$ *if* $H_1(t,x) - H_2(t,x) \leq 0]$. *Calling, for instance* $p(H_1 - H_2)$ *to this approximation, using the weak formulation (i.e. the integration by parts formula) and using that (thanks to the assumption* (23) *we get that* $p(H_1 - H_2) = p(u_1 - u_2)$ *on* Π_T,*we end by using the equation* (21) *and passing to the limit as in* [22].

Remark 3 *Notice that presentation* (37) *may be used as an approximative solutions if the functions* Φ_j, Ψ_j *may be constructed effectively.*

Remark 4 *The continuous dependence formula implies the comparison principle for* $\mathbf{W}_i$ *in the sense that if* $H(0,x)$, $u(0,s)$, $f_\Omega(t,x)$ *and* $f_\Pi(t,s)$ *are nonnegative in their respective domains, and if we assume* $g \geq 0$ *then the associated solutions satisfy that* $H, u \geq 0$ *(take one of the pair identically zero and apply the formula). We also point out that it seems possible to get this formula without the technical condition* $\mathbf{W}_t \in L^1$ *by using the notion of renormalized solutions (see references in the monograph* [3]*).*

3.3 Splitting with respect to physical process.

For numerical proposes it can be useful to take separately into account the two different processes appearing in the model.

3.3.1 Iterative process for differential equations

We propose here an algorithm which uses the splitting of the initial problem into the two following independent problems: I.*Plane filtration in the domain* $x \in \Omega/\Pi, t \in (0,T)$, $k = 1,2,...$

$$\frac{\partial H^k}{\partial t} = \nabla\left(H^k \nabla H^k\right) + f_\Omega, \quad x \in \Omega/\Pi, \tag{39}$$

$$H^k(x,0) = H_0(x), \quad x \in \Omega \tag{40}$$

$$H^k \frac{\partial H^k}{\partial n} |_{\pm} = \sigma \left(u^{k-1} - H^k{}_{\pm} \right), \quad x \in \Pi \tag{41}$$

$$\left(\sigma_1 H^k \frac{H^k \partial H^k}{\partial n} + \sigma_2 H^k \right) = g, \quad x \in \Gamma = \partial\Omega \tag{42}$$

II. *Level flow in the channel* Π

$$\frac{\partial u^k}{\partial t} = \frac{\partial}{\partial s} \left(| u^k |^{\alpha} | \frac{\partial u^k}{\partial s} |^{-1/2} \frac{\partial u^k}{\partial s} \right) + \left[H^k \frac{\partial H^k}{\partial n} \right]_{\Pi} + f_{\Pi}, x \in \Pi \tag{43}$$

$$\left(\kappa_1 \psi(s, u^k) \phi(\frac{\partial u^k}{\partial s}) + \kappa_2 u^k \right) = g, x \in \Pi \cap \Gamma. \tag{44}$$

$$u^k(x, 0) = u_0(x), \quad , \ x \in \Omega \tag{45}$$

Let us introduce the notation $h^k = H^k - H^{k-1}, \ z^k = u^k - u^{k-1}$.

Theorem 2 *Let us assume the conditions* $(32) - (34))$ *of the existence theorem. Then the "global error iterated energy"* $y_k(t) = ||h^k||^2_{2,\Omega} + ||\nabla h^k||^2_{2,\Omega_T} + ||u^k||^2_{2,\Pi} + ||u^k_s||^{3/2}_{3/2,\Pi_T}$ *satisfies the estimate*

$$y_k(t) \leq \frac{(Ct)^{k-1}}{(k-1)!} y_0(t) \to 0, \ \textit{as} \ \ k \to \infty, \ \textit{for}\ t \leq T.$$

Remark 5 *The numerical simulation of the independent problems I and II can now be obtained by well-known finite-difference or finite elements schemes.*

4 Localization Properties of Solutions

Now we use the same philosophy than some of the localization properties of solutions presented in [3]. Their proofs can be carried out with the techniques presented there. We start with the initial-boundary value problem for the uncoupled equation (equation (10) with $Q = 0$).

4.1 Pure diffusion channel level equation

Let us consider the following initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s} \left(|u|^{\alpha} \left| \frac{\partial u}{\partial s} \right|^{-1/2} \frac{\partial u}{\partial s} \right) + f_{\Pi}, s \in \Pi = [-1, 1], t \in [0, T], \tag{46}$$

$$u(i, t) = u^0(t), \ i = -1, 1, \ (\textit{or} \ \frac{\partial u(1, t)}{\partial s} = 0), \tag{47}$$

$$u(s, 0) = u_0(s), t \in]0, T[, \tag{48}$$

$$0 < \delta \leq (u^i(t), u_0(s)) \leq C_0 \tag{49}$$

4.1.1 Finite time stabilization to a non zero state

Theorem 3 *Let conditions* (49), (32) *be fulfilled and* $f_\Pi(s,t) \equiv 0$. *Then the solution of problem* (46) − (48) *becomes identically constant after a finite time* t^*, *i.e.*

$$u(s,t) \equiv u^0 \qquad \text{for } s \in [0,1], t \geq t^*.$$

Moreover if $f_\Pi \not\equiv 0$ *but*

$$\|f_\Pi(.,t)\|^{3/2}_{L^2(\Pi)} \leq \varepsilon \left(1 - \frac{t}{t_f}\right)^4_+, \tag{50}$$

for some $t_f > t^*$, *and for some suitable small constant* ε, *then the following estimate holds:*

$$\left\|u(\cdot,t) - u^0\right\|^2_{L^2(\Pi)} \leq C\left(1 - \frac{t}{t_f}\right)^4_+.$$

In particular,

$$u(s,t) \equiv u^0, \text{for any } s \in [-1,1] \text{ and } t \geq t_f.$$

In physical terms, the first assertion of the theorem means that the water level in the channel becomes constant in a finite time provided that the external source f_Π is absent (see Figure 7). If $f \not\equiv 0$ and condition (50) is fulfilled, one can find a small source intensity $\varepsilon > 0$, such that the water level in the channel stabilizes at the same instant t_f when the source disappears.

4.1.2 Wetting finite speed of propagation and formation of a wetting front. Waiting time phenomenon

We consider now local properties of weak solution of (46) with zero initial data on some subinterval $[-\rho, \rho]$ (see Figures 8, 9).

Theorem 4 (Finite speed of wetting of a dry bottom) *Let* $u(s,t) \geq 0$ *be a weak solution of equation* (46) *with* $\alpha > 1/2$ *and let*

$$f_\Pi = 0, \qquad u_0(s) = u(s,0) = 0 \quad for \quad |s| \leq \rho_0, t \in (0,T). \tag{51}$$

Then

$$u(s,t) = 0 \quad for\ |s| \leq \rho(t), \quad \theta = \theta(\alpha) > 0. \tag{52}$$

where $\rho(t)$ *is defined by the formula*

$$\rho^{1+\sigma}(t) = \rho_0^{1+\sigma} - Ct^\theta$$

with constants $C = C(C_0,\alpha)$, $\theta = \theta(\alpha)$, $\sigma = \sigma(\alpha)$. *If, additionally to* (51) *we assume that*

$$\int_{-\rho}^{\rho} |u_0(s)|^2 ds + \int_0^T \int_{-\rho}^{\rho} |f_\Pi|^2 ds dt \leq \varepsilon(\rho - \rho_0)_+^{1/(1-\nu)}, \quad \rho_0 \leq \rho,$$

then there exists $t_* \in [0, T)$ *such that*

$$u(s,t) = 0, \text{ for } s \in [-\rho_0, \rho_0] \text{ and } t \in [0, t_*].$$

4.2 Coupled channel level and surface ground water flows

Let us return to the system of equations (15), (20),(21), with $\alpha_0 = 0$, describing the mentioned coupled flows. We consider the domain $B_\rho \times (0,T), B_\rho = \{x \in \Omega |\ |x - x_0| < \rho\}$.

4.2.1 Wetting Finite Speed of Propagation. Waiting Time Phenomenon.

Theorem 5 *Let* $W = (H, u)$ *be a local weak solution of equations* (15), (20), (21) *under the assumptions*

$$H_0(x) = 0, \quad f_\Omega = 0 \quad (x,t) \in B_{\rho_0} \times [0, T),$$
$$u_0(s) = 0, \quad f_\Pi = 0 \quad (s,t) \in \Pi_{\rho_0} \times [0, T).$$

Then there exist $t_* \in (0, T)$ *and* $\rho(t)$ *such that*

$$H(x,t) = 0 \quad x \in B_{\rho(t)}, \quad u(s,t) = 0 \quad s \in \Pi_{\rho(t)}, \quad t \in [0, t_*]$$

with $\rho(t)$ *defined by the formula*

$$\rho^{1+\sigma}(t) = \rho_0^{1+\sigma} - Ct^\theta,$$

with some constant C*. If, moreover,*

$$\|H_0\|^2_{L^2(B_\rho)} + \|u_0\|^2_{L^2(\Pi_\rho)} + \int_0^T \left(\|f_\Omega\|^2_{L^2(B_\rho)} + \|f_\Pi\|^2_{L^2(\Pi_\rho)} \right) d\tau$$
$$\leq \varepsilon \left(\rho - \rho_0\right)_+^{\vartheta}, \quad \rho > \rho_0, \quad \vartheta(\alpha) > 0,$$

then there exists $t_* \in [0, T)$ *such that*

$$H(x,t) = 0 \quad x \in B_{\rho_0}, \quad u(s,t) = 0 \quad s \in \Pi_{\rho_0}, \quad \text{for any } t \in [0, t_*]$$

In terms of the original physical problem we can understand the results as follows: if the domain B_{ρ_0} was dry at the initial time i.e. the levels of the surface and ground water were zero therein, then the first assertion of the theorem gives some estimates on the location of the free boundaries (the curves and points where they are zero) generated by $H(x,t)$ and $u(s,t)$ (see Figure 10). The second assertion states that whatever the flux is outside B_{ρ_0}, this domain can only be swamped not instantaneously but after a positive finite time.

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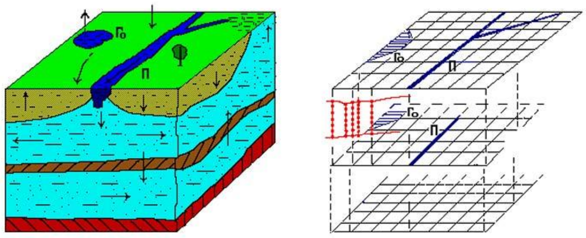

Figure 1: Scheme of interaction of underground and surface waters: a) area of modelling; b) interface of computational grids

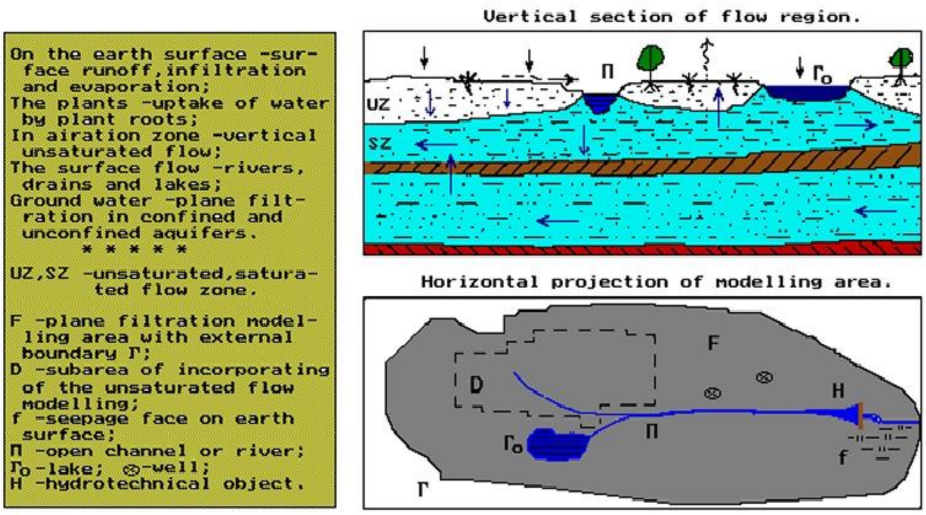

Figure 2: Vertical cross-section of the flow domain and plan view

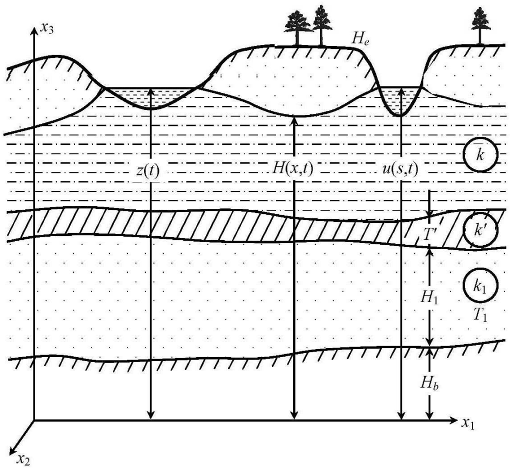

Figure 3: Vertical cross-section of the flow domain

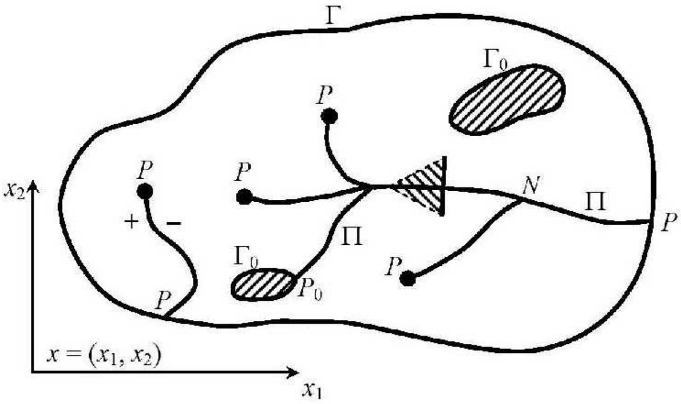

Figure 4: Plan view

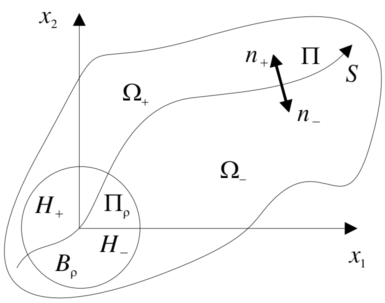

Figure 5: Plan view

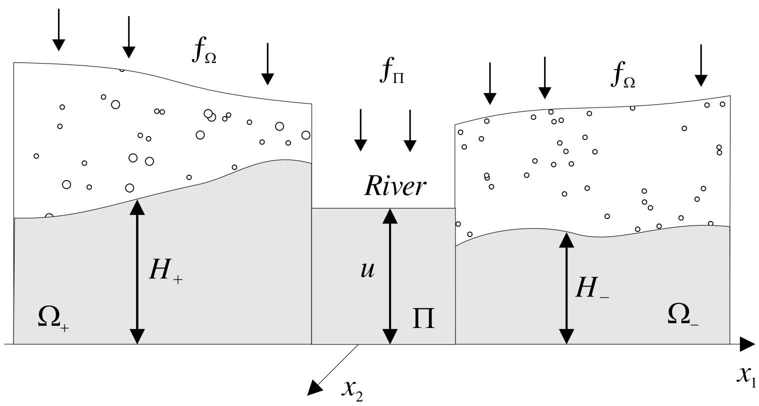

Figure 6: Vertical cross-section of the flow domain

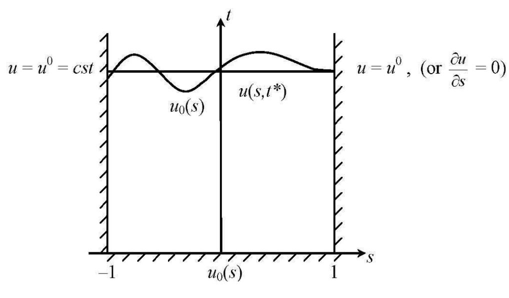

Figure 7: Stabilization to a stationary state(EDW)-vertical cross-section

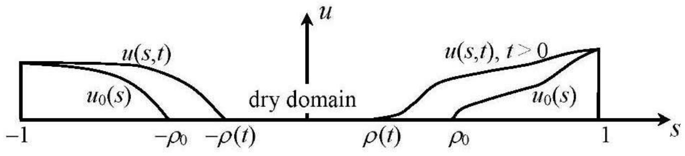

Figure 8: Finite speed of wetting(EDW)

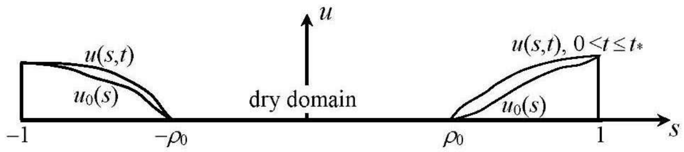

Figure 9: Waiting time of wetting(EDW)

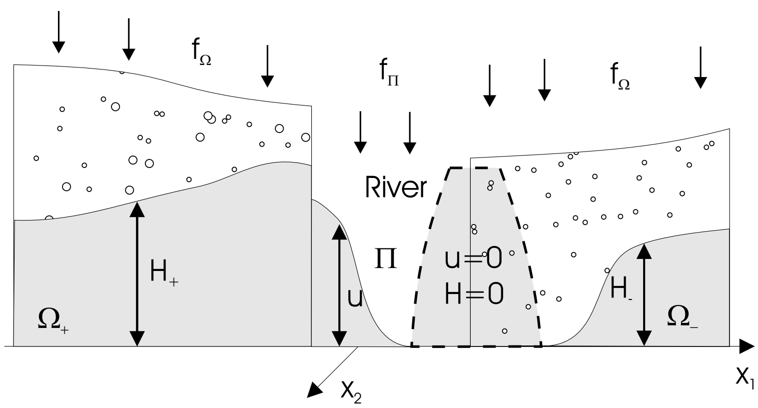

Figure 10: Finite speed of wetting(CF)