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A Finite Element Algorithm of a Nonlinear Diffusive Climate Energy Balance Model

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Abstract—We present a finite element algorithm of a climate diagnostic model that takes as a climate indicator the atmospheric sea-level temperature. This model belongs to the category of energy balance models introduced independently by the climatologists M.I. Budyko and W.D. Sellers in 1969 to study the influence of certain geophysical mechanisms on the Earth climate. The energy balance model we are dealing with consists of a two-dimensional nonlinear parabolic problem on the 2-sphere with the albedo terms formulated according to Budyko as a bounded maximal monotone graph in $\mathbb{R}^2$. The numerical model combines the first-order Euler implicit time discretization scheme with linear finite elements for space discretization, the latter is carried out for the special case of a spherical Earth and uses quasi-uniform spherical triangles as finite elements. The numerical formulation yields a nonlinear problem that is solved by an iterative procedure. We performed different numerical simulations starting with an initial datum consisting of a monthly average temperature field, calculated from the temperature field obtained from 50 years of simulations, corresponding to the period 1950–2000, carried out by the Atmosphere General Circulation Model HIRLAM.

Key words: Climate, nonlinear energy balance, finite elements.

1. *Introduction*

During recent decades there has been significant progress in climate modelling with the construction and testing of several Atmosphere-Ocean-General-Circulation-Models. These models are the ultimate tool that can be used to study and predict the Earth's climate system, in that they can include many phenomena taking part in it. However, there remain difficulties for these numerical models to be fully reliable. The first type of difficulty pertains to the lack of understanding of the physical nature of some of these phenomena such as, for example, sub-grid scale processes; so that, they have to be parameterized in order to be included in the models. However, one can argue that most of the sub-grid scale processes can be handled by direct numerical simulation (DNS) of the Navier-Stokes equations, the problem is that in the light of present and near future

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computer power such an approach is not practical for the moment. A second source of difficulty arises from the computational and numerical resources these models demand to perform well designed experiments; although one may expect that this latter problem can be partially alleviated with the continuous improvements and advances in computer technology, as well as with the development of more accurate and efficient numerical methods. In parallel with the development of general circulation models, the climatologists have developed simpler models intended to clarify the role of some phenomena, whose influence on the evolution of the climate system is considered to be very significant. This approach to the understanding of the climate phenomenology yields the so-called hierarchy of climate models. Perhaps the simplest class of models which may produce interesting results to understand the gross features of the past glacial and interglacial epochs are the so called Energy Balance Models (hereafter, EBMs) which are based on the balance between the incoming solar energy and the energy reflected to the outer space. Although simple in construction, these models may yield under different assumptions to nonlinear problems quite difficult to analyze; this being the reason why these models have caught the attention of many mathematicians. The progress of the mathematical analysis for the EBMs was a function of the different assumptions made on the spatial domain and the nonlinear terms involved in the equation. Among the many results that have appeared in the literature we mention here, in particular, the ones concerning discontinuous co-albedo functions due to Xu (1991) and Diaz (1993) for the one-dimensional case. The analysis of Diaz (1993) was extended to two dimensions, but with $c(x) \equiv 1$, in Diaz and Tello (1999) and Hetzer (1990). Many other references can be found in Diaz (1996).

As for works on the numerical approximation of EBMs, we mention the contributions of North and co-workers such as Hyde *et al.* (1990) and North and Coakley (1979), and Hetzer *et al.* (1989), where some numerical experiments were carried out. In North and co-workers model the numerical method consists of a first-order Euler implicit scheme for time discretization combined with an spectral method (Legendre polynomial expansion for latitude and trigonometric polynomial expansion for longitude) for space discretization. On the other hand, Hetzer *et al.* (1989) use a stationary quasi-linear energy balance model in their study on multiparameter sensitivity analysis of the solutions. In this model, the albedo function is continuous, while the nonlinearity originates from the radiation term which is modelled according to the Stefan-Boltzman radiation law. The model is formulated in spherical coordinates and uses second-order finite differences to discretize the diffusion terms, dealing with the singularities at the poles in an *ad hoc* manner. More recently, Bermejo *et al.* (2007) formulate and analyze a finite element model of a global nonlinear EBM of Budyko type with a nonlinear diffusion term modelled by the so-called p-Laplacian and a nonlinear discontinuous co-albedo function. The advantages of this finite element model, as well as the model of this paper, are the flexibility to use variable meshes, in particular, if one wants to properly resolve the mushy regions which appear in the transition between ice-covered and ice-free regions, and the form to avoid the singularities at the poles, which appear when the problem is formulated in spherical coordinates and discretized by grid-point methods such as finite differences, and finite elements of bounded finite volumes.

The layout of the paper is as follows. We introduce in Section 2 the model. In Section 3 we present the mathematical formulation of the model as well as mathematical properties and results concerning the existence and uniqueness of the solution. Section 4 is devoted to the numerical formulation of the model, which is carried out for the special case of a spherical Earth and uses quasi-uniform spherical triangles as finite elements. Finally, Section 5 contains numerical experiments in which we have taken as initial condition a temperature calculated by averaging 50 years of surface temperature data given by the atmospheric general circulation model HIRLAM.

2. *The Model*

Roughly speaking, the energy balance on the Earth surface is established according to the following law

$$\text{Variation of internal energy} = R_a - R_e + D, \tag{1}$$

where R_a denotes the amount of solar energy absorbed by the earth, R_e is the amount of infrared energy radiated to the space and D is a term which represents the diffusion of heat energy by atmospheric turbulence. Let $u(t, x)$ be the atmospheric sea-level temperature in Celsius degrees, i.e., $u(t, x)$ is defined on $[0, T) \times \mathcal{M}$, where $\mathcal{M}$ is a compact Riemannian manifold without boundary approximating the Earth surface; in fact, $\mathcal{M}$ is a 2-sphere of radius a. Under suitable conditions, the variation of internal energy can be expressed as $c(x)\partial u/\partial t$, where $c(x)$ is the heat capacity (we neglect the possible time dependence of c). The constitutive assumptions for the terms on the right-hand side of (1) are the following:

$$R_a = QS(t, x)\beta(x, u), \tag{2}$$

where Q is the so-called solar constant which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, Q is currently believed to be $Q = \frac{1}{4}(1360\,\mathrm{Wm}^{-2} \pm 2\,\mathrm{Wm}^{-2})$, the function $S(t, x)$ is the normalized seasonal distribution of heat flux entering the top of the atmosphere known as the insolation function. The incident solar flux at the top of the atmosphere at time t and latitude θ can be computed from celestial mechanics (see, e.g., Sellers, 1969); however, we shall use in our model the approximated formulas derived from the exact Sellers formulas by North and Coakley (1979). Specifically, in our model

$$S(t, x) = S_0(t) + S_1(t)\sin\theta + S_2(t)\left(\frac{3\sin^2\theta - 1}{2}\right), \tag{3a}$$

with

$$\begin{cases} S_0(t) = 1 + 2e\cos(2\pi t - \lambda), \\ S_1(t) = S_1[\cos 2\pi t + 2e \sin\lambda \sin 2\pi t], \\ S_2(t) = S_2[1 + 2e\cos(2\pi t - \lambda)], \end{cases} \tag{3b}$$

where θ is the latitude of the point $x \in \mathcal{M}$, e denotes the eccentricity of the earth 's orbit, presently, $e = 0.017$; λ is the angle formed by the lines connecting the Sun with the position of the Earth at the Northern Hemisphere winter solstice and the perihelion, at present $\lambda = -20°$; so that, the perihelion occurs shortly after the winter solstice in the Northern Hemisphere. The coefficients S_1 and S_2 depend upon the obliquity, δ, the present value of δ is 23.45° , so that $S_1 = -0.796$ and $S_2 = -0.477$. The unit of time t is 1 year, with $t = 0$ corresponding to the Northern Hemisphere winter solstice.

The term $\beta(x, u)$ is the so-called co-albedo function that takes values between 0 and 1. $\beta(x, u)$ represents the ratio between the absorbed solar energy and the incident solar energy at the point x on the Earth surface; obviously, $\beta(x, u)$ depends on the nature of the Earth surface. For instance, it is well known that on ice sheets $\beta(x, u)$ is considerably smaller than on the ocean surface because the white color of the ice sheets reflects a large portion of the incident solar energy, whereas the ocean, due to its dark color and high heat capacity, is able to absorb a larger amount of the incident solar energy. We further distinguish between ocean ice sheets and land ice sheets in our model. Following the approach of Budyko (1969) we take $\beta(x, u)$ as a nonlinear discontinuous function of the spatial coordinates x and the temperature u of the form given by Graves *et al.* (1993):

$$\beta(x,u) = a_0 + a_1 \sin\theta + a_2\left(\frac{3\sin^2\theta - 1}{2}\right) + a_I(u), \tag{4}$$

where the coefficients a_0, a_1 and a_2 may depend on time and represent the background albedo characterizing the U-shaped dependence of the albedo. The coefficient a_I takes care of the changes of the albedo in the presence of snow cover and is a function of the temperature u. Table 1, borrowed from Graves *et al.* (1993), shows the average values of a_0, a_1 and a_2 calculated from the monthly values of these parameters tabulated in Table 1 of Graves *et al.* (1993)

The values of a_I (u) are displayed in Table 2.

Notice that $\beta(x, u)$ is only discontinuous at the level sets $u = u_{s1}$ and $u = u_{s2}$, with $u_{s1} = -2$°C or -5°C and $u_{s2} = -7$°C or -12°C, due to the fact that a_I (u) is

Table 1

Coefficients of the co-albedo function

	Average Sky	Clear Sky
a_0	0.679	0.848
a_1	– 0.012	– 0.020
a_2	– 0.241	– 0045

Table 2

The values of $a_I(u)$

	Average Sky	Clear Sky
$a_{I(Land)}$ (u)	-0.14 if $u < -2°C$, $-[0.14, 0.0]$ if $u = -2°C$, 0.0 otherwise,	-0.50 if $u < -5°C$, $-[0.50, 0.0]$ if $u = -5°C$, 0.0 otherwise,
$a_{I(Ocean)}$(u)	-0.07 if $u < -7°C$, $[-0.07, 0.0]$if $u = -7°C$, 0.0 otherwise,	-0.25 if $u < -12°C$, $[-0.25, 0.0]$if $u = -12°C$, 0.0 otherwise,

discontinuous at these sets. Moreover, $\beta(x, u)$ is nonlinear because $a_I\ (u)$ is. To see that this statement is true, we must recall the definition of a linear function; that is, if $a_I\ (u)$ were a linear function then it would follow that given u_1 and u_2 and the real parameters l_1 and l_2, $a_I\ (l_1u_1 + l_2u_2) = l_1a_I\ (u_1) + l_2a_I\ (u_2)$, but it is obvious from the definition of $a_I\ (u)$ that the latter equality does not hold. Hence, $a_I\ (u)$ is a nonlinear function. It is worth remarking that $\beta(x, u)$ is not a single-valued function, rather, since for $u = u_{s1}$ (resp. $u = u_{s2}$) $a_I\ (u) \in [-0.14,0.0]$ or $a_I\ (u) \in [-0.5,0.0]$ (resp. $a_I\ (u) \in [-0.07,0.0]$ or $a_I\ (u) \in [-0.25,0.0]$) then for these values of u the only thing we know is that $\beta(x, u)$ is in bounded real intervals, but we do not know which points of these intervals are $\beta(x, u)$; this is the reason why we say that $\beta(x, u)$ is a multi-valued relation, or by abuse of mathematical language, it is said that $\beta(x, u)$ is a multi-valued graph. So that, it makes sense to write $z \in \beta(x, u)$ as we do below.

The term $R_e(u)$ was modelled by Budyko by performing a linear regression fitting to empirical data as

$$R_e(u) = Bu + C, \tag{5}$$

where B and C are empirical parameters relating the outgoing infrared flux to the surface temperature. According to Graves *et al*. (1993) the values that fit best the observations in a least square sense are shown in Table 3.

As for the diffusion term D, Budyko and Sellers proposed the expression

$$D = div(k(x)\nabla u),$$

where $k(x)$ is an eddy diffusion coefficient given by the formula (Graves *et al*., 1993):

Table 3

Coefficients of Budyko radiation energy $Re(u) = Bu + C$

	Average Sky	Clear Sky
$C(\mathrm{Wm}^{-2})$	212.8	249.8
$B(\mathrm{Wm}^{-2\circ}\ C^{-1})$	1.9	2.26

$$k(x) = k_0(1 + k_1 \sin^2\theta + k_2 \sin^4\theta). \tag{6}$$

The coefficients k_0, k_1 and k_2 are given in Table 4.

Finally, we show the values per unit area of the heat capacity $c(x)$. This coefficient is assumed to be a piecewise continuous function, depending on whether the local surface is land, ice or sea. See Table 5.

By substituting the above expressions into (1) we obtain the following energy balance model:

$$(P)\begin{cases} c(x)u_t - div_{\mathcal{M}}(k(x)\nabla_{\mathcal{M}}u) + Bu + C \in QS(t,x)\beta(x,u) & \text{in} \quad (0,T)\times\mathcal{M} \\ u(0,x) = u_0(x) & \text{on} \quad \mathcal{M}, \end{cases}$$

where the initial datum u_0 always will be assumed to be bounded. More precise structural assumptions to solve (P) are formulated in Section 3. A special feature of (P) is that the presence of the co-albedo function $\beta(x, u)$ may be responsible, in the case of a discontinuous function, of both the existence of free boundaries at the level sets u_{s1} and u_{s2}, and multiple solutions for certain initial conditions (even if the problem is formulated in terms of a parabolic type equation).

3. *On the Existence and Uniqueness of Solutions of the Model (P)*

To state the mathematical formulation of (P) we need to recall some basic concepts of differential geometry because the spatial domain $\mathcal{M}$ is the 2-sphere of radius a. Given an index set Λ and $\lambda \in \Lambda$, let W_λ be an open subset of $\mathcal{M}$ such that $\{W_\lambda\}_{\lambda\in\Lambda}$ is an open covering of $\mathcal{M}$, and $w_\lambda : W_\lambda \to w_\lambda(W_\lambda) \subset \mathbb{R}^2$ a homeomorphism. For $\lambda \in \Lambda$, the pair $\{W_\lambda, w_\lambda\}$ is called a chart of $\mathcal{M}$ and the family of charts $\{W_\lambda, w_\lambda\}_{\lambda\in\Lambda}$ is called an atlas of $\mathcal{M}$. Given a point $P \in W_\lambda \subset \mathcal{M}$, we set $w_\lambda(P) = (w^1_\lambda(P), w^2_\lambda(P)) = (\theta_\lambda, \varphi_\lambda) \in \mathbb{R}^2$. The

Table 4

Coefficients of the eddy diffusion coefficient

	Average Sky	Clear Sky
k_0	1.1175	1.331
k_1	− 0.957	− 2.258
k_2	0	1.616

Table 5

Heat capacity coefficient values

c_{water} (Wm^{-2} $^\circ$ C^{-1}*year*)	9.7
c_{land} (Wm$^{-2\circ}$ C^{-1}*year*)	0.016
c_{ice} (Wm$^{-2\circ}$ C^{-1}*year*)	0.10

tangent space at P is denoted by $T_P\mathcal{M}$. $T_P\mathcal{M}$ is a vector space of dimension 2 with a basis formed by the vectors $\mathbf{e_1} := \partial/\partial\theta_\lambda, \mathbf{e_2} := \partial/\partial\varphi_\lambda$. The tangent bundle $T\mathcal{M}$ is defined as $T\mathcal{M} := \cup_{P\in\mathcal{M}} T_P\mathcal{M}$. A Riemannian metric g on $\mathcal{M}$ is defined from a family of scalar products $g_P : T_P\mathcal{M} \times T_P\mathcal{M} \to \mathbb{R}$.

For a differentiable function $u : \mathcal{M} \to \mathbb{R}$ the tangent gradient $\nabla_{\mathcal{M}} u \in T\mathcal{M}$, and for $v : \mathcal{M} \to \mathcal{T}$ differentiable, the surface divergence $div_{\mathcal{M}} v \in \mathbb{R}$. We denote by $L^2(\mathcal{M})$ the set $\{u : \mathcal{M} \to \mathbb{R}$ measurable $: \int_{\mathcal{M}} |u|^2 dA < \infty\}$. This set is a Hilbert space with inner product

$$(u, v) = \int_{\mathcal{M}} uvdA$$

and norm

$$\|u\|_{L^2(\mathcal{M})} = \left(\int_{\mathcal{M}} |u|^2 dA\right)^{1/2}.$$

Analogously,

$$L^2(T\mathcal{M}) = \{X : \mathcal{M} \to T\mathcal{M} \text{ measurable} : \int_{\mathcal{M}} < X, X > dA < \infty\}.$$

Also, we shall use the spaces $L^\infty(\mathcal{M})$ and $L^\infty(T\mathcal{M})$ defined as

$$L^\infty(\mathcal{M}) = \{u : \mathcal{M} \to \mathbb{R} \text{ measurable} : \operatorname{ess}\sup_{\mathcal{M}} |u(x)| < \infty\}$$

and

$$L^\infty(T\mathcal{M}) = \{X : \mathcal{M} \to T\mathcal{M} \text{ measurable} : \operatorname{ess}\sup_{\mathcal{M}} |X(x)| < \infty\},$$

where ess sup is a shorthand notation for the essential supremum defined as

$$\operatorname{ess}\sup_{\mathcal{M}} |u(x)| = \inf\left\{\sup_{x\in\mathcal{S}} |u(x)| : \mathcal{S} \subset \mathcal{M}, \text{ with } \mathcal{M}\backslash\mathcal{S} \text{ of measure zero}\right\}.$$

We also need the Sobolev space

$$H^1(\mathcal{M}) = \{u \in L^2(\mathcal{M}) : \nabla_{\mathcal{M}} u \in L^2(T\mathcal{M})\},$$

with inner product

$$((u, v)) = \int_{\mathcal{M}} uvdA + \int_{\mathcal{M}} < \nabla_{\mathcal{M}} u, \nabla_{\mathcal{M}} v > dA$$

and norm

$$\|u\|_{H^1(\mathcal{M})} = \sqrt{((u, v))}.$$

$H^1(\mathcal{M})$ is the closure of the set of infinitely continuous functions, $C^\infty(\mathcal{M})$, in the H^1-norm. When m integer, $m > 1$, the Sobolev space of order m is the closure of $C^\infty(\mathcal{M})$ in the norm

$$\|u\|_{H^m(\mathcal{M})} = \left(\int_{\mathcal{M}} \left(\sum_{1 \le k \le m} \sum_{i_j = 1,2\ j=1,..,k} |D_{i_1} D_{i_2} ... D_{i_k} u|^2 + |u|^2 \right) dA \right)^{1/2},$$

where $D_1 = D_{\mathbf{e}_1}$ and $D_2 = D_{\mathbf{e}_2}$. When $m = 0, H^0(\mathcal{M}) = L^2(\mathcal{M})$.

Given a bounded and strictly positive function $c(x)$, and $Q > 0$, we consider the problem (P)

$$(P)\begin{cases} c(x)u_t - \mathrm{div}_{\mathcal{M}}(k(x)\nabla_{\mathcal{M}} u) + Bu + C \in QS(t,x)\beta(x,u) + f(t,x) & \text{in } (0,T) \times \mathcal{M}, \\ u(0,x) = u_0(x) & \text{on } \mathcal{M}, \end{cases}$$

under the following assumptions:
(A1) $\beta(x,\cdot)$ is a bounded maximal monotone graph of $\mathbb{R}^2$,
(A2) $f \in L^\infty((0,T) \times \mathcal{M})$,
(A3) $S : [0,T] \times \mathcal{M} \to \mathbb{R}, S \in C^1([0,T] \times \mathcal{M}), 0 < S_0 \le S(t,x) \le S_1$ $a.e. x \in \mathcal{M}$, for any $t \in [0,T]$,
(A4) $c \in L^\infty(\mathcal{M}), c(x) \ge c_0 > 0$,
(A5) $k \in C(\mathcal{M}), k(x) \ge k_0 > 0$,
(A6) $u_0 \in L^\infty(\mathcal{M})$,
(A7) $B > 0$ and $C > 0$ constants.

Note the presence of a forcing term $f(t, x)$ in the general statement of problem (P). We do not expect the existence of classical solutions to (P) due to the possible discontinuity of the co-albedo function. For this reason, we need the notion of weak solution to (P).

Definition 1. *A function $u \in C([0,T]; L^2(\mathcal{M})) \cap L^\infty((0,T) \times \mathcal{M})) \cap L^2(0,T; H^1)$ is termed a bounded weak solution of (P) if there exists $z \in L^\infty((0,T) \times \mathcal{M}), z(t,x) \in \beta(x, u(t,x))$ a.e. $(t,x) \in (0,T) \times \mathcal{M}$ such that*

$$\begin{aligned} \int_{\mathcal{M}} c(x)u(T,x)v(T,x)dA - \int_0^T \int_{\mathcal{M}} c(x)v_t(t,x)u(t,x)dAdt \\ + \int_0^T \int_{\mathcal{M}} < k(x)\nabla_{\mathcal{M}} u, \nabla_{\mathcal{M}} v > dAdt + \int_0^T \int_{\mathcal{M}} (Bu + C)vdAdt \\ = \int_0^T \int_{\mathcal{M}} QS(t,x)z(t,x)vdAdt + \int_0^T \int_{\mathcal{M}} fvdAdt \\ + \int_{\mathcal{M}} c(x)u_0(x)v(0,x)dA, \end{aligned} \tag{7}$$

$\forall v \in L^2(0,T;H^1)$ such that $v_t \in L^2(0,T;H^{-1})$. Here H^{-1} denotes the dual space of H^1.

The main results on the existence and uniqueness of bounded weak solutions to problem (P) are collected in Theorem 2 and Theorem 4; the proofs of which can be found in Bermejo *et al.* (2007).

Theorem 2. *Under the above assumptions there exists at least one bounded weak solution of (P). Moreover, if $u_0 \in H^1$ then $u_t \in L^2(0,T;L^2(\mathcal{M}))$ and* $\mathrm{div}(k(x)\nabla_{\mathcal{M}}u) \in L^2(0,T;\, L^2(\mathcal{M}))$.

Since $\beta(x, u)$ is considered to be a multi-valued graph discontinuous at the level sets $u = u_{s_1}$ and $u = u_{s_2}$, then there are cases for which problem (P), although parabolic, does not have a unique solution. Nevertheless, it can be proved, Bermejo *et al*. (2007), the uniqueness of the bounded weak solution to (P) in the class of non-degenerate functions which is introduced next.

Definition 3. *Let $u \in L^\infty(\mathcal{M})$. Given ε_0, $0 < \varepsilon_0 < 1$, for $\varepsilon \in (0, \varepsilon_0)$ and $i = 1,2$ let*

$$B_{s_i}(u, u_{s_i}; \epsilon) = \{x \in \mathcal{M} : \ |u - u_{s_i}| < \epsilon\}$$

and

$$B_{w_i}(u, u_{s_i}; \epsilon) = \{x \in \mathcal{M} : \ 0 < |u - u_{s_i}| < \epsilon\}.$$

It is said that u is a non-degenerate function in a strong (resp. weak) sense if it satisfies the following strong (resp. weak) non-degeneracy property: There exists a constant $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$area(B_{s_i}(u, u_{s_i}; \epsilon)) \leq C\epsilon \qquad (resp.area(B_{w_i}(u, u_{s_i}; \epsilon)) \leq C\epsilon).$$

Theorem 4. *Let $u_0 \in L^\infty(\mathcal{M})$. Then:*

(i) *If a bounded weak solution $u(t)$ to (P) is a strong non-degenerate function for all $t \in [0,T]$, then u is the unique bounded weak solution to (P).*

(ii) *For any $t \in (0, T]$ there is at most one bounded weak solution $u(t)$ to (P) in the class of weak non-degenerate functions.*

4. *The Numerical Model*

4.1. *Preliminaries*

We now proceed to formulate the numerical method to compute the bounded weak solution to problem (P). This method consists of a combination of $C^0 -$ finite elements for space discretization with a first-order Euler implicit scheme to discretize in time. This time scheme is also used in Hyde *et al*. (1990). We must point out that we choose the Euler implicit scheme for the main reason that our codes have been developed to integrate problem (P) when the diffusion term is also a nonlinear term modelled by the so called p-Laplacian, that is, as $\mathrm{div}_{\mathcal{M}}(|\nabla u|^{p-2}\nabla u)$, p integer > 2; and according to theoretical results of Barret and Liu (1994), and Ju (2000), one may conclude that the optimal time discretization scheme (optimality must be understood here in the sense that there is a balance between computational cost versus accuracy) combined with linear finite elements to integrate the time dependent p-Laplacian diffusion equation is the first-order Euler implicit scheme. However, we are aware that for problem (P), in which the diffusion terms are linear, it would be more convenient, as one of the reviewers has pointed out to us, to use in combination with finite elements the second- order implicit BDF2 (see Chap. III in HAIRER *et al.*, 1993) because the good properties this scheme has for stiff problems. The 2-sphere $\mathcal{M}$ is partitioned into quasi-uniform spherical triangles using the scheme of BAUMGARDNER and FREDERICKSON (1985), which consists of taking as the initial partition D_0 the spherical icosahedron and then to generate a sequence of partitions D_k, $k = 1, 2, \ldots$, by joining the mid-points on the sides of the triangles of the partition $D_{k\text{-}1}$. This procedure yields triangles with the following properties. Let N_k be the number of triangles in the partition D_k, then (a) $\mathcal{M} = \cup_{j=1}^{N_k} T_j, T_j \subset \mathcal{M}$; (b) for $i \neq j, T_i \cap T_i$ is either empty or has one vertex x_p, or T_i and T_j share a common edge γ_{ij}; (c) there exists a positive constant μ such that for all T_j, $h_j/\rho_j < \mu$, where h_j denotes the diameter of T_j and ρ_j is the diameter of the largest circle inscribed in T_j.

Following the approach of DZIUK (1988) to solve by finite elements the Poisson equation on manifolds, it is convenient to view the spherical triangles of the partition D_k of $\mathcal{M}$ as the radial projection onto $\mathcal{M}$ of 2-simplices $\Omega_j \subset \mathbb{R}^3$, such that if T_j is the image of Ω_j, then for all $j, T_j \cap \Omega_j = \{x_{1j}, x_{2j}, x_{3j}\}$, where x_{ij}, $i = 1, 2, 3$, are the vertices of both T_j and Ω_j. By analogy with the elements T_j, the simplices Ω_j form a partition D_{hk} of a polyhedron $\mathcal{M}_h$ such that

$$\mathcal{M}_h := \cup_j \Omega_j,\ \Omega_j \in D_{hk}.$$

We show in Figure 1 the initial icosahedron and the partition D_h after four refinements.

The radial projection is defined as

$$\phi : \mathcal{M}_h \to \mathcal{M}$$
$$\begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{pmatrix} \to \begin{pmatrix} \frac{\widehat{ax_1}}{\sqrt{(\widehat{x}_1)^2+(\widehat{x}_2)^2+(\widehat{x}_3)^2}} \\ \frac{\widehat{ax_2}}{\sqrt{(\widehat{x}_1)^2+(\widehat{x}_2)^2+(\widehat{x}_3)^2}} \\ \frac{\widehat{ax_3}}{\sqrt{(\widehat{x}_1)^2+(\widehat{x}_2)^2+(\widehat{x}_3)^2}}. \end{pmatrix},$$

so that, we write

$$\mathcal{M} = \cup_j \phi(\Omega_j)$$

and denote the restriction of ϕ on the element Ω_j by ϕ_j. Note that ϕ is a C^m-diffeomorphism, $m \geq 1$. We define the family of finite element spaces associated with the partitions D_{hk}.

$$\widehat{V}_h = \{\widehat{v}_h \in C^0(\mathcal{M}_h) : \widehat{v}_h|_{\Omega_j} \in P_1(\Omega_j), 1 \leq j \leq N_k\},$$

where $P_1(\Omega_j)$ is the set of polynomials of degree ≤ 1 defined on Ω_j. Let M be the global number of vertices in the partition D_k, and let $\{\alpha_l\}_{l=1}^{M}$ be the set of global basis functions for $\widehat{V}_h$, such that $\alpha_l \in \widehat{V}_h$ and at the vertex $\widehat{x}_j$ $\alpha_l(\widehat{x}_j) = \delta_{jl}$; any $\widehat{v} \in \widehat{V}_h$ can be expressed as

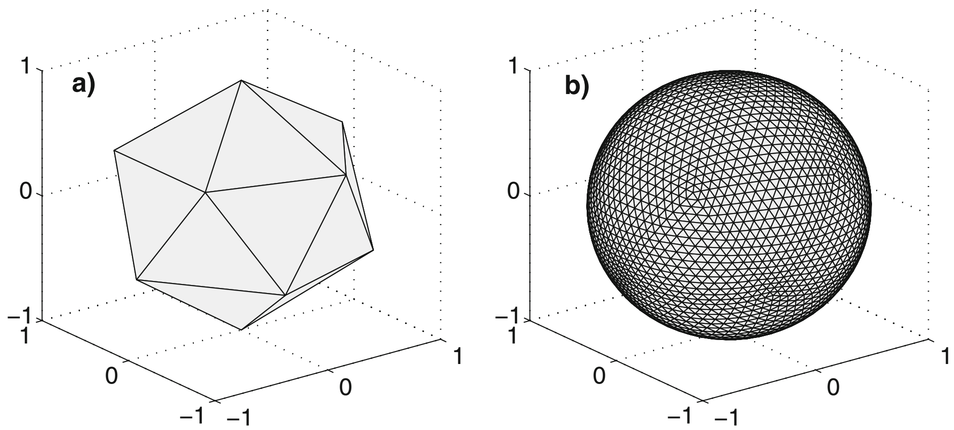

Figure 1
Initial Icosahedron and mesh after 4 refinements.

$$\widehat{v}_h(\widehat{x}) = \sum_{l=1}^{M} \widehat{v}_h(x_l)\alpha_l(\widehat{x}).$$

We define a finite element space $V_h \subset H^1(\mathcal{M})$ associated with the partition D_k via the radial ϕ − lifting as follows:

$$V_h = \{v_h \in C^0(\mathcal{M}) : \, v_h|_{T_j} = \widehat{v}_h \circ \phi_j^{-1} \quad \text{with} \quad \widehat{v}_h \in \widehat{V}_h\}.$$

The approximation spaces V_h and $\widehat{V}_h$ satisfy:
For all $u \in L^\infty((0,T)\times\mathcal{M}) \bigcap L^2(0,T;V)$, $u_t \in L^2((0,T)\times\mathcal{M})$

$$\lim_{h\to 0} \inf_{u_h \in V_h} \|u - u_h\|_{L^\infty((0,T)\times\mathcal{M})} = 0.$$

Moreover, from computational and numerical analysis points of view it is convenient to define the spaces $H^l(\mathcal{M}_h), l \geq 0$ (with the convention that for $l = 0, H^l(\mathcal{M}_h) \equiv L^2(\mathcal{M}_h)$) as

$$H^l(\mathcal{M}_h) = \{\widehat{v} : \mathcal{M}_h \to \mathbb{R} : \textit{for a.e. } x \in \mathcal{M} \textit{ and } v \in H^l(\mathcal{M}), \widehat{v} \circ \phi^{-1}(x) = v(x)\}.$$

In relation with the radial projection ϕ defined on $\mathcal{M}_h$ we have the following results (BERMEJO *et al.*, 2007):

Proposition 5. *Let J_{ϕ_j} and $J_{\phi_j^{-1}}$ denote the absolute values of the Jacobian determinants of the mappings ϕ_j and ϕ_j^{-1}, respectively. Then, for h sufficiently small there exist constants C_1 and C_2 independent of h such that*

$$\max_j \|J_{\phi_j} - 1\|_{L^\infty(\Omega_j)} \leq C_1 h^2 \textit{ and } \max_j \|J_{\phi_{j^{-1}}} - 1\|_{L^\infty(T_j)} \leq C_2 h^2.$$

Proposition 6. *For* $1 \leq p \leq \infty$ *there exist constants* c_1 *and* c_2 *such that*

$$c_1\|\widehat{v}\|_{L^p(\mathcal{M}_h)} \leq \|v\|_{L^p(\mathcal{M})} \leq c_2\|\widehat{v}\|_{L^p(\mathcal{M}_h)},$$
$$c_1\|\widehat{v}\|_{H^1(\mathcal{M}_h)} \leq \|v\|_{H^1(\mathcal{M})} \leq c_2\|\widehat{v}\|_{H^1(\mathcal{M}_h)},$$
$$|\widehat{v}|_{H^2(\mathcal{M}_h)} \leq c_2\Big(|v|_{H^2(\mathcal{M})}+h|v|_{H^1(\mathcal{M})}\Big).$$

The relevance of these results, in particular Proposition 6, lies in the fact that by virtue of it the approximation error in the family of finite element spaces $\widehat{V}_h$ is of the same order as the error in the family of spaces V_h associated to the partition D_k of spherical triangles. In terms of the numerical calculations this means that one can substitute the spherical triangles (curved triangles) by plane triangles in $\mathbb{R}^3$ and, therefore, make use of the finite element technology for plane triangles. At this point, we must say that the idea of approximating the 2-sphere by a $\mathbb{R}^3$ polyhedra of triangular faces has a long tradition in numerical computations of atmospheric flows. Just to cite a few, we mention the works of SADOURNY *et al.* (1968), and WILLIAMSON (1968) at the end of the sixties of the past century, and more recently the integration of the shallow water equations via a Lagrange-Galerkin method carried out by HEINZE and HENSE (2002), and GIRALDO and WARBURTON (2005).

Since the numerical solution to problem (P) is computed at a discrete set of time instants t_n, with $n = 0, 1, ..., N$, we choose a fixed time step Δt, such that for all n, $t_{n+1} = t_n + \Delta t$, and consider the discrete set $I_N = \{0,t_1,t_2,...,t_N = T\}$. The numerical solution to (P) is thus the map $U{:}I_N \rightarrow V_h$ such that there exists $Z^n \in L^\infty(\mathcal{M}) \bigcap V_h, Z^n \in \beta(x, U^n)$, verifying that for any $v_h \in V_h$

$$(P_{h,\Delta t})\begin{cases} \int_\mathcal{M} c\,\frac{U^n-U^{n-1}}{\Delta t} v_h dA + \int_\mathcal{M}\langle k\nabla_\mathcal{M} U^n, \nabla_\mathcal{M} v_h\rangle dA + \\ \int_\mathcal{M}(BU^n + C)v_h dA = \int_\mathcal{M} QS^n Z^n v_h dA + \int_\mathcal{M} f^n v_h dA, \end{cases}$$

where the notation $b(t_n,x) = b^n$ is used unless otherwise stated.

An important property of the finite element space V_h is that if a function $w_h \in V_h$ is an approximation to a function $w \in L^\infty((0,T) \times \mathcal{M})$ that belongs to the class of non-degenerate functions (either strong or weak), then for h sufficiently small w_h also belongs to that class. Specifically, we have the following results. For $i = 1$ and 2, let $B_{s_i}(w, u_{s_i}; \epsilon)$ and $B_{w_i}(w, u_{s_i}; \epsilon)$ be the sets introduced in Section 3, and we consider the level sets

$$A_i = \{x \in \mathcal{M} : w(t,x) = u_{s_i}\}, \qquad A_{hi} = \{x \in \mathcal{M} : w_h(t,x) = u_{s_i}\},$$
$$M_i^\pm = \{x \in \mathcal{M} : w(t,x) \gtrless u_{s_i}\} \quad \text{and} \quad M_{hi}^\pm = \{x \in \mathcal{M} : w_h(t,x) \gtrless u_{s_i}\}.$$

Note that $\mathcal{M} = A_i \cup M_i^+ \cup M_i^- = A_{hi} \cup M_{hi}^+ \cup M_{hi}^-$. It is easy to ascertain that for $z \in \beta(x,w)$ and $z_h \in \beta(x, w_h)$ it holds

$$\begin{cases} |z - z_h| \leq max|a_I(u)| & \text{if} \quad x \in A_i \cup A_{hi} \cup (M_i^+ \cap M_{hi}^-) \cup (M_i^- \cap M_{hi}^+), \\ |z - z_h| = 0 & \text{if} \quad x \in (M_i^+ \cap M_{hi}^+) \cup (M_i^- \cap M_{hi}^-). \end{cases}$$

Moreover, the following lemma can be proved (BERMEJO *et al.* 2007):

Lemma 7. *Given a function $v \in L^{\infty}((0,T)\times M)\cap L^2(0,T;V)$, and its approximation $v_h \in V_h$, for h depending on $\in$ sufficiently small the relation*

$$A_i \cup A_{hi} \cup (M_i^+ \cap M_{hi}^-) \cup (M_i^- \cap M_{hi}^+) \subset B_{s_i}(v, u_{s_i};\epsilon)$$

holds for $i = 1$ and 2. Consequently, there exists a constant $C > 0$ such that

$$area\big(A_i \cup A_{hi} \cup (M_i^+ \cap M_{hi}^-) \cup (M_i^- \cap M_{hi}^+)\big) \le C\epsilon.$$

We are now in a condition to state the result on the existence and uniqueness of the solution $\{U^n\}_1^N$ to problem $(P_{h,\Delta t})$, whose proof is given in Bermejo *et al*. (2007).

Lemma 8. *For all $n = 1,\ldots, N$, there exists a solution $U^n \in V_h$ to problem $(P_{h,\Delta t})$ which is unique in the class of strong (resp. weak) non-degenerate functions.*

An important issue when calculating a numerical solution to a model is to estimate the rate of convergence of the approximate solution to the exact one. Again, appealing to the numerical analysis employed in Bermejo *et al*. (2007) to prove its Theorem 3, we can establish the rate of convergence of U^n to $u(t_n,x)$ for all n.

Theorem 9. *Let $u(t,x)$ be the unique non-degenerate bounded weak solution to problem (P), with $u \in L^2(0,T;H^2(\mathcal{M}))$. Let $\{U^n\}_{n=1}^N$ be the unique solution to problem $(P_{h,\,\Delta t})$ such that for $n = 1, 2,\ldots, N$ and $t \in (t_{n-1}, t_n]$ we define*

$$U(t) = \frac{t - t_{n-1}}{\Delta t} U^n - \frac{t_n - t}{\Delta t} U^{n-1}.$$

Then, for Δt and h depending on $\in$ being sufficiently small, there exists a constant $C > 0$ independent of Δt and h such that

$$\|u - U\|^2_{L^{\infty}(0,T;L^2(\mathcal{M}))} \le C(\epsilon + \Delta t^2 + h^2) \tag{8}$$

4.2. The Finite element Solution

To calculate the numerical solution we recast problem $(P_{h,\Delta t})$ as follows:
Given the initial condition $U^0 \in V_h$, for $n = 1,\ldots, N$, find $U^n \in V_h$ such that for $v_h \in V_h$

$$\begin{cases} \int_{\mathcal{M}} cU^n v_h dA + \Delta t \int_{\mathcal{M}} \langle k\nabla_{\mathcal{M}} U^n, \nabla_{\mathcal{M}} v_h\rangle dA + \Delta t \int_{\mathcal{M}} (BU^n + C) v_h dA = \\ \int_{\mathcal{M}} cU^{n-1} v_h dA + \Delta t \int_{\mathcal{M}} QS^n Z^n v_h dA + \Delta t \int_{\mathcal{M}} f^n v_h dA, \end{cases} \tag{9}$$

where $Z^n \in L^{\infty}(\mathcal{M}) \bigcap V_h, Z^n \in \beta(x, U^n)$. Since U^n is unknown so is Z^n, which has to be calculated in the process of determining the solution U^n. To do so we use the following iterative procedure:

Let $Tol \in \mathbb{R}_+, 0 < Tol \ll 1$; for all $n = 1,\ldots,N$, set $W^0 = U^{n-1}$ and do:
for $k = 1, 2,\ldots$
pick up $Z^{n,k-1} \in \beta(x, W^{k-1}), Z^{n,k-1} \in V_h$ and solve

$$\begin{cases} \int_{\mathcal{M}} cW^k v_h dA + \Delta t \int_{\mathcal{M}} \langle k\nabla_{\mathcal{M}} W^k, \nabla_{\mathcal{M}} v_h \rangle dA + \Delta t \int_{\mathcal{M}} (BW^k + C) v_h dA = \\ \int_{\mathcal{M}} cU^{n-1} v_h dA + \Delta t \int_{\mathcal{M}} QS^n Z^{n,k-1} v_h dA + \Delta t \int_{\mathcal{M}} f^n v_h dA, \ \forall v_h \in V_h. \end{cases} \tag{10}$$

Stop when

$$\frac{\left\| W^k - W^{k-1} \right\|_{L^2(\mathcal{M})}}{\left\| W^0 \right\|_{L^2(\mathcal{M})}} \leq Tol$$

and then set

$$U^n(x) = W^k(x).$$

By applying the same ideas of the proof of Theorem 1 of Carl (1992) one can prove that this iterative procedure converges to U^n when $k \to \infty$.

To find out the numerical solution W^k, and therefore U^n, we approximate the triangulated 2-sphere $\mathcal{M}$ by the polyhedron $\mathcal{M}_h$ and setting, $\widehat{c}(\widehat{x}) = c \circ \phi(\widehat{x})$, $\widehat{k}(\widehat{x}) = k \circ \phi(\widehat{x})$ and $\widehat{f}^n(\widehat{x}) = f^n \circ \phi(\widehat{x})$, solve instead of (10) the following problem defined on $\mathcal{M}_h$:

For $n = 1, 2,.., N$ do:
$\widehat{W}^0(\widehat{x}) = \widehat{U}^{n-1}(\widehat{x})$
for $k = 1,2,...$
pick up $\widehat{Z}^{n,k-1} \in \beta(\widehat{x}, \widehat{W}^{k-1}), \widehat{Z}^{n,k-1} \in \widehat{V}_h$ and find $\widehat{W}^k \in \widehat{V}_h$, such that for $\widehat{v}_h \in \widehat{V}_h$

$$\begin{cases} \int_{\mathcal{M}_h} \widehat{c}\widehat{W}^k \widehat{v}_h dA_h + \Delta t \int_{\mathcal{M}_h} \widehat{k} \nabla_{\mathcal{M}_h} \widehat{W}^k \cdot \nabla_{\mathcal{M}_h} \widehat{v}_h dA_h + \Delta t \int_{\mathcal{M}_h} (B\widehat{W}^k + C) \widehat{v}_h dA_h = \\ \int_{\mathcal{M}_h} \widehat{c}\widehat{U}^{n-1} \widehat{v}_h dA_h + \Delta t \int_{\mathcal{M}_h} Q\widehat{S}^n \widehat{Z}^{n,k-1} \widehat{v}_h dA_h + \Delta t \int_{\mathcal{M}_h} \widehat{f}^n \widehat{v}_h dA_h. \end{cases} \tag{11a}$$

Stop when

$$\frac{\left\| \widehat{W}^k - \widehat{W}^{k-1} \right\|_{L^2(\mathcal{M}_h)}}{\left\| \widehat{W}^0 \right\|_{L^2(\mathcal{M}_h)}} \leq Tol$$

and set

$$\widehat{U}^n(\widehat{x}) = \widehat{W}^k(\widehat{x}), \tag{11b}$$

and for $x \in \mathcal{M}$ and $\widehat{x} \in \mathcal{M}_h$, such that $x = \phi(\widehat{x})$,

$$U^n(x) = \widehat{U}^n(\widehat{x}). \tag{11c}$$

Next, we shall describe the method to implement $\nabla_{\mathcal{M}_h} \widehat{u}_h(\widehat{x})$ for any $\widehat{u}_h(\widehat{x}) \in \widehat{V}_h$. Following Dziuk (1988) we write the tangent gradient $\nabla_{\mathcal{M}} u \in L^2(T\mathcal{M})$ when $u \in H^1(\mathcal{M})$ as

$$\nabla_{\mathcal{M}} u = \nabla u - (\overrightarrow{n}_{\mathcal{M}} \cdot \nabla u)\overrightarrow{n}_{\mathcal{M}},$$

where $\overrightarrow{n}_{\mathcal{M}}$ is the unit outward normal vector on $\mathcal{M}$ and $\nabla u = (\partial u/\partial x_i)_{i=1,2,3}$ denotes the gradient of u considered as a function of the Cartesian coordinates (x_1, x_2, x_3) referred to the Cartesian coordinate system, the origin of which is at the center of the sphere. Recalling that for $\widehat{x} \in \mathcal{M}_h, \widehat{u}(\widehat{x})$ is a lifting of $u(x)$, i.e., $x \in \mathcal{M}$ is such that $x = \phi(\widehat{x})$ and $\widehat{u}(\widehat{x}) = u \circ \phi(\widehat{x})$, then $\nabla_{\mathcal{M}} u$ will be numerically approximated by the approximation to $\nabla_{\mathcal{M}_h} \widehat{u}(\widehat{x}) \in L^2(T\mathcal{M}_h)$, the expression of which is

$$\nabla_{\mathcal{M}_h} \widehat{u}(\widehat{x}) = \nabla \widehat{u}(\widehat{x}) - (\overrightarrow{n}_{\mathcal{M}_h} \cdot \nabla \widehat{u}(x))\overrightarrow{n}_{\mathcal{M}_h} \text{ for any } \widehat{x} \in \mathcal{M}_h,$$

where $\overrightarrow{n}_{\mathcal{M}_h}$ denotes the unit outward normal vector on $\mathcal{M}_h$, which is a constant vector on each triangular face Ω_j of $\mathcal{M}_h$, defining thus a piecewise constant approximation to $\overrightarrow{n}_{\mathcal{M}}$. $\widehat{u}(\widehat{x})$ is approximated by $\widehat{u}_h(\widehat{x}) \in \widehat{V}_h$ satisfying $\widehat{u}_h(P) \mid_{\Omega_j} \in P_1(\Omega_j)$; that is

$$\widehat{u}_h(\widehat{x}) \mid_{\Omega_j} = \sum_{m=1}^{3} \widehat{U}_m \lambda_m(\widehat{x}),$$

where $\widehat{U}_m = \widehat{u}_h(\widehat{x}_m)$, and the local basis functions $\{\lambda_m(\widehat{x})\}_{m=1}^3$ are the so-called barycentric coordinates defined by the relations

$$\begin{aligned} \textstyle\sum_{m=1}^3 \widehat{x}_{mi}\lambda_m &= \widehat{x}_i, \quad \text{for } i = 1,2,3, \\ \textstyle\sum_{m=1}^3 \lambda_m &= 1 \qquad \forall P \in \Omega_j, \end{aligned}$$

here $\widehat{x}_i$ are the coordinates of any point $\widehat{x} \in \Omega_j$ and $\widehat{x}_{mi}$ are the coordinates of the vertices of Ω_j. Then, denoting by $\overrightarrow{n}_j$ the unit normal vector on Ω_j we have that for any $\widehat{x} \in \Omega_j$

$$\nabla_{\mathcal{M}_h} \widehat{u}_h(\widehat{x}) = \sum_{m=1}^{3} \widehat{U}_m \nabla \lambda_m - \left(\sum_{l=1}^{3} n_{jl} \sum_{m=1}^{3} \widehat{U}_m \frac{\partial \lambda_m}{\partial \widehat{x}_l} \right) \overrightarrow{n}_j.$$

We notice that by construction of the family of finite element spaces V_h, $\widehat{U}_m$ are also the values $u_h(x_m)$, with $x_m = \phi(\widehat{x}_m)$ being the vertices of the spherical triangles. Moreover, via the local basis functions $\{\lambda_m(\widehat{x})\}$ of the elements Ω_j we can define a set of global basis functions $\{\alpha_l(\widehat{x})\}_{l=1}^M$ for the finite element space $\widehat{V}_h$ that is characterized by the following properties: (1) For each $l, \alpha_l(\widehat{x}) \in \widehat{V}_h$; (2) for $1 \le i, l \le M, \alpha_l(\widehat{x}_i) = \delta_{il}$; (3) for $1 \le j \le N_k$, $1 \le l \le M$ and $1 \le m \le 3$, the restriction of $\alpha_l(\widehat{x})$ on the element Ω_j, i.e., $\alpha_l(\widehat{x}) \mid_{\Omega_j} = \lambda_m(\widehat{x})$ if the mesh node $\widehat{x}_l$ coincides with the m-th vertex of the Ω_j. By properties (1) and (2) the global basis functions $\alpha_l(\widehat{x})$ are piecewise linear polynomials of compact support and each element $\widehat{u}_h(\widehat{x}) \in \widehat{V}_h$ is expressed as

$$\widehat{u}_h(\widehat{x}) = \sum_{l=1}^{M} \widehat{U}_l \alpha_l(\widehat{x}).$$

By property (3) we can evaluate the domain integrals in (11a) as the sum of element integrals using the local basis functions $\{\lambda_m\}$.

Now, we calculate the integral $\int_{\mathcal{M}_h} \widehat{k}\nabla_{\mathcal{M}_h}\widehat{u}_h^n \cdot \nabla_{\mathcal{M}_h}\widehat{v}_h dA_h$ as

$$\int_{\mathcal{M}_h} \widehat{k}\nabla_{\mathcal{M}_h}\widehat{u}_h^n \cdot \nabla_{\mathcal{M}_h}\widehat{v}_h dA_h = \sum_{j=1}^{N_k}\int_{\Omega_j} \widehat{k}\nabla_{\mathcal{M}_h}\widehat{u}_h \cdot \nabla_{\mathcal{M}_h}\widehat{v}_h dA_h, \tag{12a}$$

where the element integral

$$\int_{\Omega_j} \widehat{k}\nabla_{\mathcal{M}_h}\widehat{u}_h \cdot \nabla_{\mathcal{M}_h}\widehat{v}_h dA_h = \widehat{V}S_j\widehat{U}^T,$$

with $\widehat{V} = (\widehat{V}_1, \widehat{V}_2, \widehat{V}_3)$, $\widehat{U} = (\widehat{U}_1, \widehat{U}_2, \widehat{U}_3)$, $\widehat{V}_k$ and $\widehat{U}_k$ being the values of $\widehat{V}_h$ and $\widehat{u}_h$ at the vertices of $\boldsymbol{\Omega}_j$ respectively, and S_j is the $\boldsymbol{\Omega}_j$-element symmetric matrix the entries of which are

$$s_{ik} = \int_{\Omega_j} \widehat{k}\nabla\Psi_i \cdot \nabla\lambda_k dA_h = \int_{\Omega_j} \widehat{k}(\nabla\lambda_i - (\overrightarrow{n}_j \cdot \nabla\lambda_i)\overrightarrow{n}_j) \cdot \nabla\lambda_k dA_h, \quad 1 \leq i,k \leq 3. \tag{12b}$$

Note that s_{ik} are the entries of the stiffness matrix corresponding to the two-dimensional Laplace operator minus $\int_{\Omega_j}(\overrightarrow{n}_j \cdot \nabla\lambda_i)(\overrightarrow{n}_j \cdot \nabla\lambda_k)dA_h$. We are now in a condition to describe how the evaluation of integrals of (11a) yields an algebraic system of equations the solution of which is formed by the values of $\widehat{W}^k$ at the vertices of the spherical triangles.

$$\int_{\mathcal{M}_h} \widehat{k}\nabla_{\mathcal{M}_h}\widehat{W}^k \cdot \nabla_{\mathcal{M}_h}\widehat{v}_h dA_h = \sum_{j=1}^{N_k}\int_{\Omega_j} \widehat{k}\nabla_{\mathcal{M}_h}\widehat{W}^k \cdot \nabla_{\mathcal{M}_h}\widehat{v}_h dA_h = \widehat{\mathbf{V}}^T\mathbf{S}\widehat{\mathbf{W}}^k,$$

$$\int_{\mathcal{M}_h} (\widehat{c}\widehat{W}^k + \varDelta tB\widehat{W}^k)\widehat{v}_h dA_h = \sum_{j=1}^{N_k}\int_{\Omega_j} (\widehat{c}\widehat{W}^k + \varDelta tB\widehat{W}^k)\widehat{v}_h = \widehat{\mathbf{V}}^T(\mathbf{M}_1 + \varDelta tB\mathbf{M}_2)\widehat{\mathbf{W}}^k,$$

$$\varDelta tC\int_{\mathcal{M}_h} \widehat{v}_h dA_h = \varDelta tC\sum_{j=1}^{N_k}\int_{\Omega_j} \widehat{v}_h dA_h = \varDelta tC\widehat{\mathbf{V}}^T\mathbf{L},$$

$$\int_{\mathcal{M}_h} \widehat{c}\widehat{U}^{n-1}\widehat{v}_h dA_h = \sum_{j=1}^{N_k}\int_{\Omega_j} \widehat{c}\widehat{U}^{n-1}\widehat{v}_h dA_h = \widehat{\mathbf{V}}^T\mathbf{M}_1\widehat{\mathbf{U}}^{n-1},$$

$$\varDelta tQ\int_{\mathcal{M}_h} \widehat{S}^n\widehat{Z}^{n,k-1}\widehat{v}_h dA_h = \Delta tQ\sum_{j=1}^{N_k}\int_{\Omega_j} \widehat{S}^n\widehat{Z}^{n,k-1}\widehat{v}_h dA_h = \varDelta tQ\widehat{\mathbf{V}}^T \cdot \widehat{\mathbf{Z}}^{n,k-1},$$

$$\varDelta t\int_{\mathcal{M}_h} \widehat{f}^n\widehat{v}_h dA_h = \Delta t\sum_{j=1}^{N_k}\int_{\Omega_j} \widehat{f}^n\widehat{v}_h dA_h = \Delta t\widehat{\mathbf{V}}^T \cdot \mathbf{F}^n.$$

In these formulas the M-dimensional vector $\widehat{\mathbf{V}}^T := (\widehat{V}_1, \ldots \widehat{V}_M)$, $\widehat{V}_i$ being the value of $\widehat{V} \in \widehat{V}_h$ at the mesh point x_i. Similarly, $\widehat{\mathbf{W}}^k := (\widehat{W}_1^k, \ldots \widehat{W}_M^k)^T$, $\widehat{\mathbf{W}}^{k-1} := (\widehat{W}_1^{k-1}, \ldots \widehat{W}_M^{k-1})^T$ and $\widehat{\mathbf{U}}^{n-1} := (\widehat{U}_1^{n-1}, \ldots \widehat{U}_M^{n-1})^T$. **S,** $\mathbf{M}_1$ and $\mathbf{M}_2$ are sparse symmetric $M \times M$ matrices obtained by assembling the corresponding element matrices. Thus,

$$\mathbf{S} = \bigcup_{j=1}^{N_k} S_j$$

where S_j is the element matrix whose entries are given by (12b).

$$\mathbf{M}_1 = \bigcup_{j=1}^{N_k} M_{1j}, \mathbf{M}_2 = \bigcup_{j=1}^{N_k} M_{2j} \quad \text{and} \quad \mathbf{L} = \bigcup_{j=1}^{N_k} L_j$$

where M_{1j}, M_{2j} and L_j are element matrices with entries

$$\begin{cases} m_{1ik} = \int_{\Omega_j} \widehat{c}\lambda_i\lambda_k dA_h, \\ m_{2ik} = \int_{\Omega_j} \lambda_i\lambda_k dA_h, & (1 \leq i, k \leq 3). \\ l_{ii} = \int_{\Omega_j} \lambda_i dA_h, & l_{ij} = 0 \,\text{when}\, i \neq j \end{cases}$$

The M-dimensional vector $\widehat{\mathbf{Z}}^{n,k-1} = (\widehat{Z}_1^{n,k-1}, \ldots, \widehat{Z}_M^{n,k-1})^T$ is obtained by assembling the element vectors $\widehat{\mathbf{Z}}_j^{n,k-1}$:

$$\widehat{\mathbf{Z}}^{n,k-1} = \bigcup_{j=1}^{N_k} \widehat{\mathbf{Z}}_j^{n,k-1},$$

the entries of $\widehat{\mathbf{Z}}_j^{n,k-1}$ being given by

$$\widehat{z}_l^{n,k-1} = \int_{\Omega_j} \widehat{S}^n \widehat{Z}^{n,k-1} \lambda_l dA_h, \quad 1 \leq l \leq 3.$$

Likewise, the vector $\mathbf{F}^n = (F_1^n, \ldots, F_M^n)^T$ is obtained by assembling of the element vectors $\mathbf{F}_j^n$ the entries of which are the values of the integrals

$$\int_{\Omega_j} \widehat{f}^n \lambda_k dA_h, \quad 1 \leq k \leq 3.$$

We use the 7 points Hammer quadrature rule for triangles, which is exact for polynomials of degree 5, to calculate the integrals because the expressions for $S(t,x), \widehat{Z}^{n,k-1}$ and $\widehat{k}(\widehat{x})$ give integrands that are polynomials of degree 4.

Important features that make this formulation attractive for computations are the absence of the so-called "pole problem" and the discretization of the Laplace-Beltrami operator (i.e., the Laplace operator defined on an $(d-1)$-dimensional manifold in $\mathbb{R}^d$) can be managed with the computer codes developed for the Laplace operator in a Cartesian coordinate system.

The algebraic version of the iteration algorithm is then:

Iteration algorithm (*algebraic version*)

For $n = 1, 2, \ldots, N$ *do:*

$\widehat{\mathbf{W}}^0 = \widehat{\mathbf{U}}^{n-1}$

for $k = 1, 2, \ldots$ *do:*

calculate

$$\widehat{Z}^{n,k-1} \in \beta(\widehat{x}, \widehat{W}^{k-1}), \widehat{Z}^{n,k-1} \in \widehat{V}_h$$

and

$$\widehat{\mathbf{Z}}^{n,k-1} = \int_{\mathcal{M}_h} \widehat{S}^n \widehat{Z}^{n,k-1} \widehat{v}_h dA_h \quad \forall \widehat{v}_h \in \widehat{V}_h,$$

then solve

$$\widehat{\mathbf{W}}^k = \mathbf{M}_1 \widehat{\mathbf{U}}^{n-1} + \Delta t (Q \widehat{\mathbf{Z}}^{n,k-1} + \mathbf{F}^n) - \Delta t C \mathbf{L}. \tag{13}$$

Stop when

$$\frac{\left\| \widehat{\mathbf{W}}^k - \widehat{\mathbf{W}}^{k-1} \right\|_{l^2}}{\left\| \widehat{\mathbf{W}}^0 \right\|_{l^2}} \leq Tol$$

and set

$$\begin{cases} \widehat{W}^k(\widehat{x}) = \sum_{l=1}^M \widehat{W}_l^k \alpha_l(\widehat{x}) \\ \widehat{\mathbf{U}}^n = \widehat{\mathbf{W}}^k \end{cases} \tag{14}$$

5. *Numerical Experiments*

Starting with an initial condition that we may consider representative of the present climate temperature, we shall run our model to predict the seasonal evolution of the surface temperature as well as the influence of the concentration of CO_2 on the increase of such a temperature. All the numerical experiments are performed under the hypothesis of *average sky* and with the co-albedo coefficients a_0, a_1 and a_2 being piecewise monthly constants; the values of which are borrowed from Table 1 of GRAVES *et al.* (1993) .

The initial condition is obtained by averaging for every month of the year the surface temperature data given by the general circulation model HIRLAM from the year 1950 up to the year 2000. Figure 2 represents the distribution of the initial temperature which corresponds to December. The computational mesh consists of 20480 triangles and 10242 mesh points, which means an average $h = 0.0431$ rads $\simeq 260$ Kms. We calculate the numerical solution taking a time step length $\Delta t = 0.01 = 3.6$ days, and solving (13) with a tolerance of 0.001. Since $S(t, x)$ depends periodically on time with a period of one year, then after an initial transient state the solution of the model will also be periodic because the coefficients of our model do not depend on time (e.g., BADII and DIAZ 1999). This can be seen in Figure 7 where we represent the evolution of the temperature at a point near Madrid (Spain) under different concentrations of CO_2 in the atmosphere, see equation (15) below.

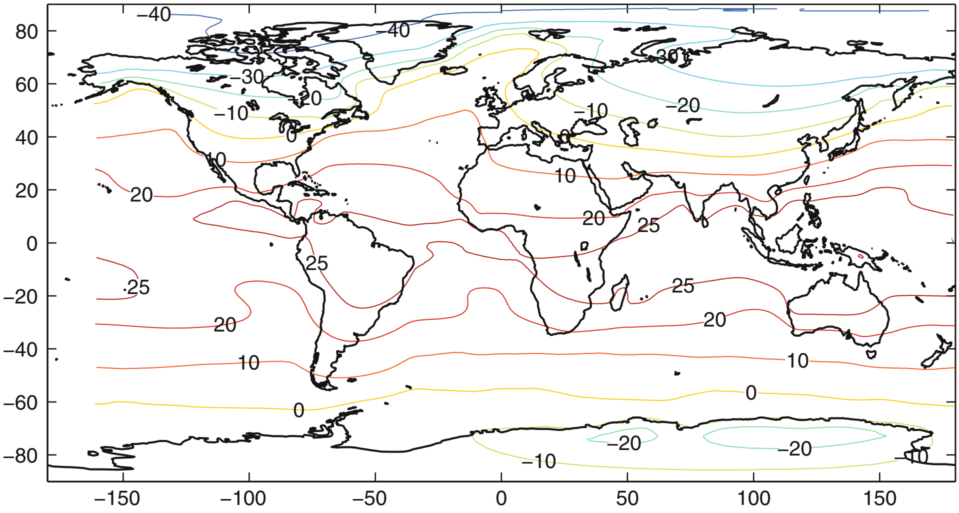

Figure 2
Distribution of temperature at time $t = 0$.

We have noted that the transient period of the model, also known as the spin-up period, is about 9 years. After this, the solution becomes periodic with a period of about 1 year as long time numerical experiments (40 years) have shown. It seems that this periodic state is stable for the parameters used in our calculations. This is the reason we

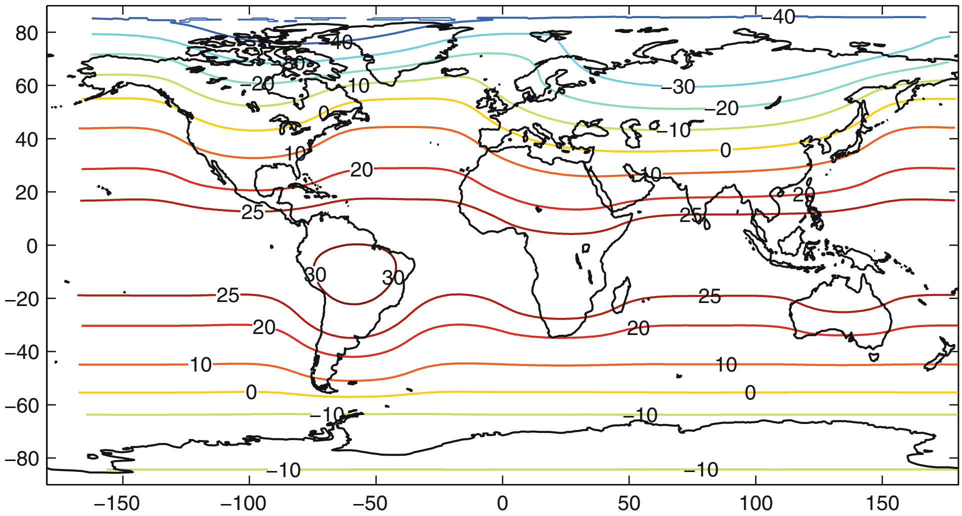

Figure 3
Distribution of January average temperature.

have presented the results for 10 years of simulations. Figure 3 shows the distribution of January temperature in the stationary periodic regime. Figure 4 displays the $-2°C$ snow lines for the Northern and Southern Hemispheres in January.

Figures 5 shows the distribution of temperature for the month of July, whereas the snow line for this month in both hemispheres is represented in Figure 6.

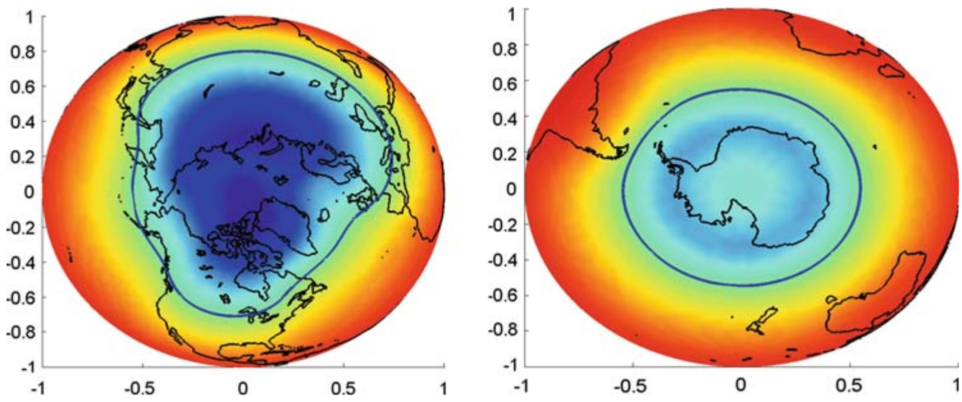

Figure 4
$-$ 2°C January snow line. Left: Northern Hemisphere; right: Southern Hemisphere.

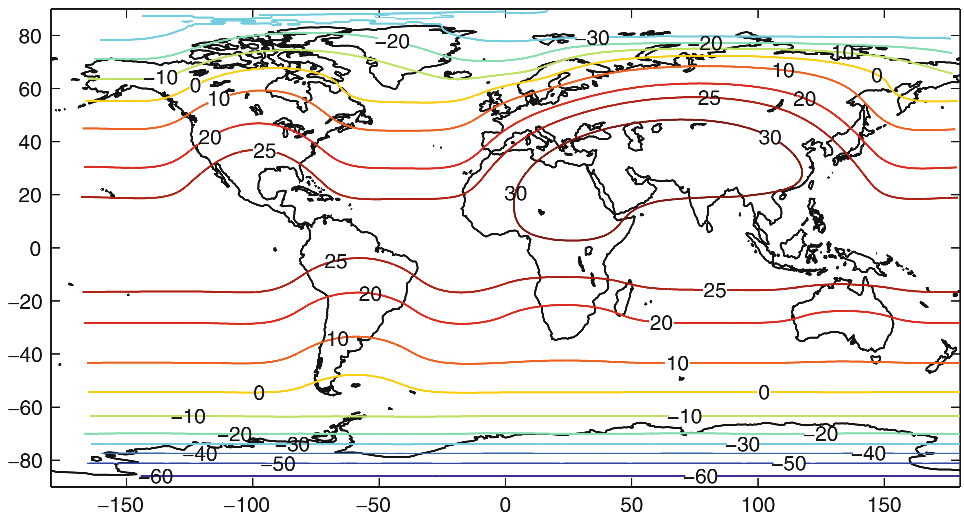

Figure 5
Distribution of average temperature for July.

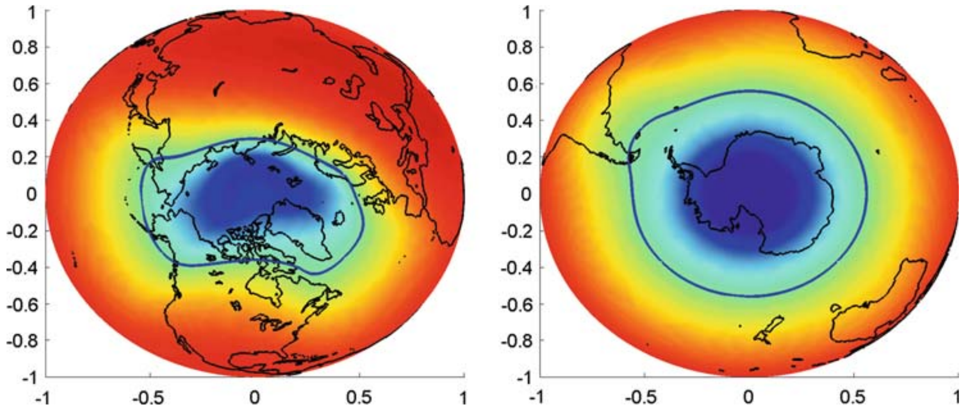

Figure 6
– 2°C July snow line. Left: Northern Hemisphere; right: Southern Hemisphere.

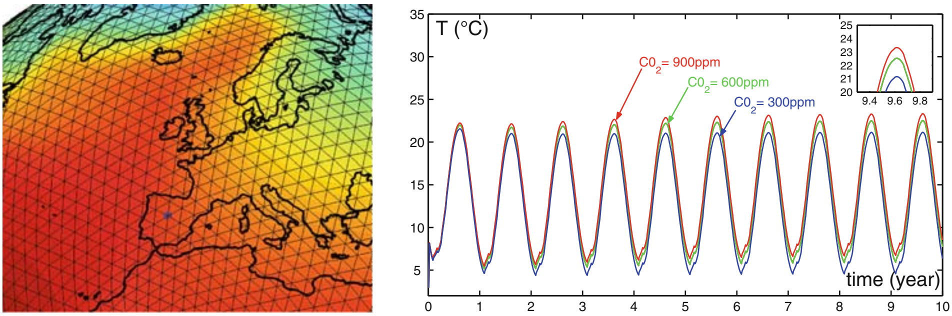

Figure 7
CO_2 influence on temperature at a point near Madrid. The box, showns the temperature corresponding to the month of July.

One can also simulate with our simple EBM the influence of CO_2 on the increase of temperature. We do so by considering that the concentration of CO_2 plays the role of an a additional forcing term $f(t,x)$ in the governing equation. Following Myhre *et al.* (1998) we model such a forcing as

$$f(t,x) = 5.35ln\left(\frac{C}{C_0}\right)\beta(x,u), \tag{15}$$

where $C_0 = 300$ ppm represents the concentration of CO_2 of preindustrial times and C is the value of concentration of CO_2 different of 300. Figure 7 displays the influence of the concentration of CO_2 on the temperature at a point near Madrid (Spain). We note that doubling the levels of CO_2 will produce an increase in the July and January average temperatures larger than 1.5°C.

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