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Mathematical and numerical analysis of a nonlinear diffusive climate energy balance model

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ABSTRACT

The purpose of this paper is to carry out the mathematical and numerical analysis of a two-dimensional nonlinear parabolic problem on a compact Riemannian manifold without boundary, which arises in the energy balance for the averaged surface temperature. We use a possibly quasi-linear diffusion operator suggested by P.H. Stone in 1972. The modelling of the Budyko discontinuous coalbedo is formulated in terms of a bounded maximal monotone graph of $\mathbb{R}^2$. The existence of global solutions is proved by applying a fixed point argument. Since the uniqueness of solutions may fail for the case of discontinuous coalbedo, we introduce the notion of non-degenerate solutions and show that the problem has at most one solution in this class of functions. The numerical analysis is carried out for the special case of a spherical Earth and uses quasi-uniform spherical triangles as finite elements. We study the existence, uniqueness and stability of the approximate solutions. We also show results of some long-term numerical experiments.

1. Introduction

We present the mathematical and numerical analysis of a climate diagnostic model that takes as climate indicator the atmospheric sea-level temperature. Such a model belongs to the category of global energy balance models introduced independently by Budyko [7] and Sellers [27] in 1969 to study the influence of certain geophysical mechanisms on the Earth climate. A detailed derivation of the averaged balance equation, involving possibly memory terms, can be found, for instance, in Díaz and Hetzer [15]. Nevertheless, in the present paper we shall deal with a simplified model avoiding such nonlocal terms. Due to that, the nonlinear partial differential equation of our model can be presented by means of a simplified modelling argument. Roughly speaking, the energy balance on the Earth surface is established according to the following law

$$\text{Variation of internal energy} = R_a - R_e + D, \tag{1}$$

where R_a denotes the amount of solar energy absorbed by the Earth, R_e is the amount of energy radiated to the space and D is a term which represents the diffusion of heat energy by atmospheric turbulence. Let $u(t, x)$ be the atmospheric sea-level temperature in Kelvin degrees, i.e.$u(t, x)$ is defined on $[0, T) \times \mathcal{M}$, where $\mathcal{M}$ is a compact Riemannian manifold without boundary approximating the Earth surface. Under suitable conditions, the variation of internal energy can be expressed as

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$c(x)\frac{\partial u}{\partial t}$, where $c(x)$ is the heat capacity (we neglect the possible time dependence of c). The constitutive assumptions for the terms on the right-hand side of (1) are the following:

$$R_a = QS(t,x)\beta(x,u), \tag{2}$$

where Q is the so-called solar constant which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, Q is currently believed to be $Q = \frac{1}{4}(1370\ \mathrm{Wm}^{-2} \pm 2\ \mathrm{Wm}^{-2})$, the function $S(t,x)$ is known as the insolation function given by the distribution of incident solar radiation at the top of the atmosphere. When the averaging time is of the order of one year or longer, there exists a constant $S_0 > 0$ such that for all t and x, $S(t,x) \geq S_0$. The term $\beta(x,u)$ is the so-called coalbedo function that takes values between 0 and 1. $\beta(x,u)$ represents the ratio between the absorbed solar energy and the incident solar energy at the point x on the Earth surface. Obviously, $\beta(x,u)$ depends on the nature of the Earth surface. For instance, it is well known that on ice sheets $\beta(x,u)$ is much smaller than that on the ocean surface because the white color of the ice sheets reflects a large portion of the incident solar energy, whereas the ocean, due to its dark color and high heat capacity, is able to absorb a larger amount of the incident solar energy. In our model, $\beta(x,u)$ is given by a nonlinear discontinuous function as proposed by Budyko [7],

$$\beta(x,u) = \begin{cases} \beta_i & \text{for } u < u_s, \\ \beta_w & \text{for } u > u_s. \end{cases} \tag{3}$$

Here u_s denotes the assumed "ice margin" temperature, β_i is the coalbedo value for ice regions, and β_w is the value for the rest of ice free surfaces. We point out that Sellers proposed in [27] the modelling of the coalbedo function as a continuous function (even piecewise differentiable with respect to u) reaching the above values when $u < u_s - \varepsilon$ and $u > u_s + \varepsilon$, respectively, for some small $\varepsilon > 0$.

The term $R_e(u)$ was modelled by Budyko by performing a linear regression fitting to empirical data as

$$R_e(u) = Bu + C, \tag{4}$$

where B and C are given constants. On the contrary, Sellers suggested in [27] that R_e must be expressed according to the Stefan–Boltzmann law $R_e = \sigma u^4$, where σ is called emissivity constant.

As for the diffusion term D, independently of linear diffusion operators (see, e.g., [19]), P.H. Stone proposed in [28] that a better way to account for the effect of large scale atmospheric circulation is an eddy diffusive approximation such as

$$D(u) = \mathrm{div}(k(x,u,\nabla u)\nabla u),$$

where $k(x,u,\nabla u)$ is a non linear eddy diffusion coefficient. In particular, he proposed the expression $k = b(x)|\nabla u|$. In our model, we generalize Stone's approach to represent the eddy diffusive terms by setting $k(x,u,\nabla u) = k(x)|\nabla u|^{p-2}$, with $p \geq 2$ and $k(x) > k_0 > 0$. This allows us to unifying the results concerning both the linear diffusion case ($p = 2$), proposed by Budyko and Sellers, and the nonlinear diffusion ($p = 3$) proposed by Stone.

By substituting the above expressions into (1) we obtain the following energy balance model:

$$(P)\begin{cases} c(x)u_t - \mathrm{div}(k(x)|\nabla u|^{p-2}\nabla u) + \mathcal{G}(x,u) \in QS(t,x)\beta(x,u) + f(t,x) & \text{in } (0,T)\times\mathcal{M} \\ u(0,x) = u_0(x) & \text{on } \mathcal{M}, \end{cases}$$

where the initial datum u_0 will be always assumed to be bounded. Here, we have taken $R_e(u) = \mathcal{G}(x,u) - f(t,x)$, where $\mathcal{G}(x,u)$ is a strictly increasing function (which includes the two alternative choices mentioned above) in u and $f(t,x)$ is a forcing term. More precise structural assumptions to solve (P) are formulated in Section 2. A special feature of (P) is that the presence of the coalbedo function $\beta(x,u)$ may be responsible, in the case of a discontinuous function, of both the existence of a free boundary at the level set $u = u_s$ and multiple solutions for certain initial conditions (even if the problem is formulated in terms of a parabolic type equation).

The progress of the mathematical analysis for problem (P) was function of the different assumptions made on the spatial domain and the nonlinear terms involved in the equation. Among the many results that have appeared in the literature we mention here, specially, the ones concerning with discontinuous coalbedo functions due to Xu [31] and Díaz [12] for the one-dimensional case. The analysis of Díaz [12] was extended to two dimensions, but with $c(x) \equiv 1$, in Dí az and Tello [18]. Hetzer [20] considered a two-dimensional Sellers model that corresponds to a formulation of (P) in which $p = 2$, $\beta(x,u)$ is locally Lipschitz and the radiation term R_e is expressed by the Stefan–Boltzmann law. Many other references can be found in Díaz [13]. More recently, Díaz, Hetzer and Tello [16] have considered an energy balance model with hysteresis.

As for works on the numerical approximation of (P) we mention the contributions of North and Coakley [24] and Hetzer, Jarausch and Mackens [21], where some numerical experiences were carried out. Here, we make a more theoretical approach trying to obtain some optimal error estimates. In doing so, we point out that some important difficulties are posed by the presence of the nonlinear terms $\mathrm{div}(k|\nabla u|^{p-2}\nabla u)$ and $\beta(x,u)$ on the left- and right-hand sides of (P), respectively. When $p > 2$, it is well known that, in general, the possible solution does not belong to $W^{2,p}(\Omega)$ (see [8] for an illustrative example). So that, this is a barrier to achieve optimality in the space error estimate when one uses linear finite elements. However, inspired by the techniques of Rulla [25] (see also Savaré [26] and Lippold [23] for related analysis to compute optimal error estimates in time in evolution inequalities), we have been able to obtain an optimal time error estimate under mild regularity assumptions. Our estimate improves previous time estimates for the parabolic p-Laplacian for $p > 2$ obtained by Barret and Liu [2] and Wei [30].

The layout of the paper is as follows. For the sake of completeness, we introduce in Section 2 some notations and preliminaries of the analysis on manifolds and state the theorems on the existence and uniqueness of weak solutions of the model. The numerical formulation of (P) is carried out for the special case of a spherical Earth and uses quasi-uniform spherical triangles as finite elements. The study of the existence, uniqueness and stability of the approximate solutions is presented in Sections 3–5 respectively. We mention that the results obtained in these sections seem to be new even for the linear diffusion case $p = 2$. The detailed proof of the results stated in Section 2 is given in Section 6. This extends, in different ways, the previous results of [18]. Finally, Section 7 contains some numerical experiences.

2. On the existence and uniqueness of solutions

We recall the expression of the diffusion operator $D(u)$ in $\mathcal{M}$. To do so we recall some basic concepts of Differential Geometry following the monograph of Aubin [1]. Given an index set Λ and $\lambda \in \Lambda$, let W_λ be an open subset of $\mathcal{M}$ such that $\{W_\lambda\}_{\lambda\in\Lambda}$ is an open covering of $\mathcal{M}$ and $w_\lambda : W_\lambda \to w_\lambda(W_\lambda) \subset \mathbb{R}^2$ a homeomorphism. For $\lambda \in \Lambda$, the pair $\{W_\lambda, w_\lambda\}$ is called a chart of $\mathcal{M}$ and the family of charts $\{W_\lambda, w_\lambda\}_{\lambda\in\Lambda}$ is called an atlas of $\mathcal{M}$.

Given a point $P \in W_\lambda \subset \mathcal{M}$, we set $w_\lambda(P) = (w^1_\lambda(P), w^2_\lambda(P)) = (\theta_\lambda, \varphi_\lambda) \in \mathbb{R}^2$. The tangent space at P is denoted by $T_P\mathcal{M}$. $T_P\mathcal{M}$ is a vector space of dimension 2 with a basis formed by the vectors $\mathbf{e_1} := \frac{\partial}{\partial\theta_\lambda}$, $\mathbf{e_2} := \frac{\partial}{\partial\varphi_\lambda}$. The tangent bundle $T\mathcal{M}$ is defined by $T\mathcal{M} := \cup_{p\in\mathcal{M}} T_P\mathcal{M}$. A Riemannian metric g on $\mathcal{M}$ is defined from a family of scalar products $g_P : T_P\mathcal{M} \times T_P\mathcal{M} \to \mathbb{R}$.

Let $(\theta_\lambda, \varphi_\lambda)$ be the coordinate framework in $w_\lambda(W_\lambda) \subset \mathbb{R}^2$ and let α_λ be a partition of unity subordinate to the covering W_λ. Then, we assume that $g = \sum \alpha_\lambda g^\lambda$ is a Riemannian metric on $\mathcal{M}$ with g^λ defined over each local chart. Given $p \in W_\lambda \subset \mathcal{M}$, the set $\{\mathbf{e_1} := \frac{\partial}{\partial\theta_\lambda}, \mathbf{e_2} := \frac{\partial}{\partial\varphi_\lambda}\}$ is a basis for the tangent space $T_p\mathcal{M}$. For a differentiable function $u : \mathcal{M} \to \mathbb{R}$ we define $\mathrm{grad}_\mathcal{M} u \in T_p\mathcal{M}$ by

$$\mathrm{grad}_\mathcal{M} u = g^{ij}\frac{\partial u}{\partial y_j}\mathbf{e_i},$$

where g^{ij} are the elements of the inverse matrix of (g_{ij}), $y_1 = \theta$ and $y_2 = \varphi$. Let $X : \mathcal{M} \to T\mathcal{M}$, the divergence of X is defined as

$$\mathrm{div}_\mathcal{M} X = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial y_i}\left(X^i\sqrt{\det g}\right).$$

Finally, given a bounded and strictly positive function $k(x)$ and $u : \mathcal{M} \to \mathbb{R}$, the diffusion operator $D(u)$ is defined by

$$D(u) = \mathrm{div}(k(x)|\nabla u|^{p-2}\nabla u) = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial y_i}\left(\sqrt{\det g}\,k\left|g^{kl}\frac{\partial u}{\partial y_l}\mathbf{e_k}\right|^{p-2} g^{ij}\frac{\partial u}{\partial y_j}\right),$$

here $y_1 = \theta_\lambda, y_2 = \varphi_\lambda$, $|\cdot| = g(\cdot,\cdot)^{\frac{1}{2}}$, and g^{ij} are the coefficients of the inverse matrix of $g^\lambda = (g_{ij})$. For $p = 2$ and $k(x) \equiv 1$ the above expression coincides with the Laplace–Beltrami operator,

$$\Delta u = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial y_i}\left(g^{ij}\sqrt{\det g}\frac{\partial u}{\partial y_j}\right).$$

As usual (see e.g. Aubin [1] and Chavel [9]), given $p > 1$ we denote by $L^p(\mathcal{M})$ the set $\{u : \mathcal{M} \to \mathbb{R}$ measurable : $\int_\mathcal{M} |u|^p \mathrm{d}A < \infty\}$ where $\mathrm{d}A = \sum_{\lambda\in\Lambda} \alpha_\lambda \sqrt{\det g^\lambda}\mathrm{d}\theta_\lambda \mathrm{d}\varphi_\lambda$. This set is a Banach space with the norm

$$\left(\int_\mathcal{M} |u|^p \mathrm{d}A\right)^{\frac{1}{p}} = \left(\sum_{\lambda\in\Lambda}\int_{w_\lambda(W_\lambda)} \alpha_\lambda |u(w_\lambda^{-1}(\theta_\lambda,\varphi_\lambda))|^p\sqrt{\det g^\lambda}\mathrm{d}\theta_\lambda\mathrm{d}\varphi_\lambda\right)^{\frac{1}{p}}.$$

Analogously,

$$L^p(T\mathcal{M}) = \left\{X : \mathcal{M} \to T\mathcal{M} \text{ measurable} : \int_\mathcal{M} < X, X >^{\frac{p}{2}} \mathrm{d}A < \infty\right\},$$

where $<,>$ denotes the inner product in the tangent space. We introduce the functional space

$$V = \{u \in L^2(\mathcal{M}) : \nabla u \in L^p(T\mathcal{M})\}, \quad p \geq 2,$$

which is a Banach space with the usual norm.

Given a bounded and strictly positive function $c(x)$, $p \geq 2$ and $Q > 0$, we consider the problem

$$(P)\begin{cases} c(x)u_t - \mathrm{div}(k(x)|\nabla u|^{p-2}\nabla u) + \mathcal{G}(x,u) \in QS(t,x)\beta(x,u) + f(t,x) & \text{in } (0,T)\times\mathcal{M}, \\ u(0,x) = u_0(x) \quad \text{on } \mathcal{M}, \end{cases}$$

under the following assumptions:

($\mathrm{H_G}$) $\mathcal{G} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and strictly increasing for the argument $s \in \mathbb{R}$, such that $|\mathcal{G}(x,s)| \geq C|s|^r$ for some $r \geq 1$ and constant $C > 0$,

(H_β) $\beta(x,\cdot)$ is a bounded maximal monotone graph of $\mathbb{R}^2$,
(H_f) $f \in L^\infty((0,T)\times \mathcal{M})$,
(H_M) $\mathcal{M}$ is a C^∞ 2-D connected compact oriented Riemannian manifold without boundary. The gradient and divergence operators defined on $\mathcal{M}$ are to be understood in the sense of the Riemannian metric over $\mathcal{M}$,
(H_S) $S : [0,T]\times\mathcal{M}\to\mathbb{R}$, $S\in C^1([0,T]\times\mathcal{M})$, $0 < S_0 \le S(t,x) \le S_1$ *a.e.* $x\in\mathcal{M}$, for any $t\in[0,T]$,
(H_c) $c \in L^\infty(\mathcal{M})$, $c(x) \ge c_0 > 0$,
(H_k) $k \in C(\mathcal{M})$, $k(x) \ge k_0 > 0$,
(H_0) $u_0 \in L^\infty(\mathcal{M})$.

Note that $\mathcal{G}(s) = Cs$ corresponds to Budyko model [7], whereas $\mathcal{G}(s) = C|s|^3 s$ corresponds to Sellers model [27].

We do not expect the existence of classical solutions to (P) due to the possible discontinuity of the coalbedo function and the degeneracy of the diffusion operator. For this reason, we need the notion of weak solution to (P).

Definition 1. A function $u \in C([0,T];L^2(\mathcal{M})) \cap L^\infty((0,T)\times(\mathcal{M})) \cap L^p(0,T;V)$ is termed a bounded weak solution of (P) if there exists $z \in L^\infty((0,T)\times\mathcal{M})$, $z(t,x)\in\beta(x,u(t,x))$ a.e. $(t,x)\in(0,T)\times\mathcal{M}$ such that

$$\int_{\mathcal{M}} c(x)u(T,x)v(T,x)\mathrm{d}A - \int_0^T\int_{\mathcal{M}} c(x)v_t(t,x)u(t,x)\mathrm{d}A\mathrm{d}t + \int_0^T\int_{\mathcal{M}}\langle k(x)|\nabla u|^{p-2}\nabla u, \nabla v\rangle \mathrm{d}A\mathrm{d}t$$
$$+ \int_0^T\int_{\mathcal{M}}\mathcal{G}(u)v\mathrm{d}A\mathrm{d}t = \int_0^T\int_{\mathcal{M}} QS(t,x)z(t,x)v\mathrm{d}A\mathrm{d}t + \int_0^T\int_{\mathcal{M}} fv\mathrm{d}A\mathrm{d}t + \int_{\mathcal{M}} c(x)u_0(x)v(0,x)\mathrm{d}A, \tag{5}$$

$\forall v \in L^p(0,T;V)$ such that $v_t \in L^{p'}(0,T;V')$.

The main results on the existence and uniqueness of bounded weak solutions to problem (P) are collected in Theorems 1 and 2, the proofs of which are presented in Section 6.

Theorem 1. *Under the above assumptions there exists at least one bounded weak solution of (P). Moreover, if $u_0 \in V$ then $u_t \in L^2(0,T;L^2(\mathcal{M}))$ and $\mathrm{div}(k(x)|\nabla u|^{p-2}\nabla u) \in L^2(0,T;L^2(\mathcal{M}))$.* □

Remark 1. We point out that from the last assertion of Theorem 1 we can conclude that $u \in L^2(0,T;W^{1+s,p}(\mathcal{M}))$ for some real s, $0 < s < 1$ (see the references in chapter 4 of the monograph Díaz [11]).

Since $\beta(x,u)$ is considered to be a multi-valued graph, then there are cases for which problem (P), although parabolic, has not a unique solution. Nevertheless, we shall prove the uniqueness of the bounded weak solution to (P) in the class of non-degenerate functions which is introduced next.

Definition 2. Let $u \in L^\infty(\mathcal{M})$. Given ϵ_0, $0 < \epsilon_0 < 1$, for $\epsilon \in (0,\epsilon_0)$ let

$$B_s(u,u_s;\epsilon) = \{x\in\mathcal{M} : |u-u_s| < \epsilon\}$$

and

$$B_w(u,u_s;\epsilon) = \{x\in\mathcal{M} : 0 < |u-u_s| < \epsilon\}.$$

It is said that u is a non-degenerate function in a strong (resp. weak) sense if it satisfies the following strong (resp. weak) non-degeneracy property: there exists a constant $C > 0$ such that for any $\epsilon \in (0,\epsilon_0)$

$$\mathrm{meas}(B_s(u,u_s;\epsilon)) \le C\epsilon \qquad (\text{resp. } \mathrm{meas}(B_w(u,u_s;\epsilon)) \le C\epsilon).$$

In Section 6 we shall prove the following result concerning the uniqueness of non-degenerate solutions:

Theorem 2. *Let $u_0 \in L^\infty(\mathcal{M})$. Then:*

(i) *If a bounded weak solution $u(t)$ to (P) is a strong non-degenerate function for all $t \in [0,T]$, then u is the unique bounded weak solution to (P).*
(ii) *For any $t \in (0,T]$ there is at most one bounded weak solution $u(t)$ to (P) in the class of weak non-degenerate functions.*

3. On the numerical approximation: Preliminaries

We now proceed to formulate and analyze a numerical method to compute the bounded weak solution to problem (P). This method consists of a combination of C^0- finite elements for space discretization with a first-order Euler implicit scheme to discretize in time. Hereafter we shall assume that $\mathcal{M}$ is the 2-sphere of radius $a = 1$ that is partitioned into quasi-uniform spherical triangles. A simple method to construct this partition, introduced by Baumgardner et al. [3], consists of taking as the initial partition D_0 the spherical icosahedron and then to generate a sequence of partitions $D_k,\ k = 1, 2, \ldots,$ by joining the midpoints on the sides of the triangles of the partition D_{k-1}. This procedure yields triangles with the following properties [4]: Let N_k be the number of triangles in the partition D_k, then (a)$\mathcal{M} = \cup_{j=1}^{N_k} T_j,\ T_j \subset \mathcal{M}$; (b) for $i \neq j$, $T_i \cap T_i$ is either empty or has one vertex x_p, or T_i and T_j share a common edge γ_{ij}; (c) there exist positive constants ν and μ such that for all T_j, $\frac{h}{h_j} \leq \nu$ and $\frac{h_j}{\rho_j} < \mu$, where h_j denotes the diameter of T_j and ρ_j is the diameter of the largest circle inscribed in T_j given in radians, and $h = \max(h_j)$.

Following the approach of [17] to solve by finite elements the Poisson equation on manifolds, it is convenient to view the spherical triangles of the partition D_k of $\mathcal{M}$ as the radial projection onto $\mathcal{M}$ of 2-simplices $\Omega_j \subset \mathbb{R}^3$, such that if T_j is the image of Ω_j, then for all j, $T_j \cap \Omega_j = \{x_{1j}, x_{2j}, x_{3j}\}$, where x_{ij}, $i = 1, 2, 3$, are the vertices of both T_j and Ω_j. By analogy with the elements T_j, the simplices Ω_j form a partition D_{hk} of a polyhedron $\mathcal{M}_h$ such that

$$\mathcal{M}_h := \cup_j \Omega_j, \quad \Omega_j \in D_{hk}.$$

The radial projection is defined as

$$\phi : \mathcal{M}_h \to \mathcal{M}$$
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \to \begin{pmatrix} \dfrac{ax'_1}{\sqrt{(x'_1)^2 + (x'_2)^2 + (z'_3)^2}} \\ \dfrac{ax'_2}{\sqrt{(x'_1)^2 + (x'_2)^2 + (x'_3)^2}} \\ \dfrac{ax'_3}{\sqrt{(x'_1)^2 + (x'_2)^2 + (x'_3)^2}}. \end{pmatrix}$$

So that, we write

$$\mathcal{M} = \cup_j \phi(\Omega_j)$$

and denote the restriction of ϕ on the element Ω_j by ϕ_j. Note that ϕ is a C^m diffeomorphism, $m \geq 1$. We define the family of finite element spaces associated to the partitions D_{hk}.

$$\widehat{V}_h = \{\widehat{v}_h \in C^0(\mathcal{M}_h) : \widehat{v}_h|_{\Omega_j} \in P_1(\Omega_j),\ 1 \leq j \leq N_k\},$$

where $P_1(\Omega_j)$ is the set of polynomials of degree ≤ 1 defined on Ω_j. Let M be the global number of vertices in the partition D_k, and let $\{\alpha_l\}_{l=1}^{M}$ be the set of global basis functions for $\widehat{V}_h$, such that $\alpha_l \in \widehat{V}_h$ and at the vertex x_j $\alpha_l(x_j) = \delta_{jl}$; any $\widehat{v} \in \widehat{V}_h$ can be expressed as

$$\widehat{v}_h(x) = \sum_{l=1}^{M} \widehat{v}_h(x_l)\alpha_l(x).$$

We define a finite element space $V_h \subset W^{1,p}(\mathcal{M})$ associated to the partition D_k via the radial $\phi-$lifting as follows:

$$V_h = \{v_h \in C^0(\mathcal{M}) : v_h|_{T_j} = \widehat{v}_h \circ \phi_j^{-1} \text{with}\ \widehat{v}_h \in \widehat{V}_h\}.$$

The approximation spaces V_h and $\widehat{V}_h$ satisfy:

Property 1. *For all $u \in L^\infty((0, T) \times \mathcal{M}) \bigcap L^p(0, T; V)$, $u_t \in L^2((0, T) \times \mathcal{M})$*

$$\lim_{h\to 0} \inf_{u_h \in V_h} \|u - u_h\|_{L^\infty((0,T)\times\mathcal{M})} = 0.$$

Moreover, from computational and numerical analysis points of view (see Section 7 for further details) it is convenient to define the spaces $W^{l,p}(\mathcal{M}_h),\ l \geq 0$ and $1 \leq p \leq \infty$ (with the convention that for $l = 0,\ W^{l,p}(\mathcal{M}_h) \equiv L^p(\mathcal{M}_h)$) as

$$W^{l,p}(\mathcal{M}_h) = \{\widehat{v} : \mathcal{M}_h \to \mathbb{R} : \text{for a.e.}\ x \in \mathcal{M} \quad \text{and} \quad v \in W^{l,p}(\mathcal{M}), \quad \widehat{v} \circ \phi^{-1}(x) = v(x)\}.$$

In relation with the radial projection ϕ defined on $\mathcal{M}_h$ we have the following result.

Proposition 1. *Let J_{ϕ_j} and $J_{\phi_j^{-1}}$ denote the absolute values of the Jacobian determinants of the mappings ϕ_j and ϕ_j^{-1} respectively. Then, for h sufficiently small there exist constants C_1 and C_2 independent of h such that*

$$\max_j \|J_{\phi_j} - 1\|_{L^\infty(\Omega_j)} \leq C_1 h^2 \ \textit{and} \ \max_j \|J_{\phi_{j^{-1}}} - 1\|_{L^\infty(T_j)} \leq C_2 h^2.$$

Proof. Let $T_j \subset \mathcal{M} = \phi_j(\Omega_j)$, $\Omega_j \subset \mathcal{M}_h$, and let $P^* \in T_j = \phi_j(P)$, $P \in \Omega_j$. It follows from elemental geometrical considerations that $\mathrm{d}A_h = J_{\phi_j}\mathrm{d}A = \cos\delta \mathrm{d}A$, where δ is the angle between the vector radius OP^* and the unit outward normal vector $\overrightarrow{n}_{jh}$ to Ω_j at P. Furthermore, $\delta \le diam(T_j) \le h$, so that $\|J_{\phi_j} - 1\|_{L^\infty(\Omega_j)} \le \max|\cos\delta - 1| \le C_1 h^2$. Analogously, $\mathrm{d}A = J_{\phi_{j-1}}\mathrm{d}A_h = \frac{\mathrm{d}A_h}{\cos\delta}$. Therefore, $\|J_{\phi_j^{-1}} - 1\|_{L^\infty(T_j)} \le C_2 h^2$. $\square$

Using this result and the arguments of the proof of Lemma 3 of [17] it is easy to show the following inequalities.

Proposition 2. *For $1 \le p \le \infty$ there exist constants c_1 and c_2 independent of h such that*

$$c_1 \|\widehat{v}\|_{L^p(\mathcal{M}_h)} \le \|v\|_{L^p(\mathcal{M})} \le c_2 \|\widehat{v}\|_{L^p(\mathcal{M}_h)},$$
$$c_1 \|\widehat{v}\|_{W^{1,p}(\mathcal{M}_h)} \le \|v\|_{W^{1p}(\mathcal{M})} \le c_2 \|\widehat{v}\|_{W^{1,p}(\mathcal{M}_h)},$$
$$|\widehat{v}|_{W^{2,p}(\mathcal{M}_h)} \le c_2 \left(|v|_{W^{2,p}(\mathcal{M})} + h\,|v|_{W^{1,p}(\mathcal{M})}\right).$$

The relevance of these results, in particular Proposition 2, lies in the fact that by virtue of it the approximation error in the family of finite element spaces $\widehat{V}_h$ is of the same order as the error in the family of spaces V_h associated to the partition D_k of spherical triangles. In terms of the numerical calculations this means that one can substitute the spherical triangles (curved triangles) by plane triangles in $\mathbb{R}^3$; and, therefore, make use of the finite element technology for plane triangles. To estimate the error of the numerical solution we shall use the linear interpolation operators $\widehat{I}_h : W^{1+s,p}(\mathcal{M}_h) \to \widehat{V}_h$ and $I_h : W^{1+s,p}(\mathcal{M}) \to V_h$, which by virtue of the compact imbeddings $W^{1+s,p}(\mathcal{M}_h) \hookrightarrow C^0(\mathcal{M}_h)$ and $W^{1+s,p}(\mathcal{M}) \hookrightarrow C^0(\mathcal{M})$, respectively, for $p \ge 2$, are well defined by $\widehat{I}_h u(x_i) = I_h u(x_i) = u(x_i)$, x_i being a mesh point of $\mathcal{M}$ and $\mathcal{M}_h$. Next, let $u \in W^{1+s,p}(\mathcal{M})$, and consequently $\widehat{u} \in W^{1+s,p}(\mathcal{M}_h)$, on the account that $W^{1+s,p}(\mathcal{M}) = \left[W^{1,p}(\mathcal{M}), W^{2,p}(\mathcal{M})\right]_{s,p}$, respectively $W^{1+s,p}(\mathcal{M}_h) = \left[W^{1,p}(\mathcal{M}_h), W^{2,p}(\mathcal{M}_h)\right]_{s,p}$, the interpolation theory in Sobolev spaces of integer order [8] together with the results of linear operator interpolation theory in Banach spaces [5] yields

$$\|\widehat{u} - I_h\widehat{u}\|_{L^q(\mathcal{M}_h)} \le Ch^{d(\frac{1}{q}-\frac{1}{p})+1+s}\,|\widehat{u}|_{W^{1+s,p}(\mathcal{M}_h)},$$

where d denotes the dimension of the space (here $d = 2$) and $|\cdot|_{W^{1+s,p}(\mathcal{M}_h)}$ is the seminorm in $W^{1+s,p}(\mathcal{M}_h)$ [8]. Then, by virtue of Proposition 2 it follows that

$$\|u - I_h u\|_{L^q(\mathcal{M})} \le Ch^{d(\frac{1}{q}-\frac{1}{p})+1+s}\,|u|_{W^{1+s,p}(\mathcal{M})}. \tag{6}$$

Since the numerical solution to problem (P) is computed at a discrete set of time instants t_n, with $n = 0, 1, \ldots, N$, we choose a fixed time step Δt, such that for all n, $t_{n+1} = t_n + \Delta t$, and consider the discrete set $I_N = \{0, t_1, t_2, \ldots, t_N = T\}$. A map $U : I_N \to V_h$ is the numerical solution to (P) if there exists $Z^n \in L^\infty(\mathcal{M})$, $Z^n \in \beta(x, U^n)$, such that that for any $v_h \in V_h$

$$(P_{h,\Delta t})\begin{cases} \displaystyle\int_{\mathcal{M}} c\frac{U^n - U^{n-1}}{\Delta t} v_h \mathrm{d}A + \int_{\mathcal{M}} \left\langle k\left|\nabla U^n\right|^{p-2}\nabla U^n, \nabla v_h\right\rangle \mathrm{d}A \\ \displaystyle\quad + \int_{\mathcal{M}} \mathcal{G}(x, U^n) v_h \mathrm{d}A = \int_{\mathcal{M}} QS^n Z^n v_h \mathrm{d}A + \int_{\mathcal{M}} f^n v_h \mathrm{d}A, \end{cases}$$

wherethe notation $b(x, t_n) = b^n$ is used unless otherwise stated. For numerical analysis purposes it is also convenient to introduce the semidiscrete bounded weak solution to (P) as a map $u_h : [0, T] \to V_h$, $u_h \in L^p(0, T; V_h) \cap C([0, T]; L^2(\mathcal{M})) \cap L^\infty((0, T) \times \mathcal{M})$ such that there exists $z_h \in L^\infty((0, T) \times \mathcal{M})$, $z_h \in \beta(x, u_h)$, verifying that for all $v_h \in V_h$

$$(P_h)\begin{cases} \displaystyle\int_{\mathcal{M}} c u_{ht} v_h \mathrm{d}A + \int_{\mathcal{M}} \langle k|\nabla u_h|^{p-2}\nabla u_h, \nabla v_h\rangle \mathrm{d}A + \int_{\mathcal{M}} \mathcal{G}(x, u_h) v_h \mathrm{d}A = \int_{\mathcal{M}} QS z_h v_h \mathrm{d}A + \int_{\mathcal{M}} f v_h \mathrm{d}A \quad a.e.t \in (0, T], \\ u_h(0) = u_{0h} \end{cases}$$

where u_{0h} is the approximation to u_0 in V_h.

Inspired by the methodology employed in the analysis of the continuous problem (P) we study the existence and uniqueness of the solutions to problems (P_h) and $(P_{h,\Delta t})$ respectively. Therefore, we proceed with the analysis of stability and convergence of the semidiscrete and fully discrete solutions. However, before doing so we need some preliminary results which are stated now. First, adapting the arguments of Lemma 2.2 and Lemma 2.3 of Barrett and Liu [2] to our problem we have the following result.

Lemma 1. (A) *For all $p \in [2, \infty)$ and $\delta \ge 0$, there exist positive constants M_1 and M_2 such that for all $\psi, \mu \in T_P\mathcal{M}$ or $\mathcal{M}$*

$$\left||\psi|^{p-2}\psi - |\mu|^{p-2}\mu\right| \le M_1|\psi - \mu|^{1-\delta}(|\psi| + |\mu|)^{p-2+\delta}, \tag{7}$$

$$g_P\left(|\psi|^{p-2}\psi - |\mu|^{p-2}\mu, \psi - \mu\right) \ge M_2|\psi - \mu|^{2+\delta}(|\psi| + |\mu|)^{p-2-\delta} \tag{8}$$

and

$$|\psi|^p + p g_P\left(|\psi|^{p-2}\psi, \mu - \psi\right) \le |\mu|^p, \tag{9}$$

where $|\cdot| = \sqrt{g_P(\cdot,\cdot)}$.

(B) *For all* $p \in (1, \infty)$ *there exists a constant* k_0 *such that for all* $a, \ \sigma_1, \sigma_2 \geq 0$ *and for all* $k \in (0, k_0)$

$$(a+\sigma_1)^{p-2}\,\sigma_1\sigma_2 \leq k\,(a+\sigma_1)^{p-2}\,\sigma_1^2 + K(k^{-1})\,(a+\sigma_2)^{p-2}\,\sigma_2^2. \tag{10}$$

Second, let the operator $\mathcal{A} : V \to V'$ be defined as

$$\langle \mathcal{A}u, v\rangle_{V'\times V} = \int_{\mathcal{M}} \left\langle k\,|\nabla u|^{p-2}\,\nabla u, \nabla v\right\rangle \mathrm{d}A,$$

the continuity and monotonicity bounds of this operator for $p \geq 2$ are (see for instance Díaz [11] and Chow [10]):

$$\|\mathcal{A}u - \mathcal{A}v\|_{V'} \leq M_1\,\|u-v\|_V\,(\|u\| + \|u\|)^{p-2}$$

and

$$\langle \mathcal{A}u - \mathcal{A}u, u-v\rangle_{V'\times V} \geq M_2\,\|\nabla(u-v)\|^p_{L^p(\mathcal{M})}\,.$$

For a.e. t in $[0, T]$, let $w(t, x) \in L^\infty(\mathcal{M})$ and let w_h be an approximation to w, for instance, the finite element approximation in V_h. If w belongs to the class of non-degenerate functions, either strong or weak, then, for h sufficiently small, its approximation $w_h \in V_h$ also belongs to this class. Specifically, we have the following results. Let $B_s(w, u_s; \epsilon)$ and $B_w(w, u_s; \epsilon)$ be the sets introduced in Section 2, and we consider the level sets

$$\begin{aligned} &A = \{x \in \mathcal{M} : w(t,x) = u_s\}, \qquad A_h = \{x \in \mathcal{M} : w_h(t,x) = u_s\},\\ &M^{\pm} = \{x \in \mathcal{M} : w(t,x) \gtrless u_s\} \quad \text{and} \quad M_h^{\pm} = \{x \in \mathcal{M} : w_h(t,x) \gtrless u_s\}. \end{aligned}$$

We note that $\mathcal{M} = A \cup M^+ \cup M^- = A_h \cup M_h^+ \cup M_h^-$. It is a simple matter to ascertain that for $z \in \beta(x, w)$ and $z_h \in \beta(x, w_h)$ it holds

$$\begin{cases} |z - z_h| \leq |\beta_w - \beta_i| & \text{if } x \in A \cup A_h \cup (M^+ \cap M_h^-) \cup (M^- \cap M_h^+),\\ |z - z_h| = 0 & \text{if } x \in (M^+ \cap M_h^+) \cup (M^- \cap M_h^-). \end{cases}$$

The following lemma states that the non-degeneracy property is also satisfied by the finite element approximation.

Lemma 2. *Given a strong non-degenerate function* $v \in L^\infty((0, T) \times M) \cap L^p(0, T; V)$, $p \geq 2$, *and its approximation* $v_h \in V_h$, *for* h *depending on* ϵ *sufficiently small the relation*

$$A \cup A_h \cup (M^+ \cap M_h^-) \cup (M^- \cap M_h^+) \subset B_s(v, u_s; \epsilon)$$

holds. Consequently, there exists a constant $C > 0$ *such that*

$$meas\left(A \cup A_h \cup (M^+ \cap M_h^-) \cup (M^- \cap M_h^+)\right) \leq C\epsilon.$$

Proof. It is clear that $A \subset B_s(v, u_s; \epsilon)$. From the inequalities

$$v_h - |v_h - v| \leq v \leq v_h + |v_h - v| \quad \text{for a.e. } x \in \mathcal{M}$$

it follows by virtue of Property 1 that $A_h \subset B_s(v, u_s; \epsilon)$ for h depending on ϵ sufficiently small. Next, we have to prove that if $x \in M^+ \cap M_h^-$ then $x \in B_s(v, u_s; \epsilon)$. It is easy to see by using Property 1 that for any $x \in M^+ \cap M_h^-$ the inequalities

$$u_s < v < \epsilon + v_h < u_s + \epsilon$$

hold; hence x is in $B_s(v, u_s; \epsilon)$ and this means that $M^+ \cap M_h^- \subset B_s(v, u_s; \epsilon)$. Likewise, it is easy to ascertain that if $x \in M^- \cap M_h^+$ then $v_h > v$; consequently, the inequalities

$$u_s - \epsilon < v < u_s$$

hold; this implies that x is also in $B_s(v, u_s; \epsilon)$ and, therefore $M^- \cap M_h^+ \subset B_s(v, u_s; \epsilon)$. Similar results hold for $B_w(v, u_s; \epsilon)$. $\square$

4. Existence and uniqueness of the approximate solutions

We now turn our attention to establish the existence and uniqueness of the solutions to problems (P_h) and $(P_{h,\Delta t})$ with the hypothesis $(\mathrm{H_G})$ of Section 2 restricted to the cases $\mathcal{G}(s) = Cs$ (Budyko model) and $\mathcal{G}(s) = C\,|s|^3\,s$ (Sellers model), because these are the cases of interest to climatologists.

First, we note that since V_h is both a subset of V and of $L^\infty(\mathcal{M})$, then we can argue as in Section 6 to establish the existence of at least one $u_h \in L^p(0, T; V_h) \cap C([0, T]; L^2(\mathcal{M})) \cap L^\infty((0, T) \times \mathcal{M})$ as solution of (P_h). As for the uniqueness of the semidiscrete bounded weak solution, we note that since $V_h \subset L^\infty(\mathcal{M})$ the technique used in Section 6 to prove the uniqueness of the continuous solution is also valid for the semidiscrete case. We have thus the following result

Lemma 3. *For any $u_{h0} \in V_h$ and $\forall T > 0$, there exists at least one $u_h \in L^2(0, T; V_h)$ which is a semidiscrete bounded weak solution of (P_h). Furthermore, for h depending on ϵ sufficiently small, u_h is unique.*

In fact, the same kind of argument shall be used to prove the uniqueness of the solution to $(P_{h,\Delta t})$, see Lemma 5. Next, we prove some a priori bounds which are needed for the error estimates of the approximate solutions.

Lemma 4. *The unique semidiscrete bounded weak solution $u_h \in L^p(0, T; V_h) \cap C([0, T]; L^2(\mathcal{M})) \cap L^\infty((0, T) \times \mathcal{M})$ satisfies the following a priori bounds:*

(i) *There exists a positive constant K_0 such that for a.e. $t \in (0, T]$,*

$$\|u_h(t)\|_{L^2(\mathcal{M})} \leq K_0. \tag{11}$$

(ii) *There exist positive constants K_1 and K_2 such that*

$$\begin{aligned} &\|u_{ht}\|^2_{L^2(0,T;L^2(\mathcal{M}))} + K_1(\|\nabla u_h(T)\|^p_{L^p(\mathcal{M})} + \int_{\mathcal{M}} G(x, u_h(T))\mathrm{d}A) \\ &\quad \leq K_2\|S\|^2_{L^2(0,T;L^2(\mathcal{M}))} + \frac{1}{c_0^2}\|f\|^2_{L^2(0,T;L^2(\mathcal{M}))} + \frac{2k_0}{c_0 p}\|\nabla u_h(0)\|^p_{L^p(\mathcal{M})} + \frac{1}{c_0}\int_{\mathcal{M}} G(x, u_h(0))\mathrm{d}A, \end{aligned} \tag{12}$$

where

$$G(x, v) := \int_0^v \mathcal{G}(x, s)ds,$$

$K_1 = \frac{2}{c_0}\min(\frac{k_0}{p}, 1)$ *and* $K_2 = (\frac{\beta_w Q}{c_0})^2$.

Proof. To prove the uniform stability estimate (11), we take $v_h = u_h$ in problem (P_h) and use assumptions $(\mathrm{H_G})$, (H_f), (H_S) and (H_c) to get

$$\begin{aligned} &c_0\frac{\mathrm{d}}{\mathrm{d}t}\|u_h(t)\|^2_{L^2(\mathcal{M})} + k_0\|\nabla u_h(t)\|^p_{L^p(\mathcal{M})} + C\|u_h(t)\|^r_{L^r(\mathcal{M})} \\ &\quad \leq Q|\mathcal{M}|^{\frac{1}{2}}\|S\|_{L^\infty((0,T)\times\mathcal{M})}\|z_h\|_{L^\infty((0,T)\times\mathcal{M})}\|u_h(t)\|_{L^2(\mathcal{M})} + \|f\|_{L^\infty((0,T)\times\mathcal{M})}\|u_h(t,\cdot)\|_{L^2(\mathcal{M})}, \end{aligned}$$

where $r = 2$ (Budyko model) or $r = 5$ (Sellers model). By the imbedding $L^r(\mathcal{M}) \subset L^2(\mathcal{M})$ for $r \geq 2$ it follows that there exist positive constants γ and K such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_h(t)\|_{L^2(\mathcal{M})} + \gamma\|u_h(t)\|^{r-1}_{L^2(\mathcal{M})} \leq K,$$

where $K = c_0^{-1}Q|\mathcal{M}|^{\frac{1}{2}}\|S\|_{L^\infty((0,T)\times\mathcal{M})}\|z_h\|_{L^\infty((0,T)\times\mathcal{M})} + c_0^{-1}\|f\|_{L^\infty((0,T)\times\mathcal{M})}$ and $\gamma = 2Cc_0^{-1}$. Next, by virtue of a version of the Gronwall inequality (Ju [22]) it follows the result with the constant K_0 given by

$$K_0 = \max\left(\|u_{0h}\|_{L^2(\mathcal{M})}, \left(\frac{K}{\gamma}\right)^{\frac{1}{r-1}}\right).$$

To prove the second estimate we take v_h as $u_{ht}(t, \cdot)$ in (P_h) and integrate in time to obtain

$$\begin{aligned} &\int_0^T\int_{\mathcal{M}} c|u_{ht}(t)|^2\mathrm{d}A\mathrm{d}t + \frac{1}{p}\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}} k|\nabla u_h(t)|^p\mathrm{d}A\mathrm{d}t + \int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}} G(x, u_h(t))\mathrm{d}A\mathrm{d}t \\ &\quad = \int_0^T\int_{\mathcal{M}} QS(t, x)z_h(t)u_{ht}(t)\mathrm{d}A\mathrm{d}t + \int_0^T\int_{\mathcal{M}} f(t, x)u_{ht}(t)\mathrm{d}A\mathrm{d}t. \end{aligned}$$

Then, the estimate follows by virtue of assumptions $(\mathrm{H_G})$–(H_k) and applying the Young inequality to the last term on the right-hand side. □

Remark 2. When $p > 3$, it is possible to improve (11) by applying Lemma 2.6 of Ju [22], but for our purposes the bound (11) is sufficient.

Lemma 5. *For all $n = 1, \ldots, N$, there exists a solution $U^n \in V_h$ to problem $(P_{h,\Delta t})$ which is unique in the class of strong (resp. weak) non-degenerate functions defined in Section 2.*

Proof. We note that from $(P_{h,\Delta t})$ it follows that

$$\int_{\mathcal{M}} cU^n v_h \mathrm{d}A + \Delta t \left(\int_{\mathcal{M}} \left\langle k|\nabla U^n|^{p-2}\nabla U^n, \nabla v_h \right\rangle \mathrm{d}A + \int_{\mathcal{M}} \mathcal{G}(x, U^n) v_h \mathrm{d}A \right)$$
$$= \int_{\mathcal{M}} cU^{n-1} v_h \mathrm{d}A + \Delta t \left(\int_{\mathcal{M}} QS(x,t) Z^n v_h \mathrm{d}A + \int_{\mathcal{M}} f^n v_h \mathrm{d}A \right). \tag{13}$$

The existence of U^n can be obtained by different methods (minima of functional, super and subsolutions, etc) as indicated in Theorem 2 of [14]. To prove the uniqueness in the class of non-degenerate functions we follow the arguments of Section 6, but in order to do so we need to introduce piecewise lineal functions in t, which are constructed with the help of the fully discrete solution $\{U^n\}$. Thus, let us assume that the approximate initial condition U^0 satisfies the strong non-degeneracy condition (similarly, we can also assume that U^0 satisfies the weak non-degeneracy condition) and consider that at time t_1 there exist $U^1 \in V_h$ and $V^1 \in V_h$, both being solutions to $(P_{h,\Delta t})$ corresponding to $Z^1 \in \beta(U^1)$ and $\overline{Z}^1 \in \beta(V^1)$ respectively. For $t \in (t_0, t_1]$ we define

$$U(t) = \frac{t - t_0}{\Delta t} U^1 + \frac{t_1 - t}{\Delta t} U^0, \quad U(t) \in V_h, \tag{14}$$

and

$$V(t) = \frac{t - t_0}{\Delta t} V^1 + \frac{t_1 - t}{\Delta t} U^0, \quad V(t) \in V_h, \tag{15}$$

so that the following relations hold

$$\begin{aligned} &U(t) - V(t) = \frac{t - t_0}{\Delta t}\left(U^1 - V^1\right), \\ &\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{U^1 - U^0}{\Delta t} \quad \text{and} \quad \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{V^1 - U^0}{\Delta t}. \end{aligned} \tag{16}$$

Then, $(P_{h,\Delta t})$ yields

$$\int_{\mathcal{M}} c\frac{\mathrm{d}}{\mathrm{d}t}(U - V) v_h \mathrm{d}A + \int_{\mathcal{M}} \left\langle k\left(|\nabla U^1|^{p-2}\nabla U^1 - |\nabla V^1|^{p-2}\nabla V^1\right), \nabla v_h \right\rangle \mathrm{d}A$$
$$+ \int_{\mathcal{M}} \left(\mathcal{G}(x, U^1) - \mathcal{G}(x, V^1)\right) v_h \mathrm{d}A = \int_{\mathcal{M}} QS(t,x)(Z^1 - \overline{Z}^1) v_h \mathrm{d}A. \tag{17}$$

Setting $v_h = U(t) - V(t)$ and arguing as in the proof of Theorem 2 in Section 6 it follows that for $p \geq 2$ there exist C^*_δ and C^0 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U(t) - V(t)\|^2_{L^2(\mathcal{M}_\delta)} \leq C^*_\delta \|U(t) - V(t)\|^p_{L^\infty(\mathcal{M}_\delta)} + C_0\|U(t) - V(t)\|^2_{L^2(\mathcal{M}_\delta)},$$

where we take the scaling parameter δ sufficiently small to make $C^*_\delta \leq 0$. So that, by the Gronwall inequality it follows that $U^1 = V^1$. Arguing by induction, we extend this reasoning to any $t \in (t_{n-1}, t_n]$, $n = 2, \ldots, N$, and prove that $U^n = V^n$. Hence, it follows the uniqueness of the fully discrete solution $\{U^n\}_{n=1}^N$ to problem $(P_{h,\Delta t})$ for $p > 2$. The case $p = 2$ can be treated similarly using the arguments of Section 6. $\square$

Next, we proceed to prove a priori bounds for the fully discrete solution $\{U^n]_{n=1}^N$.

Lemma 6. *The unique solution $\{U^n\}_{n=1}^N$ to problem $(P_{h,\Delta t})$ satisfies the following a priori bounds:*

(i) *There exists a positive constant K_0 such that for $n = 1, \ldots, N$,*

$$\|U^n\|_{L^2(\mathcal{M})} \leq K_0. \tag{18}$$

(ii) *For $n = 1, \ldots, N$ it holds*

$$\Delta t \sum_{n=1}^N \left\| \frac{U^n - U^{n-1}}{\Delta t} \right\|^2_{L^2(\mathcal{M})} + \frac{2}{pc_0}\left\|\nabla U^N\right\|^p_{L^p(\mathcal{M})} + \frac{2}{c_0}\int_{\mathcal{M}} G(x, U^N)\mathrm{d}A$$
$$\leq \frac{K_1 \Delta t}{2c_0^2}\sum_{n=1}^N \left\|S^n\right\|^2_{L^2(\mathcal{M})} + \frac{2\Delta t}{c_0^2}\sum_{n=1}^N \left\|f^n\right\|^2_{L^2(\mathcal{M})} + \frac{2}{pc_0}\left\|\nabla U^0\right\|^p_{L^p(\mathcal{M})} + \frac{2}{c_0}\int_{\mathcal{M}} G(x, U^0)\mathrm{d}A. \tag{19}$$

(iii) *There exists a positive constant C independent of h and Δt such that*

$$\sum_{n=1}^{N}\int_{\mathcal{M}}\left\langle k\left(\left|\nabla U^n\right|^{p-2}\nabla U^n-\left|\nabla U^{n-1}\right|^{p-2}\nabla U^{n-1}\right),\nabla\left(\frac{U^n-U^{n-1}}{\Delta t}\right)\right\rangle \mathrm{d}A$$
$$+\sum_{n=1}^{N}\int_{\mathcal{M}}\left(\mathcal{G}(x,U^n)-\mathcal{G}(x,U^{n-1})\right)\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A+\max_{n=1,\dots N}\left\|\frac{U^n-U^{n-1}}{\Delta t}\right\|^2_{L^2(\mathcal{M})}\le C \tag{20}$$

Proof. (i) To prove (18) we take $v_h=U^n$ in $(P_{h,\Delta t})$ and obtain

$$\int_{\mathcal{M}}c|U^n|^2\mathrm{d}A+\Delta t\int_{\mathcal{M}}k|\nabla U^n|^p\mathrm{d}A+\Delta t\int_{\mathcal{M}}\mathcal{G}(x,U^n)U^n\mathrm{d}A$$
$$\le|\mathcal{M}|^{\frac12}\Delta tQ\|S\|_{L^\infty((0,T)\times\mathcal{M})}\|\sqrt{c^{-1}}Z^n\|_{L^\infty((0,T)\times\mathcal{M})}\|\sqrt{c}U^n\|_{L^2(\mathcal{M})}$$
$$+|\mathcal{M}|^{\frac12}\Delta t\|\sqrt{c^{-1}}f^n\|_{L^\infty((0,T)\times\mathcal{M})}\|\sqrt{c}U^n\|_{L^2(\mathcal{M})}+\int_{\mathcal{M}}cU^nU^{n-1}\mathrm{d}A.$$

By virtue of assumptions $(\mathrm{H_G})$, (H_f), $(\mathrm{H_S})$ and (H_c), neglecting the second term on the left-hand side, which is positive, using the imbedding of $L^r(\mathcal{M})\subset L^2(\mathcal{M})$ for $r\ge2$ and applying the Cauchy inequality to the last term on the right-hand side it follows that

$$\frac{\|\sqrt{c}U^n\|_{L^2(\mathcal{M})}-\|\sqrt{c}U^{n-1}\|_{L^2(\mathcal{M})}}{\Delta t}+\gamma\|\sqrt{c}U^n\|^{r-1}_{L^2(\mathcal{M})}\le K,$$

where

$$K=|\mathcal{M}|^{\frac12}\{Q\|S\|_{L^\infty((0,T)\times\mathcal{M})}\|\sqrt{c^{-1}}Z^n\|_{L^\infty((0,T)\times\mathcal{M})}+\|\sqrt{c^{-1}}f^n\|_{L^\infty((0,T)\times\mathcal{M})}\}.$$

Applying the discrete version of the Gronwall inequality used in the proof of Lemma 4 (Ju [22]) yields the bound (18).

(ii) To obtain the bound (19) we set, for $n=1,\dots,N$, $v_h=\frac{U^n-U^{n-1}}{\Delta t}$ in $(P_{h,\Delta t})$ and take into account the following inequalities:

(I_1) By virtue of (9)

$$\int_{\mathcal{M}}\left\langle k|\nabla U^n|^{p-2}\nabla U^n,\nabla(\frac{U^n-U^{n-1}}{\Delta t})\right\rangle\mathrm{d}A\ge\frac{1}{p\Delta t}\int_{\mathcal{M}}k(|\nabla U^n|^p-|\nabla U^{n-1}|^p)\mathrm{d}A. \tag{21}$$

(I_2) By virtue of assumption $(\mathrm{H_G})$

$$\int_{\mathcal{M}}\mathcal{G}(x,U^n)\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A\ge\frac{1}{\Delta t}\int_{\mathcal{M}}\int_{U^{n-1}}^{U^n}\mathcal{G}(x,s)ds\mathrm{d}A$$
$$=\frac{1}{\Delta t}\int_{\mathcal{M}}(G(x,U^n)-G(x,U^{n-1}))\mathrm{d}A. \tag{22}$$

(I_3) By the Cauchy inequality,

$$\left|\int_{\mathcal{M}}QS^nZ^n\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A\right|\le\frac{c_0}{4}\left|\left|\frac{U^n-U^{n-1}}{\Delta t}\right|\right|^2_{L^2(\mathcal{M})}+\frac{K_1}{c_0}\left|\left|S^n\right|\right|^2_{L^2(\mathcal{M})} \tag{23}$$

where $K_1=(\beta_w^2Q^2)$.

(I_4) Finally, applying once more the Cauchy inequality it follows that

$$\left|\int_{\mathcal{M}}f^n\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A\right|\le\frac{c_0}{4}\left|\left|\frac{U^n-U^{n-1}}{\Delta t}\right|\right|^2_{L^2(\mathcal{M})}+\frac{1}{c_0}\left|\left|f^n\right|\right|^2_{L^2(\mathcal{M})}. \tag{24}$$

Substituting (21)–(24) in $(P_{h,\Delta t})$ we obtain:

$$\frac{c_0}{2}\left|\left|\frac{U^n-U^{n-1}}{\Delta t}\right|\right|^2_{L^2(\mathcal{M})}+\frac{1}{p\Delta t}\int_{\mathcal{M}}k(|\nabla U^n|^p-|\nabla U^{n-1}|^p)\mathrm{d}A$$
$$+\frac{1}{\Delta t}\int_{\mathcal{M}}(G(U^n)-G(U^{n-1}))\mathrm{d}A\le\frac{K_1}{c_0}\left|\left|S^n\right|\right|^2_{L^2(\mathcal{M})}+\frac{1}{c_0}\left|\left|f^n\right|\right|^2_{L^2(\mathcal{M})}.$$

Multiplying by Δt on both sides of this inequality and summing up for $n=1,\dots,N$ yields the inequality (19).

(iii) From $(P_{h,\Delta t})$ it follows that

$$\begin{aligned}
&\int_{\mathcal{M}} c\left(\frac{U^n-U^{n-1}}{\Delta t}-\frac{U^{n-1}-U^{n-2}}{\Delta t}\right)\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A\\
&\quad+\int_{\mathcal{M}}\left\langle k\left(\left|\nabla U^n\right|^{p-2}\nabla U^n-\left|\nabla U^{n-1}\right|^{p-2}\nabla U^{n-1}\right),\nabla\left(\frac{U^n-U^{n-1}}{\Delta t}\right)\right\rangle\mathrm{d}A\\
&\quad+\int_{\mathcal{M}}\left(\mathcal{G}(x,U^n)-\mathcal{G}(x,U^{n-1})\right)\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A\\
&=\int_{\mathcal{M}} Q\left(S^nZ^n-S^{n-1}Z^{n-1}\right)\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A+\int_{\mathcal{M}}\left(f^n-f^{n-1}\right)\frac{U^n-U^{n-1}}{\Delta t}\mathrm{d}A.
\end{aligned}$$

To obtain (20) we sum this expression from $n=2,\ldots,N$ and use: (1) the relation $(a-b)a=\frac{1}{2}\left(a^2-b^2+(a-b)^2\right)$ as well as the Cauchy and the Young inequalities, (2) the result (19), (3) the fact that Z^n is bounded for all n, and (4) the assumptions (H_f) and (H_S). □

5. Error analysis

Our next concern is to estimate the rate of convergence of the fully discrete solution. We do this by splitting the proof into two stages. In the first one, we estimate the rate of convergence of the semidiscrete solution u_h to u, and devote the second stage to estimate the rate of convergence of the fully discrete solution U^n to $u_h(t_n)$. In the development of the proofs of theorems and lemmata that follow in this section, we shall use, unless otherwise stated, the letter C to denote generic positive constants which are independent of h and Δt; in general, the values of such constants are different at the different places of appearance.

Theorem 3. *Let $u(t,x)$ and $u_h(t,x)$ be the unique non-degenerate bounded weak solutions to problems (P) and (P_h) respectively, with $u\in L^2(0,T;W^{1+s,p}(\mathcal{M}))$. Let $\{U^n\}_{n=1}^N$ be the unique solution to problem $(P_{h,\Delta t})$ such that for $n=1,2,\ldots,N$ and $t\in(t_{n-1},t_n]$ we define*

$$U(t)=\frac{t-t_{n-1}}{\Delta t}U^n-\frac{t_n-t}{\Delta t}U^{n-1}.$$

Then, for Δt and h depending on ϵ sufficiently small, there exists a constant $C>0$ independent of Δt and h such that

$$\|u-U\|^2_{L^\infty(0,T;L^2(\mathcal{M}))}\ \le\ C(\epsilon^{q_1}+\Delta t^2+h^{\frac{2q}{r}}) \tag{25}$$

where $q_1=\min(1,2/r)$ and $q=\min\left(2s,\left(\frac{p-2}{p}\right)+1+s\right)$ with $r=2$ or 5.

Proof. We set $u(t,x)-U(t,x)=(u(t,x)-u_h(t,x))+(u_h(t,x)-U(t,x))$. By the triangle inequality it follows that

$$\|u-U\|_{L^\infty(0,T;L^2(\mathcal{M}))}\le\|u-u_h\|_{L^\infty(0,T;L^2(\mathcal{M}))}+\|u_h-U\|_{L^\infty(0,T;L^2(\mathcal{M}))}.$$

The terms on the right-hand side of this inequality are bounded by Theorems 4 and 5 below as

$$\|u-u_h\|^2_{L^\infty(0,T;L^2(\mathcal{M}))}\le C(\epsilon+h^q)^{\frac{2}{r}} \tag{26}$$

and

$$\|u_h-U\|^2_{L^\infty(0,T;L^2(\mathcal{M}))}\le C(\Delta t^2+\epsilon), \tag{27}$$

respectively. So that it remains to prove the bounds (26) and (27). This is done in what follows. □

5.1. Rate of convergence of $u_h(t,x)$ to $u(t,x)$

Theorem 4. *As in Theorem 3, let $u(t,x)$ and $u_h(t,x)$ be the unique non-degenerate bounded weak solutions to problems (P) and (P_h) respectively. Then, for a.e. $t\in(0,T]$ there exist positive constants M_1^*, M_2^* and K such that*

$$\begin{aligned}
&\|u(t)-u_h(t)\|^2_{L^2(\mathcal{M})}+M_1^*\int_{\mathcal{M}}k\left(|\nabla u(t)|+|\nabla u((t)-u_h(t))|\right)^{p-2}|\nabla(u(t)-u_h(t))|^2\,\mathrm{d}A\\
&\quad+M_2^*\int_{\mathcal{M}}k\left(|u(t)|+|u(t)-u_h(t)|\right)^{r-2}|u(t)-u_h(t)|^2\,\mathrm{d}A\le\max(\|u(0)-u_h(0)\|^2_{L^2(\mathcal{M})},\quad K(\epsilon+h^q)^{2/r}),
\end{aligned} \tag{28}$$

where $q = \min\left(2s, \left(\frac{p-2}{p}\right) + 1 + s\right)$ *(recall that* $r = 2$ *or* 5*, and* $p \geq 2$*) and by approximation theory with* $u_h(0) = I_h u(0)$

$$\|u(0) - u_h(0)\|^2_{L^2(\mathcal{M})} \leq Ch^{4(\frac{1}{2}-\frac{1}{p})+2s+2} |u(0)|_{W^{1+s,p}(\mathcal{M})}.$$

Proof. Subtracting (P_h) from the expression that follows after multiplying (P) by $v_h \in V_h$ and integrating by parts, and decomposing

$$\int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))\, v_h \mathrm{d}A = \alpha_1 \int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))\, v_h \mathrm{d}A + \alpha_2 \int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))\, v_h \mathrm{d}A,$$
$$\text{with } 0 \leq \alpha_1, \alpha_2 \leq 1 \text{ and } \alpha_1 + \alpha_2 = 1,$$

one obtains that

$$\int_{\mathcal{M}} c \frac{\mathrm{d}}{\mathrm{d}t}(u - u_h) v_h \mathrm{d}A + \int_{\mathcal{M}} \left\langle k \left(|\nabla u|^{p-2} \nabla u - |\nabla u_h|^{p-2} \nabla u_h \right), \nabla v_h \right\rangle \mathrm{d}A$$
$$+ \alpha_1 \int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))\, v_h \mathrm{d}A + \alpha_2 \int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))\, v_h \mathrm{d}A = \int_{\mathcal{M}} QS(t,x)(z - z_h) v_h \mathrm{d}A.$$

Next, we choose $v_h(t) = (w_h(t) - u(t)) + (u(t) - u_h(t))$, with $w_h(t) \in V_h$, and apply inequality (8) of Lemma 1-(A), with $\delta = 0$ for the second and third terms and $\delta = r - 2$ for the fourth term on the left-hand side respectively, together with the elementary inequality $\frac{1}{2}(|a| + |b|) < |a - b| + |b| < 2(|a| + |b|)$, a, b real numbers. The result that follows is that there are constants M, C_1 and C_2 independent of h and Δt such that

$$\frac{c_0}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u - u_h\|^2_{L^2(\mathcal{M})} + M \int_{\mathcal{M}} k \left(|\nabla u| + |\nabla(u - u_h)| \right)^{p-2} |\nabla(u - u_h)|^2 \mathrm{d}A$$
$$+ C_1 \int_{\mathcal{M}} (|u| + |u - u_h|)^{r-2} |u - u_h|^2 \mathrm{d}A + C_2 \|u - u_h\|^r_{L^r(\mathcal{M})} \leq \sum_{1=1}^{4} |R_1(t)|, \tag{29}$$

where

$$R_1(t) = \int_{\mathcal{M}} QS(t,x)(z - z_h)(u - u_h) \mathrm{d}A,$$
$$R_2(t) = \int_{\mathcal{M}} \left\langle k \left(|\nabla u|^{p-2} \nabla u - |\nabla u_h|^{p-2} \nabla u_h \right), \nabla(w_h - u) \right\rangle \mathrm{d}A + (\alpha_1 + \alpha_2) \int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))\,(w_h - u) \mathrm{d}A,$$
$$R_3(t) = \int_{\mathcal{M}} QS(t,x)(z - z_h)(w_h - u) \mathrm{d}A,$$
$$R_4(t) = \int_{\mathcal{M}} c \frac{\mathrm{d}}{\mathrm{d}t}(u - u_h)(w_h - u) \mathrm{d}A.$$

To estimate $R_1(t)$ we make use of the Young inequality and hypothesis $(\mathrm{H_S})$ to get

$$|R_1(t)| \leq \varepsilon_2 Q^{r'} \int_{\mathcal{M}} |S(t,x)|^{r'} |(z - z_h)|^{r'} \mathrm{d}A + \varepsilon_1 \|u - u_h\|^r_{L^r(\mathcal{M})},$$

where $\varepsilon_1 = \frac{C_2}{2}$, $\varepsilon_2 = \frac{1}{r'}\left(\frac{rC_2}{2}\right)^{\frac{1}{1-r}}$ and $r' = \frac{r}{r-1}$. By virtue of Lemma 2 and hypothesis $(\mathrm{H_S})$ it follows that there exists a constant C such that

$$\int_{\mathcal{M}} |S(t,x)|^{r'} |(z - z_h)|^{r'} \mathrm{d}A \leq C(\beta_w - \beta_i)^{r'} \epsilon \|S\|^{r'}_{L^\infty((0,T)\times\mathcal{M})}.$$

Hence

$$|R_1(t)| \leq \varepsilon_2 C Q^{r'} (\beta_w - \beta_i)^{r'} \epsilon \|S\|^{r'}_{L^\infty((0,T)\times\mathcal{M})} + \frac{C_2}{2} \|u - u_h\|^r_{L^r(\mathcal{M})}.$$

As for the term $R_2(t)$ we have that

$$|R_2(t)| \leq \left| \int_{\mathcal{M}} \left\langle k \left(|\nabla u|^{p-2} \nabla u - |\nabla u_h|^{p-2} \nabla u_h \right), \nabla(w_h - u) \right\rangle \mathrm{d}A \right| + \left| \int_{\mathcal{M}} (\mathcal{G}(x,u) - \mathcal{G}(x,u_h))(w_h - u) \mathrm{d}A \right|.$$

We bound the first term on the right hand side using inequality (7) of Lemma 1-(A) with $\delta = 0$ and obtain

$$\left| \int_{\mathcal{M}} \left\langle k \left(|\nabla u|^{p-2} \nabla u - |\nabla u_h|^{p-2} \nabla u_h \right), \nabla(w_h - u) \right\rangle \right| \mathrm{d}A \leq K_2 \int_{\mathcal{M}} k \left(|\nabla u| + |\nabla u_h| \right)^{p-2} |\nabla(u - u_h)|\, |\nabla(u - w_h)| \mathrm{d}A.$$

Next, using again the inequality $\frac{1}{2}(|a|+|b|) \leq |a-b|+|b|$, a and b real numbers, and the inequality (10) of Lemma 1-(B), it follows that there exist positive constants c_1 and $C(c_1^{-1})$ independent of h and Δt such that

$$\int_{\mathcal{M}} (k(|\nabla u|+|\nabla u_h|))^{p-2} |\nabla(u-u_h)| |\nabla(u-w_h)| \, dA \leq c_1 \int_{\mathcal{M}} k(|\nabla u|+|\nabla(u-u_h)|)^{p-2} |\nabla(u-u_h)|^2 \, dA$$
$$+ C(c_1^{-1}) \int_{\mathcal{M}} k(|\nabla u|+|\nabla(u-w_h)|)^{p-2} |\nabla(u-w_h)|^2 \, dA.$$

In this inequality we choose c_1 such that $c_1 K_2 < M$, M being the constant multiplying the second term on the right-hand side of (29). To bound the second term of $R_2(t)$ we follow the same approach. Thus, by virtue of Lemma 1-A with $\delta = 0$

$$\int_{\mathcal{M}} |(\mathcal{G}(x,u)-\mathcal{G}(x,u_h))(w_h-u)| \, dA \leq K_3 \int_{\mathcal{M}} (|u|+|u_h|)^{r-2} |u-u_h| |w_h-u| \, dA.$$

But as above there exist c_2 and $C(c_2^{-1})$ such that

$$\int_{\mathcal{M}} (|u|+|u_h|)^{r-2} |u-u_h| |w_h-u| \, dA \leq c_2 \int_{\mathcal{M}} (|u|+|u-u_h|)^{r-2} |u-u_h|^2 \, dA$$
$$+ C(c_2^{-1}) \int_{\mathcal{M}} (|u|+|u-w_h|)^{r-2} |u-w_h|^2 \, dA.$$

Similarly, in this inequality c_2 has been chosen such that $c_2 K_3 < C_1$, C_1 being the constant multiplying the third term on the right-hand side of (29). Putting all pieces together, we bound $R_2(t)$ as

$$|R_2(t)| \leq c_1 K_2 \int_{\mathcal{M}} k(|\nabla u|+|\nabla(u-u_h)|)^{p-2} |\nabla(u-u_h)|^2 \, dA + c_2 K_3 \int_{\mathcal{M}} (|u|+|u-u_h|)^{r-2} |u-u_h|^2 \, dA$$
$$+ C(c_1^{-1}) \int_{\mathcal{M}} k(|\nabla u|+|\nabla(u-w_h)|)^{p-2} |\nabla(u-w_h)|^2 \, dA + C(c_2^{-1}) \int_{\mathcal{M}} (|u|+|u-w_h|)^{r-2} |u-w_h|^2 dA.$$

To estimate $R_3(t)$ we apply the same technique as we did for $R_1(t)$; so that, there exist constants C such that

$$|R_3| \leq Q^2 C(\beta_w - \beta_i)^2 \epsilon \, \|S\|^2_{L^\infty((0,T)\times\mathcal{M})} + C \, \|u-w_h\|^2_{L^2(\mathcal{M})}.$$

The term $R_4(t)$ is bounded by virtue of Lemma 6, $u_t \in L^2(0,T; L^2(\mathcal{M}))$ and the Cauchy–Schwarz inequality. Thus, there exists a constant C such that

$$|R_4(t)| \leq C \, \|u-w_h\|_{L^2(\mathcal{M})}.$$

Collecting the estimates for $R_1(t)$, $R_2(t)$, $R_3(t)$ and $R_4(t)$ and applying Hölder inequality to the terms of the bound of $|R_2(t)|$ multiplied by $C(c_1^{-1})$ and $C(c_2^{-1})$ yields

$$\frac{d}{dt} \|u-u_h\|^2_{L^2(\mathcal{M})} + M_1^* \int_{\mathcal{M}} k(|\nabla u|+|\nabla(u-u_h)|)^{p-2} |\nabla(u-u_h)|^2 \, dA$$
$$+ M_2^* \int_{\mathcal{M}} (|u|+|u-u_h|)^{r-2} |u-u_h|^2 \, dA + c_0^{-1} C_2 \, \|u-u_h\|^r_{L^r(\mathcal{M})}$$
$$\leq C \left(\|u-w_h\|_{L^2(\mathcal{M})} + \|u-w_h\|^2_{L^2(\mathcal{M})} \right) + \epsilon K_4 \left(\|S\|^2_{L^\infty((0,T)\times\mathcal{M})} + \|S\|^{r'}_{L^\infty((0,T)\times\mathcal{M})} \right)$$
$$+ K_6 \, \|u-w_h\|^2_{L^r(\mathcal{M})} + K_5 \, \|u-w_h\|^2_{W^{1,p}(\mathcal{M})}.$$

Next, assuming that $u(t) \in W^{1+s,p}(\mathcal{M})$ for a.e. t, we take $w_h(t) = I_h u(t)$ and $u_{h0} = I_h u_0$, so that the error estimate (6) and the Gronwall inequality (as in Lemma 4-(i)) yield the result. □

5.2. *Rate of convergence of $U^n(x)$ to $u_h(t_n, x)$*

To estimate the rate of convergence of $U^n(x)$ to $u_h(t_n, x)$ we shall assume some extra regularity on f and S; specifically,

(H_f^*) $f \in C^{0,1}([0,T]; L^2(\mathcal{M}))$,

(H_S^*) $S_{tt} \in L^2(0,T; L^2(\mathcal{M}))$.

Then following Rulla's approach [25] we are able to achieve an optimal order of convergence in time for the regularity conditions of u and u_t assumed in Theorem 1. Now, in order to simplify some expressions which appear in the development of our arguments, we introduce a new notation. Thus, for $t \in (t_{n-1}, t_n]$, $n = 1, \dots, N$, we set

$$\begin{cases} U(t) := \dfrac{t - t_{n-1}}{\Delta t} U^n + \dfrac{t_n - t}{\Delta t} U^{n-1}, \quad \widehat{U}(t) := U^n, \quad \widehat{f}(t) := f(t_n), \quad \widehat{S}(t) := S(t_n), \\ E(t) := u_h(t) - U(t), \quad \widehat{E}(t) := u_h(t) - \widehat{U}(t), \quad \widehat{Z}(t) := Z^n, \\ p^{\Delta t}(t) := \dfrac{t_n - t}{\Delta t}. \end{cases}$$

In preparation for the proof of Theorem 5 we state some auxiliary lemmata.

Lemma 7. (i) *There exists a positive constant C independent of Δt such that*

$$\int_0^{t_n} \|U_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t \le C. \tag{30}$$

(ii)

$$u_{ht} \in L^\infty(0, T; L^2(\mathcal{M})). \tag{31}$$

Proof. (i) By virtue of Lemma 6-(ii) we have that

$$\int_0^{t_n} \|U_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\| \frac{U^j - U^{j-1}}{\Delta t} \right\|^2_{L^2(\mathcal{M})} = \left[\Delta t \sum_{j=1}^{j=n} \left\| \frac{U^j - U^{j-1}}{\Delta t} \right\|^2_{L^2(\mathcal{M})} \right] \le C$$

(ii) From Lemma 6 it follows that $\widehat{U}(t)$ and $U_t(t)$ are in $L^\infty(0, T; L^2(\mathcal{M}))$, and $U(t) \in C\left([0, T], L^2(\mathcal{M})\right)$. Furthermore,

$$\left\|U(t) - \widehat{U}(t)\right\|^2_{L^\infty(0,T;L^2(\mathcal{M}))} \le \Delta t^2 \|U_t\|^2_{L^\infty(0,T;L^2(\mathcal{M}))} \le C\Delta t^2. \tag{32}$$

Next, we wish to prove that when $N \to \infty$, $U(t) \to u_h(t)$ and $U_t(t) \to u_{ht}(t)$ for a.e.$t \in (0, T)$. To do so we define two uniform partitions $\{t_n\}_{n=0}^N$ and $\{t_j\}_{j=0}^J$ in $[0, T]$ such that $\Delta t_N = \frac{T}{N}$, $\Delta t_J = \frac{T}{J}$, and set as above

$$\widehat{U}_N(t) = U^n \text{ for } t \in (t_{n-1}, t_n],$$
$$U_N(t) = \frac{t_n - t}{\Delta t_N} U^{n-1} + \frac{t - t_{n-1}}{\Delta t_N} U^{n-1} \quad \text{for } t \in (t_{n-1}, t_n], n = 1, 2, \dots N$$

and

$$\widehat{U}_J(t) = U^j \text{ for } t \in (t_{j-1}, t_j],$$
$$U_J(t) = \frac{t_j - t}{\Delta t_J} U^{j-1} + \frac{t - t_{j-1}}{\Delta t_J} U^{j-1} \quad \text{for } t \in (t_{j-1}, t_j], j = 1, 2, \dots, J.$$

From $(P_{h,\Delta t})$ it follows that for all $t \in (t_{n-1}, t_n] \cap (t_{j-1}, t_j]$, $0 \le n \le N$ and $0 \le j \le J$,

$$\int_{\mathcal{M}} c\left(U_N(t) - U_J(t)\right)_t v_h \mathrm{d}A + \int_{\mathcal{M}} \left\langle k\left(\left|\nabla \widehat{U}_N\right|^{p-2} \widehat{U}_N - \left|\nabla \widehat{U}_J\right|^{p-2} \widehat{U}_J\right), \nabla v_h \right\rangle \mathrm{d}A$$
$$+ \int_{\mathcal{M}} \left(\mathcal{G}(x, U^n) - \mathcal{G}(x, U^j)\right) v_h \mathrm{d}A = \int_{\mathcal{M}} Q\left(S^n Z^n - S^j Z^j\right) v_h \mathrm{d}A + \int_{\mathcal{M}} \left(f^n - f^j\right) v_h \mathrm{d}A. \tag{33}$$

Setting $v_h = U_N(t) - U_J(t) = \left(U_N(t) - \widehat{U}_N(t)\right) - \left(U_J(t) - \widehat{U}_J(t)\right) + \left(\widehat{U}_N(t) - \widehat{U}_J(t)\right)$ and taking into account that $V_h \subset W^{1,p}(\mathcal{M})$, we have by virtue of the monotonicity bound that

$$c_0 \frac{\mathrm{d}}{\mathrm{d}t} \left\|U_N(t) - U_J(t)\right\|^2 + M_2 \left\|\nabla\left(\widehat{U}_N - \widehat{U}_J\right)\right\|^p_{L^p(\mathcal{M})} + M_2 \left\|\widehat{U}_N - \widehat{U}_J\right\|^r_{L^2(\mathcal{M})} \le \sum_{k=1}^{5} |R_{hk}(t)| \tag{34}$$

where

$$R_{h1}(t) = \int_{\mathcal{M}} \left\langle k\left(\left|\nabla \widehat{U}_N\right|^{p-2} \widehat{U}_N - \left|\nabla \widehat{U}_J\right|^{p-2} \widehat{U}_J\right), \nabla\left(U_N - \widehat{U}_N\right) - \nabla\left(U_J - \widehat{U}_J\right) \right\rangle \mathrm{d}A,$$
$$R_{h2}(t) = \int_{\mathcal{M}} \left(\mathcal{G}(x, \widehat{U}_N) - \mathcal{G}(x, \widehat{U}_J)\right) \left((U_N - \widehat{U}_N) - (U_J - \widehat{U}_J)\right) \mathrm{d}A,$$
$$R_{h3}(t) = \int_{\mathcal{M}} Q(S^n - S^j) Z^n (U_N - U_J) \mathrm{d}A,$$
$$R_{h4}(t) = \int_{\mathcal{M}} Q S^j (Z^n - Z^j)(U_N - U_J) \mathrm{d}A,$$

and

$$R_{h5}(t) = \int_{\mathcal{M}} \left(f^n - f^j\right)(U_N - U_J)\mathrm{d}A.$$

To bound $R_{h1}(t)$ we shall introduce the discrete operator $\mathcal{A}_h : V_h \to V_h$, which is defined as

$$(\mathcal{A}_h v_h, w_h) = \langle \mathcal{A} v_h, w_h\rangle_{V'\times V} = \int_{\mathcal{M}} k\left\langle |\nabla v_h|^{p-2}\nabla v_h, \nabla w_h\right\rangle \mathrm{d}A \quad \forall w_h \in V_h. \tag{35}$$

From $(P_{h,\Delta t})$ we have that for $n = 1, 2, \ldots, N$

$$\left(\mathcal{A}_h \widehat{U}_N, w_h\right) = -\int_{\mathcal{M}} \frac{U^n - U^{n-1}}{\Delta t} w_h \mathrm{d}A - \int_{\mathcal{M}} \mathcal{G}(x, \widehat{U}_N) w_h \mathrm{d}A + \int_{\mathcal{M}} QS^n Z^n w_h \mathrm{d}A + \int_{\mathcal{M}} f^n w_h \mathrm{d}A,$$

then, with $w_h = \mathcal{A}_h \widehat{U}_N$, it follows by virtue of Lemma 6 that there is a constant C such that

$$\left\|\mathcal{A}_h \widehat{U}_N\right\|_{L^2(\mathcal{M})} \le C. \tag{36}$$

We are now in a position to estimate $R_{h1}(t)$ as $t \in (t_{n-1}, t_n] \cap (t_{j-1}, t_j]$. Noting that

$$R_{h1}(t) = \left(\mathcal{A}_h \widehat{U}_N - \mathcal{A}_h \widehat{U}_J, \left(U_N - \widehat{U}_N\right) - \left(U_J - \widehat{U}_J\right)\right)$$

we have by virtue of (32) and (36) that

$$|R_{h1}(t)| \le TC\left(\frac{1}{N} + \frac{1}{J}\right). \tag{37}$$

To estimate the term $R_{h2}(t)$ we have that

$$|R_{h2}(t)| \le C\int_{\mathcal{M}} \left(\left|\widehat{U}_N\right|^{r-1} + \left|\widehat{U}_J\right|^{r-1}\right)\left(\left|U_N - \widehat{U}_N\right| + \left|U_J - \widehat{U}_J\right|\right)\mathrm{d}A,$$

then by Hölder inequality it follows that

$$|R_{h2}(t)| \le C\left(\left\|\widehat{U}_N(t)\right\|^{r-1}_{L^r(\mathcal{M})} + \left\|\widehat{U}_J(t)\right\|^{r-1}_{L^r(\mathcal{M})}\right)\left(\left\|U_N(t) - \widehat{U}_N(t)\right\|_{L^r(\mathcal{M})} + \left\|U_J(t) - \widehat{U}_J(t)\right\|_{L^r(\mathcal{M})}\right).$$

Using an inverse estimate for functions of the finite element space V_h between the norms $L^r(\mathcal{M})$ and $L^2(\mathcal{M})$ (see [8]), Lemma 6-(i) and (32) it follows that for fixed h

$$|R_{h2}(t)| \le TCh^{-(1-1/r)}\left(\frac{1}{N} + \frac{1}{J}\right). \tag{38}$$

The assumption (H_S) and Lemma 6-(i) lead to

$$|R_{h3}(t)| \le \left|\int_{\mathcal{M}} Q(S^n - S^j)Z^n(U_N - U_J)\mathrm{d}A\right| \le C_1 L_S|t_n - t_j|\left\|U_N(t) - U_J(t)\right\|_{L^2(\mathcal{M})} \le TC\left(\frac{1}{N} + \frac{1}{J}\right). \tag{39}$$

To estimate $R_{h4}(t)$ we set

$$R_{h4}(t) = \int_{\mathcal{M}} QS^j(Z^n - Z^j)\left[(U_N - \widehat{U}_N) - (U_J - \widehat{U}_J) + (\widehat{U}_N - \widehat{U}_J)\right]\mathrm{d}A.$$

Then by virtue of (32) it follows that

$$|R_{h4}(t)| \le TC\left(\frac{1}{N} + \frac{1}{J}\right) + \int_{\mathcal{M}} \left|QS^j(Z^n - Z^j)\right|\left|\widehat{U}_N - \widehat{U}_J\right|\mathrm{d}A.$$

Using Lemma 11 and an inverse estimate for the elements of V_h we have that

$$\begin{aligned}\int_{\mathcal{M}} \left|QS^j(Z^n - Z^j)\right|\left|\widehat{U}_N - \widehat{U}_J\right|\mathrm{d}A &\le C\left\|\widehat{U}_N - \widehat{U}_J\right\|^2_{L^\infty(\mathcal{M})} \le Ch^{-2}\left\|\widehat{U}_N - \widehat{U}_J\right\|^2_{L^2(\mathcal{M})}\\ &\le Ch^{-2}\left(\left\|U_N - \widehat{U}_N\right\|^2_{L^2(\mathcal{M})} + \left\|U_J - \widehat{U}_J\right\|^2_{L^2(\mathcal{M})} + \left\|U_N - U_J\right\|^2_{L^2(\mathcal{M})}\right).\end{aligned}$$

Hence for fixed h

$$|R_{h4}(t)| \le TCh^{-2}\left(\frac{1}{N} + \frac{1}{J}\right) + Ch^{-2}\left\|U_N(t) - U_J(t)\right\|^2_{L^2(\mathcal{M})}. \tag{40}$$

The Lipschitz continuity of $f(t)$ leads to

$$|R_{h5}(t)| \le \left| \int_{\mathcal{M}} \left(f^n - f^j\right)(U_N - U_J)\mathrm{d}A \right| \le CL|t_n - t_j| \left\| U_N(t) - U_J(t) \right\|_{L^2(\mathcal{M})} \le TC\left(\frac{1}{N} + \frac{1}{J}\right). \tag{41}$$

From the estimates (37)–(41) it follows that there exists a positive constant C independent of Δt_N, Δt_K and h such that

$$c_0 \frac{\mathrm{d}}{\mathrm{d}t} \left\| U_N(t) - U_J(t) \right\|^2 \le TC(h^{-2} + h^{-(1-1/r)})\left(\frac{1}{N} + \frac{1}{J}\right) + Ch^{-2}\left\| U_N(t) - U_J(t) \right\|^2_{L^2(\mathcal{M})}$$

The Gronwall inequality, with $U_N(0) = U_J(0)$, yields

$$c_0 \left\| U_N(T) - U_J(T) \right\|^2 \le TCh^{-2}\left(e^{Ch^{-2}T} - 1\right)\left(\frac{1}{N} + \frac{1}{J}\right).$$

Hence, $\{U_N(t)\}$ is a Cauchy sequence in $L^2(\mathcal{M}) \cap V_h$ that converges to $u_h \in C([0, T], L^2(\mathcal{M}))$. Furthermore, by virtue of (32) the sequence $\{\widehat{U}_N(t)\}$ converges to $u_h(t)$ in the $L^2(\mathcal{M})$ norm. In fact, the sequence $\{\widehat{U}_N(t)\}$ also converges to $u_h(t)$ in $W^{1,p}(\mathcal{M})$ because $u_h(t) \in V_h \subset W^{1,p}(\mathcal{M})$ and by the monotone inequality we have that

$$\begin{aligned} M_2 \left\| \nabla\left(\widehat{U}_N(t) - \widehat{U}_J(t)\right) \right\|^p_{L^p(\mathcal{M})} &\le \left| \left(\mathcal{A}_h\widehat{U}_N(t) - \mathcal{A}_h\widehat{U}_J(t), \widehat{U}_N(t) - \widehat{U}_J(t)\right) \right| \\ &\le \left(\left\| \mathcal{A}_h\widehat{U}_N(t) \right\|_{L^2(\mathcal{M})} + \left\| \mathcal{A}_h\widehat{U}_N(t) \right\|_{L^2(\mathcal{M})} \right) \left\| \widehat{U}_N(t) - \widehat{U}_J(t) \right\|_{L^2(\mathcal{M})} \le TC\left(\frac{1}{N} + \frac{1}{J}\right). \end{aligned}$$

Next, we prove that

$$\lim_{N\to\infty} \left(\mathcal{A}_h\widehat{U}_N(t), v_h\right) = \left(\mathcal{A}_h u_h(t), v_h\right), \quad \forall v_h \in V_h, \tag{42}$$

uniformly in $(0, T]$. Since both $\widehat{U}_N(t)$ and $u_h(t)$ are in V_h, then by virtue of (36), with $v_h = \widehat{U}_N(t) - u_h(t)$, it follows that

$$\left| \left(\mathcal{A}_h\widehat{U}_N(t) - \mathcal{A}_h u_h(t), \widehat{U}_N(t) - u_h(t)\right) \right| \le C \left\| \widehat{U}_N(t) - u_h(t) \right\|_{L^2(\mathcal{M})}.$$

Hence, as $N \to \infty$ the result (42) holds. Similarly, arguing as in the proof of the bound (38) it is proven that

$$\lim_{N\to\infty} \int_{\mathcal{M}} \mathcal{G}(x, \widehat{U}_N(t)) v_h \mathrm{d}A = \int_{\mathcal{M}} \mathcal{G}(x, u_h(t)) v_h \mathrm{d}A \quad \forall v_h \in V_h. \tag{43}$$

We also prove that if the sequences $\{\widehat{U}_N(t)\}$ and $u_h(t)$ satisfy the strong (resp. weak) non-degeneracy property then

$$\lim_{N\to\infty} \int_{\mathcal{M}} \beta(x, \widehat{U}_N(t)) v_h \mathrm{d}A = \int_{\mathcal{M}} \beta(x, u_h(t)) v_h \mathrm{d}A \quad \forall v_h \in V_h. \tag{44}$$

Setting $v_h = \widehat{U}_N(t) - u_h(t)$ and applying Lemma 11 it follows that

$$\left| \int_{\mathcal{M}} \left(\beta(x, \widehat{U}_N(t)) - \beta(x, u_h(t))\right)\left(\widehat{U}_N(t) - u_h(t)\right) \mathrm{d}A \right| \le C \left\| \widehat{U}_N(t) - u_h(t) \right\|^2_{L^\infty(\mathcal{M})}.$$

The result (44) follows by making use of the inverse estimate between the norms $L^\infty(\mathcal{M})$ and $L^2(\mathcal{M})$ for the elements of V_h. Now, considering problem $(P_{h,\Delta t})$ and the assumptions (H_S) and (H_f^*), together with (42)–(44), it follows that

$$\lim_{N\to\infty} \int_{\mathcal{M}} \frac{\mathrm{d}U_N(t)}{\mathrm{d}t} v_h \mathrm{d}A + \int_{\mathcal{M}} \left\langle k \left|\nabla u_h\right|^{p-2} \nabla u_h, \nabla v_h \right\rangle \mathrm{d}A + \int_{\mathcal{M}} \mathcal{G}(x, u_h) v_h \mathrm{d}A = \int_{\mathcal{M}} QSz_h v_h \mathrm{d}A + \int_{\mathcal{M}} f v_h \mathrm{d}A \quad \forall v_h \in V_h.$$

The sequence $\{\frac{dU_N(t)}{\mathrm{d}t}\} \in V_h$ is uniformly bounded in $L^2(\mathcal{M})$ according to Lemma 6-(iii), then there is $u_h^*(t) \in V_h$ such that for a.e. $t \in (0, T]$, $\{\frac{dU_N(t)}{\mathrm{d}t}\} \rightharpoonup u_h^*(t)$. Hence,

$$\int_{\mathcal{M}} u_h^*(t) v_h \mathrm{d}A + \int_{\mathcal{M}} \left\langle k \left|\nabla u_h\right|^{p-2} \nabla u_h, \nabla v_h \right\rangle \mathrm{d}A + \int_{\mathcal{M}} \mathcal{G}(x, u_h) v_h \mathrm{d}A = \int_{\mathcal{M}} QSz_h v_h \mathrm{d}A + \int_{\mathcal{M}} f v_h \mathrm{d}A \quad \forall v_h \in V_h.$$

And from the uniqueness of the solution of (P_h) in the class of non-degenerate functions we conclude that $u_h^*(t) = u_{ht}(t)$. Moreover, since $\left\| u_h^*(t) \right\|_{L^\infty(0,T;L^2(\mathcal{M}))}$ is bounded so is $\left\| u_{h_t}(t) \right\|_{L^\infty(0,T;L^2(\mathcal{M}))}$. □

Lemma 8. *Let $f(t, x)$ satisfy (H_f^*). Then, for all $n = 1, \ldots, N$, it holds*

$$\int_0^{t_n} \|f(t) - \widehat{f}(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t \le \frac{\Delta t^2}{3} t_n L, \tag{45}$$

where L is the Lipschitz constant for $f(t, x)$, $t \in [0, T]$.

Proof. We have that

$$\begin{aligned}\int_0^{t_n} \|f(t)-\widehat{f}(t)\|^2_{L^2(\mathcal{M})}\mathrm{d}t &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|f(t)-\widehat{f}(t)\|^2_{L^2(\mathcal{M})}\mathrm{d}t \\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \Delta t^2 p^{\Delta t}(t)^2 \|\frac{f(t_j-\Delta t p^{\Delta t}(t))-f^j}{\Delta t p^{\Delta t}(t)}\|^2_{L^2(\mathcal{M})}\mathrm{d}t \\ &\le \sum_{j=1}^{n}\Delta t^2 L^2 \int_{t_{j-1}}^{t_j}\frac{\left(t_j-t\right)^2}{\Delta t^2}\mathrm{d}t = \frac{1}{3}\Delta t^2 t_n L^2. \quad \square\end{aligned}$$

Lemma 9. *There exist some positive constants C and γ_1 independent of Δt and h, such that for all $n = 1, \dots, N$,*

$$\Delta t \left| \int_0^{t_n} \left(\|u_{ht}(t)\|^2_{L^2(\mathcal{M})} - \|U_t(t)\|^2_{L^2(\mathcal{M})}\right)\mathrm{d}t\right| \le C\Delta t^2 + C\Delta t \sum_{j=1}^{n}\left\|E(t_{j-1})\right\|^2_{L^2(\mathcal{M})} + 2\gamma_1 \left\|E(t_n)\right\|^2_{L^2(\mathcal{M})}. \tag{46}$$

Since the proof of this lemma is long and rather technical, we postpone its presentation to the end of the Section after the proof of the next theorem.

Theorem 5. *There exist some positive constants C and C_1 independent of Δt and h such that for the solutions $u_h(t,x)$ and $\{U^n(x)\}_{n=1}^N$ of (P_h) and $(P_{h,\Delta t})$ respectively, the following error estimate holds:*

$$\begin{aligned}&\|u_h(t_n)-U(t_n)\|^2_{L^2(\mathcal{M})} + C\int_0^{t_n}\int_{\mathcal{M}} k\left(|\nabla u_h| + \left|\nabla(u_h-\widehat{U})\right|\right)^{p-2}\left|\nabla(u-\widehat{U})\right|^2 \mathrm{d}A\mathrm{d}t \\ &\quad + C\int_0^{t_n}\int_{\mathcal{M}}\left(|u_h| + \left|u_h-\widehat{U}\right|\right)^{r-2}\left|u_h-\widehat{U}\right|^2\mathrm{d}A\mathrm{d}t + C\frac{\Delta t}{2}\int_0^{t_n}\|(u_h(t)-U(t))_t\|^2_{L^2(\mathcal{M})}\mathrm{d}t \le C_1(\epsilon+\Delta t^2).\end{aligned} \tag{47}$$

Proof. From (P_h) and $(P_{h,\Delta t})$, with $v_h \in V_h$, it follows that for a.e. $t \in (0,T]$

$$\begin{aligned}&\int_{\mathcal{M}} c\frac{\mathrm{d}}{\mathrm{d}t}(u_h-U)v_h\mathrm{d}A + \int_{\mathcal{M}}\left\langle k\left(|\nabla u_h|^{p-2}\nabla u_h - |\nabla\widehat{U}|^{p-2}\nabla\widehat{U}\right), \nabla v_h\right\rangle\mathrm{d}A \\ &\quad + \int_{\mathcal{M}}\left(\mathcal{G}(u_h)-\mathcal{G}(\widehat{U})\right)v_h\mathrm{d}A = \int_{\mathcal{M}} QS(z_h-\widehat{Z})v_h\mathrm{d}A + \int_{\mathcal{M}}(f-\widehat{f})v_h\mathrm{d}A.\end{aligned} \tag{48}$$

Choosing

$$v_h = u_h(t)-\widehat{U}(t) = (u_h(t)-U(t)) + (U(t)-\widehat{U}(t)) = E(t) - (E(t)-\widehat{E}(t))$$

and performing similar operations as in Section 5.1, we obtain that

$$\begin{aligned}&\frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E(t)\|^2_{L^2(\mathcal{M})} + M\int_{\mathcal{M}} k\left(|\nabla u_h| + \left|\nabla(u_h-\widehat{U})\right|\right)^{p-2}\left|\nabla(u-\widehat{U})\right|^2\mathrm{d}A \\ &\quad + C_1\int_{\mathcal{M}}\left(|u_h| + \left|u_h-\widehat{U}\right|\right)^{r-2}\left|u_h-\widehat{U}\right|^2\mathrm{d}A + C_2\left\|\widehat{E(t)}\right\|^r_{L^r(\mathcal{M})} \\ &\le \int_{\mathcal{M}} E_t(E-\widehat{E})\mathrm{d}A + \int_{\mathcal{M}} QS(z_h-\widehat{Z})(u_h-\widehat{U})\mathrm{d}A + \int_{\mathcal{M}}\left(f-\widehat{f}\right)\widehat{E}\mathrm{d}A.\end{aligned} \tag{49}$$

We bound the terms on the right side of this inequality. To do so with the first term we note that

$$E(t) = \widehat{E}(t) + \Delta t p^{\Delta t}(t)U_t(t);$$

hence, we can set

$$\begin{aligned}\int_{\mathcal{M}} E_t(E-\widehat{E})\mathrm{d}A &= \Delta t p^{\Delta t}(t)\int_{\mathcal{M}}(u_{ht}-U_t)U_t\mathrm{d}A \\ &= \frac{\Delta t p^{\Delta t}(t)}{2}\left(\|u_{ht}(t)\|^2_{L^2(\mathcal{M})} - \|U_t(t)\|^2_{L^2(\mathcal{M})} - \|E_t(t)\|^2_{L^2(\mathcal{M})}\right),\end{aligned} \tag{50}$$

where the relation $2(a-b)b = a^2 - b^2 - (a-b)^2$, a and b real numbers, has been used to obtain the right side of (50). To bound the term

$$\int_{\mathcal{M}} QS(z_h-\widehat{Z})(u_h-\widehat{U})\mathrm{d}A$$

we use the same technique as for the term R_1 in the proof of Theorem 4. Thus, applying Lemma 2 with $w = u_h(t)$ and $w_h = \widehat{U}(t)$ and the Young inequality it follows that

$$\left| \int_{\mathcal{M}} QS(z_h - \widehat{Z})(u_h - \widehat{U})\mathrm{d}A \right| \leq \varepsilon_2 C Q^{r'} (\beta_w - \beta_i)^{r'} \epsilon \, \|S(t)\|^{r'}_{L^{r'}(\mathcal{M})} + \frac{C_2}{2} \left\| \widehat{E}(t) \right\|^{r}_{L^r(\mathcal{M})}. \tag{51}$$

Finally, applying the Cauchy inequality yields

$$\left| \int_{\mathcal{M}} \left(f - \widehat{f}\right) \widehat{E} \mathrm{d}A \right| \leq K' \left\| \widehat{E}(t) \right\|^2_{L^2(\mathcal{M})} + K \left\| f(t) - \widehat{f}(t) \right\|^2_{L^2(\mathcal{M})}, \tag{52}$$

where K and K' are positive constants independent of Δt and h. Using (51) and (52), and on the account of

$$\widehat{E}(t) = E(t) - \Delta t p^{\Delta t}(t) U_t(t),$$

it follows from (49), with $E_0 = 0$, that

$$\begin{aligned}
&\frac{c_0}{2} \|E(t_n)\|^2_{L^2(\mathcal{M})} + M \int_0^{t_n} \int_{\mathcal{M}} k \left(|\nabla u_h| + \left|\nabla(u_h - \widehat{U})\right|\right)^{p-2} \left|\nabla(u - \widehat{U})\right|^2 \mathrm{d}A\mathrm{d}t \\
&\quad + C_1 \int_0^{t_n} \int_{\mathcal{M}} \left(|u_h| + \left|u_h - \widehat{U}\right|\right)^{r-2} \left|u_h - \widehat{U}\right|^2 \mathrm{d}A\mathrm{d}t + \Delta t \int_0^{t_n} p^{\Delta t}(t) \|E_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t \\
&\quad \leq \overline{C} \left\{ \begin{array}{l} \displaystyle\int_0^{t_n} \|E(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t + \epsilon \int_0^{t_n} \|S(t)\|^{r'}_{L^{r'}(\mathcal{M})} \mathrm{d}t + \Delta t^2 \int_0^{t_n} \|U_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t \\ \displaystyle + \int_0^{t_n} \left\|f(t) - \widehat{f}(t)\right\|^2_{L^2(\mathcal{M})} \mathrm{d}t + \Delta t \int_0^{t_n} p^{\Delta t}(t) \left(\|u_{ht}(t)\|^2_{L^2(\mathcal{M})} - \|U_t(t)\|^2_{L^2(\mathcal{M})} \right) \mathrm{d}t \end{array} \right\},
\end{aligned} \tag{53}$$

where $\overline{C} = \max\left(1, K', K, K_1, \varepsilon_2 C Q^{r'} (\beta_w - \beta_i)^{r'}\right)$. Next, borrowing the arguments of Rulla [25] to our context, and considering that $U_t(t)$ is constant in each interval $(t_{j-1}, t_j]$, so that the last term of (53) is equal to $\Delta t \int_0^{t_n} \left(p^{\Delta t}(t) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} - \frac{1}{2} \|U_t(t)\|^2_{L^2(\mathcal{M})}\right) \mathrm{d}t$, we will work out this latter term and $\Delta t \int_0^{t_n} p^{\Delta t}(t) \|E_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t$ to obtain the term

$$\Delta t \int_0^{t_n} \left(\|u_{ht}(t)\|^2_{L^2(\mathcal{M})} - \|U_t(t)\|^2_{L^2(\mathcal{M})} \right) \mathrm{d}t,$$

which is bounded by Lemma 9. Thus, since $p^{\Delta t}(t) \geq 1/2$ for each interval $(t_{j-1}, t_{j-1/2}]$ it follows that

$$\Delta t \int_0^{t_n} p^{\Delta t}(t) \|E_t(t)\|^2_{L^2(\mathcal{M})} \geq \frac{\Delta t}{2} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j-1/2}} \|E_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t. \tag{54}$$

On the other hand, changing the variable $t \to t + \frac{\Delta t}{2}$ and taking into account that U_t is constant in $[t_{j-1}, t_j]$ it follows that

$$\begin{aligned}
\frac{\Delta t}{2} \sum_{j=1}^{n} \int_{t_{j-1/2}}^{t_j} \|E_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t &= \frac{\Delta t}{2} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j-1/2}} \left\| E_t\left(t + \frac{\Delta t}{2}\right) \right\|^2_{L^2(\mathcal{M})} \mathrm{d}t \\
&= \frac{\Delta t}{2} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j-1/2}} \left\| u_{ht}\left(t + \frac{\Delta t}{2}\right) - U_t \right\|^2_{L^2(\mathcal{M})} \mathrm{d}t \leq \Delta t \int_0^{t_n} p^{\Delta t}(t) \, \|(\overline{u}_h(t) - U(t))_t\|^2_{L^2(\mathcal{M})} \mathrm{d}t,
\end{aligned} \tag{55}$$

where $\overline{u}_h(t) = u_h(t + \frac{\Delta t}{2})$. Introducing these changes in (53) one has

$$\begin{aligned}
&\frac{c_0}{2} \|E(t_n)\|^2_{L^2(\mathcal{M})} + M \int_0^{t_n} \int_{\mathcal{M}} k \left(|\nabla u_h| + \left|\nabla(u_h - \widehat{U})\right|\right)^{p-2} \left|\nabla(u - \widehat{U})\right|^2 \mathrm{d}A\mathrm{d}t \\
&\quad + C_1 \int_0^{t_n} \int_{\mathcal{M}} \left(|u_h| + \left|u_h - \widehat{U}\right|\right)^{r-2} \left|u_h - \widehat{U}\right|^2 \mathrm{d}A\mathrm{d}t + \frac{\Delta t}{2} \int_0^{t_n} \|E_t(t)\|^2_{L^2(\mathcal{M})} \mathrm{d}t \\
&\quad \leq \overline{C} \left\{ \begin{array}{l} \displaystyle \Delta t \int_0^{t_n} \left(p^{\Delta t}(t) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} + \Delta t \|U_t(t)\|^2_{L^2(\mathcal{M})} - \frac{1}{2} \|U_t(t)\|^2_{L^2(\mathcal{M})} \right) \mathrm{d}t \\ \displaystyle + \Delta t \int_0^{t_n} p^{\Delta t}(t) \, \|(\overline{u}_h(t) - U(t))_t\|^2_{L^2(\mathcal{M})} \mathrm{d}t \\ \displaystyle + \int_0^{t_n} \left\|f(t) - \widehat{f}(t)\right\|^2_{L^2(\mathcal{M})} \mathrm{d}t + \epsilon \int_0^{t_n} \|S(t)\|^{r'}_{L^{r'}(\mathcal{M})} \mathrm{d}t. \end{array} \right\}.
\end{aligned} \tag{56}$$

Setting $\overline{E}_t(t) = (\overline{u}_h(t) - U(t))_t$, $\overline{f}(t) = f(t + \frac{\Delta t}{2})$, $\overline{S}(t) = S(t + \frac{\Delta t}{2})$ and repeating the steps from (48) to (53) it follows that there exists a positive constant $\overline{C}_1$, independent of Δt and h, such that

$$\Delta t \int_0^{t_n} p^{\Delta t}(t) \, \|(\overline{u}_h(t) - U(t))_t\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$
$$\leq \overline{C}_1 \left\{ \begin{array}{l} \Delta t \displaystyle\int_0^{t_n} \left(p^{\Delta t}(t) \, \|\overline{u}_{ht}(t)\|^2_{L^2(\mathcal{M})} + \Delta t \, \|U_t(t)\|^2_{L^2(\mathcal{M})} - \frac{1}{2} \, \|U_t(t)\|^2_{L^2(\mathcal{M})} \right) \mathrm{d}t \\ + \displaystyle\int_0^{t_n} \left\|\overline{S}(t) - \widehat{S}(t)\right\|^2_{L^2(\mathcal{M})} \mathrm{d}t + \int_0^{t_n} \left\|\overline{f}(t) - \widehat{f}(t)\right\|^2_{L^2(\mathcal{M})} \mathrm{d}t + \epsilon \int_0^{t_n} \left\|\overline{S}(t)\right\|^{r'}_{L^{r'}(\mathcal{M})} \mathrm{d}t \end{array} \right\}. \quad (57)$$

Next, noting that for all n

$$\int_0^{t_n} p^{\Delta t}(t) \, \|\overline{u}_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} p^{\Delta t}(t) \left\| u_{ht}\left(t + \frac{\Delta t}{2}\right) \right\|^2_{L^2(\mathcal{M})} \mathrm{d}t = \sum_{j=1}^{n} \int_{t_{j-1/2}}^{t_{j+1/2}} p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$
$$= \int_0^{t_n} p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t - \int_0^{\Delta t/2} p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$
$$+ \int_{t_n}^{t_n + \Delta t/2} p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t,$$

and taking into account Lemma 7-(ii) and the fact that for $t \in [t_n, t_n + \frac{\Delta t}{2}]$, $p^{\Delta t}(t - \frac{\Delta t}{2}) = \frac{t_n + \frac{\Delta t}{2} - t}{\Delta t} \geq 0$, it follows that there exists a positive constant C such that

$$\int_0^{t_n} p^{\Delta t}(t) \, \|\overline{u}_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t \leq \int_0^{t_n} p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t + C\Delta t. \quad (58)$$

From this inequality and (57) one gets that the first plus the second terms on the right-hand side of (56) yield the term

$$\Delta t \int_0^{t_n} \left(\left(p^{\Delta t}(t) + p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} + 2\Delta t \, \|U_t(t)\|^2_{L^2(\mathcal{M})} - \|U_t(t)\|^2_{L^2(\mathcal{M})} \right) \mathrm{d}t + C\Delta t^2$$

Here, to estimate the first term of the integrand we note that for $t \in (0, T]$

$$p^{\Delta t}(t) + p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) = \frac{1}{2} + p^{\Delta t/2}(t),$$

so that

$$\Delta t \int_0^{t_n} \left(p^{\Delta t}(t) + p^{\Delta t}\left(t - \frac{\Delta t}{2}\right) \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$
$$= \Delta t \int_0^{t_n} \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t + \Delta t \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(p^{\Delta t/2}(t) - \frac{1}{2} \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t.$$

But

$$\Delta t \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(p^{\Delta t/2}(t) - \frac{1}{2} \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t = \Delta t \sum_{j=1}^{n} \frac{2}{\Delta t} \int_{t_{j-1}}^{t_{j-1/4}} \left(t_j - \frac{\Delta t}{4} - t \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$
$$- \Delta t \sum_{j=1}^{n} \frac{2}{\Delta t} \int_{t_{j-1/4}}^{t_j} \left(t + \frac{\Delta t}{4} - t_j \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$
$$\leq \Delta t \sum_{j=1}^{n} \frac{2}{\Delta t} \int_{t_{j-1}}^{t_{j-1/4}} \left(t_j - \frac{\Delta t}{4} - t \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t$$

because $t + \frac{\Delta t}{4} - t_j \geq 0$ for $t \in [t_{j-1/4}, t_j]$. Now, by virtue of Lemma 7-(ii) and noting that

$$\int_{t_{j-1}}^{t_{j-1/4}} \left(t_j - \frac{\Delta t}{4} - t \right) \mathrm{d}t = \frac{9}{32} \Delta t^2,$$

it follows that there exists a positive constant C such that

$$\Delta t \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(p^{\Delta t/2}(t) - \frac{1}{2} \right) \|u_{ht}(t)\|^2_{L^2(\mathcal{M})} \, \mathrm{d}t \leq C\Delta t^2.$$

Hence, putting all pieces together yields

$$\Delta t\int_0^{t_n}\left(p^{\Delta t}(t)+p^{\Delta t}\left(t-\frac{\Delta t}{2}\right)\right)\|u_{ht}(t)\|^2_{L^2(\mathcal{M})}\,\mathrm{d}t\le\Delta t\int_0^{t_n}\|u_{ht}(t)\|^2_{L^2(\mathcal{M})}\,\mathrm{d}t+C\Delta t^2.$$

Finally, to obtain the inequality (47) we collect all these bounds on the right-hand side of (56), and apply the Gronwall inequality and Lemmata 7, 8 and 9 with γ_1 sufficiently small such that $1-2\gamma_1>0$. □

5.3. Proof of Lemma 9

To prove Lemma 9 we start estimating $\|u_{ht}(t)\|^2_{L^2(\mathcal{M})}$ and $\|U_t(t)\|^2_{L^2(\mathcal{M})}$. Thus, from (P_h) one obtains that

$$\|u_{ht}(t)\|^2_{L^2(\mathcal{M})}=-\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}}k|\nabla u_h(t)|^p\mathrm{d}A-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}}G(x,u_h(t))\mathrm{d}A+\int_{\mathcal{M}}QS\frac{\mathrm{d}\varphi(u_h(t))}{\mathrm{d}t}\mathrm{d}A+\int_{\mathcal{M}}f(t)u_{ht}(t)\mathrm{d}A, \tag{59}$$

where we recall that $\varphi(\cdot)$ is a real convex proper continuous function such that $\beta(x,v)=\partial\varphi(x,v)$. Analogously, from $(P_{h,\Delta t})$ it follows that for $t\in(t_{j-1},t_j],\ j=1,2,\ldots,N$,

$$\begin{aligned}\|U_t(t)\|^2_{L^2(\mathcal{M})}&=-\int_{\mathcal{M}}\langle k|\nabla U^j|^{p-2}\nabla U^j,\nabla\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\rangle\mathrm{d}A-\int_{\mathcal{M}}\mathcal{g}(x,U^j)\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\mathrm{d}A\\&\quad+\int_{\mathcal{M}}Q\widehat{S}(t)Z^j\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\mathrm{d}A+\int_{\mathcal{M}}\widehat{f}(t)\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\mathrm{d}A.\end{aligned}$$

But making use of the inequalities (21) and (22), we have

$$\begin{aligned}\|U_t\|^2_{L^2(\mathcal{M})}&\le-\frac{1}{p\Delta t}\int_{\mathcal{M}}k(|\nabla U^j|^p-|\nabla U^{j-1}|^p)\mathrm{d}A-\frac{1}{\Delta t}\int_{\mathcal{M}}(G(x,U^j)-G(x,U^{j-1}))\mathrm{d}A\\&\quad+\int_{\mathcal{M}}Q\widehat{S}(t)Z^j\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\mathrm{d}A+\int_{\mathcal{M}}\widehat{f}(t)\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\mathrm{d}A\end{aligned} \tag{60}$$

So that, (59) and (60) yield

$$\begin{aligned}&\Delta t\int_0^{t_n}\left(\|U_t(t)\|^2_{L^2(\mathcal{M})}-\|u_{ht}(t)\|^2_{L^2(\mathcal{M})}\right)\mathrm{d}t\le-\Delta t\left[\sum_{j=1}^n\frac{1}{p\Delta t}\int_{\mathcal{M}}\left(\int_{t_{j-1}}^{t_j}k(|\nabla U^j|^p-|\nabla U^{j-1}|^p)\mathrm{d}t\right)\mathrm{d}A\right]\\&\quad-\Delta t\left[\sum_{j=1}^n\frac{1}{\Delta t}\int_{\mathcal{M}}\left(\int_{t_{j-1}}^{t_j}(G(x,U^j)-G(x,U^{j-1}))\mathrm{d}t\right)\mathrm{d}A\right]+\frac{\Delta t}{p}\int_{\mathcal{M}}k\left(|\nabla u_h^n|^p-|\nabla u_{h0}|^p\right)\mathrm{d}A\\&\quad+\Delta t\int_{\mathcal{M}}(G(x,u_h^n)-G(x,u_{h0}))\mathrm{d}A-\Delta t\int_{\mathcal{M}}\left[Q\left(S^n\varphi(u_h^n)-S^0\varphi(u_{h0})\right)-Q\int_0^{t_n}\varphi(u_h(t))S_t\mathrm{d}t\right]\mathrm{d}A\\&\quad+\Delta t\sum_{j=1}^n\int_{\mathcal{M}}Q\int_{t_{j-1}}^{t_j}\widehat{S}(t)Z^j\left(\frac{U^j-U^{j-1}}{\Delta t}\right)\mathrm{d}t\mathrm{d}A-\Delta t\left[\int_{\mathcal{M}}\int_0^{t_n}f(t)u_{ht}(t)\mathrm{d}t\mathrm{d}A-\int_{\mathcal{M}}\int_0^{t_n}\widehat{f}(t)U_t(t)\mathrm{d}t\mathrm{d}A\right]\\&\equiv(T_1+T_2+\cdots+T_7).\end{aligned} \tag{61}$$

We bound the terms $(T_1)-(T_4)$. Since ∇U^j (resp. ∇U^{j-1}) and $G(x,U^j)$ (resp. $G(x,U^{j-1})$) are constant with respect to t in $[t_{j-1},t_j]$, then

$$\sum_{j=1}^n\int_{t_{j-1}}^{t_j}k(|\nabla U^j|^p-|\nabla U^{j-1}|^p)\mathrm{d}t=\Delta tk\left(|\nabla U^n|^p-|\nabla U^{j0}|^p\right)$$

and

$$\sum_{j=1}^n\int_{t_{j-1}}^{t_j}(G(x,U^j)-G(x,U^{j-1}))\mathrm{d}t=\Delta t\left(G(x,U^n)-G(x,U^0)\right).$$

Hence, taking $U^0=u_{h0}$ it follows that

$$T_1+T_2+T_3+T_4=\frac{\Delta t}{p}\int_{\mathcal{M}}k(|\nabla u_h^n|^p-|\nabla U^n|^p)\mathrm{d}A+\Delta t\int_{\mathcal{M}}\left(G(x,u_h^n)-G(x,U^n)\right)\mathrm{d}A.$$

Next, using (9) and $(\mathcal{H}_G)$ yields

$$T_1+T_2+T_3+T_4\le\Delta t\int_{\mathcal{M}}\langle k|\nabla U^n|^{p-2}\nabla U^n,\nabla(u_h^n-U^n)\rangle\mathrm{d}A+\Delta t\int_{\mathcal{M}}\mathcal{g}(x,U^n)(u_h^n-U^n)\mathrm{d}A$$

and by virtue of $(P_{h,\Delta t})$, with $v_h = u_h^n - U^n$,

$$T_1 + T_2 + T_3 + T_4 \le \Delta t \int_{\mathcal{M}} QS^n Z^n (u_h^n - U^n)\mathrm{d}A + \Delta t \int_{\mathcal{M}} \left(f^n - \frac{U^n - U^{n-1}}{\Delta t} \right) (u_h^n - U^n)\mathrm{d}A.$$

Applying the Cauchy inequality to bound the first term on the right-hand side of this inequality yields

$$\Delta t \left| \int_{\mathcal{M}} QS^n Z^n (u_h^n - U^n)\mathrm{d}A \right| \le \frac{\Delta t^2}{\gamma_1} Q \beta_w^2 \int_{\mathcal{M}} |S^n|^2 \mathrm{d}A + \Delta t \gamma_1 \frac{1}{2\Delta t} \|u_h^n - U^n\|^2_{L^2(\mathcal{M})},$$

where γ_1 is a constant sufficiently small. Similarly, by virtue of Lemmata 4 and 6 we can bound the second term as

$$\Delta t \left| \int_{\mathcal{M}} \left(f^n - \frac{U^n - U^{n-1}}{\Delta t} \right) (U^n - u_h^n)\mathrm{d}A \right| \le \frac{\Delta t^2}{\gamma_1} \left\| f^n - \frac{U^n - U^{n-1}}{\Delta t} \right\|^2_{L^2(\mathcal{M})} + \Delta t \gamma_1 \frac{1}{2\Delta t} \|u_h^n - U^n\|^2_{L^2(\mathcal{M})}.$$

Thus, collecting these two bounds, we have that there is a constant C such that

$$|T_1 + T_2 + T_3 + T_4| \le C\Delta t^2 + \gamma_1 \|E(t_n)\|^2_{L^2(\mathcal{M})}. \tag{62}$$

To estimate $T_5 + T_6$ we take into account that $\varphi(\cdot)$ is a convex function and that $Z^j \in \beta(U^j) = \partial\varphi(U^j)$. Therefore, considering T_6, we have by virtue of the definition of $\partial\varphi$

$$\Delta t \int_{\mathcal{M}} Q \int_{t_{j-1}}^{t_j} \widehat{S}(t) Z^j \left(\frac{U^j - U^{j-1}}{\Delta t} \right) \mathrm{d}t\mathrm{d}A \ge \Delta t \int_{\mathcal{M}} Q \int_{t_{j-1}}^{t_j} \widehat{S} \left(\frac{\varphi(U^j) - \varphi(U^{j-1})}{\Delta t} \right) \mathrm{d}t\mathrm{d}A.$$

Since $\varphi(U^j)$ (resp. $\varphi(U^{j-1})$) are constants in $(t_{j-1}, t_j]$, and using the notation

$$\widetilde{S}_t = \frac{S^j - S^{j-1}}{\Delta t}, \quad j = 1, \ldots, N,$$

it follows that

$$\begin{aligned} &\Delta t \sum_{j=1}^{n} \int_{\mathcal{M}} Q \int_{t_{j-1}}^{t_j} \widehat{S}(t) \left(\frac{\varphi(U^j) - \varphi(U^{j-1})}{\Delta t} \right) \mathrm{d}t\mathrm{d}A \\ &\quad = \Delta t \int_{\mathcal{M}} Q \left(S^n \varphi(U^n) - S^0 \varphi(U^0) \right) \mathrm{d}A - \Delta t \int_{\mathcal{M}} Q \left(\Delta t \sum_{j=1}^{n} \widetilde{S}_t \varphi(U^{j-1}) \right) \mathrm{d}A. \end{aligned}$$

Hence,

$$T_5 + T_6 \le \Delta t \int_{\mathcal{M}} QS^n (\varphi(U^n) - \varphi(u_h^n))\mathrm{d}A - \Delta t \int_{\mathcal{M}} Q \left(\Delta t \sum_{j=1}^{n} \widetilde{S}_t \varphi(U^{j-1}) - \int_0^{t_n} \varphi(u_h(t)) S_t \mathrm{d}t \right) \mathrm{d}A. \tag{63}$$

We bound the first term on the right-hand side of this inequality by appealing to the convexity of φ and using the fact that $z_h \in \beta(x, u_h^n) = \partial\varphi(x, u_h^n)$. Thus

$$\Delta t \int_{\mathcal{M}} QS^n (\varphi(U^n) - \varphi(u_h^n))\mathrm{d}A \le \Delta t \int_{\mathcal{M}} QS^n Z^n (U^n - u_h^n)\mathrm{d}A,$$

and by the Cauchy inequality we can find constants C and γ_1 sufficiently small such that

$$\left| \Delta t \int_{\mathcal{M}} QS^n Z^n (U^n - u_h^n)\mathrm{d}A \right| \le \Delta t^2 C \|S^n\|^2_{L^2(\mathcal{M})} + \frac{\gamma_1}{2} \|E(t_n)\|^2_{L^2(\mathcal{M})}. \tag{64}$$

To estimate the second term on the right side of (63) we use (H_S^*) and carry out the following decomposition

$$\begin{aligned} &\Delta t \int_{\mathcal{M}} Q \left(\Delta t \sum_{j=1}^{n} S_t \varphi(U^{j-1}) - \int_0^{t_n} \varphi(u_h(t)) S_t \mathrm{d}t \right) \mathrm{d}A = \Delta t \int_{\mathcal{M}} Q \left(\Delta t \sum_{j=1}^{n} (\widetilde{S}_t(x) - S_t(x, t_{j-1})) \varphi(U^{j-1}) \right) \mathrm{d}A \\ &\quad + \Delta t \int_{\mathcal{M}} Q \left(\Delta t \sum_{j=1}^{n} S_t(x, t_{j-1}) (\varphi(U^{j-1}) - \varphi(u_h^{j-1})) \right) \mathrm{d}A \\ &\quad + \Delta t \int_{\mathcal{M}} Q \left(\Delta t \sum_{j=1}^{n} S_t(x, t_{j-1}) \varphi(u_h^{j-1}) - \int_0^{t_n} \varphi(u_h(t)) S_t \mathrm{d}t \right) \equiv (A_1 + A_2 + A_3). \end{aligned} \tag{65}$$

Then, noting that

$$\Delta t\sum_{j=1}^{n}(\widetilde{S}_t(x)-S_t(x,t_{j-1}))\varphi(U^{j-1})=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(\widetilde{S}_t(x)-S_t(x,t_{j-1}))\varphi(U^{j-1})\mathrm{d}t,$$

we have, by virtue of a Taylor expansion of $\widetilde{S}_t$ and the Cauchy inequality, that A_1 is bounded as

$$A_1\le K|\mathcal{M}|^{\frac{1}{2}}\max_{1\le i\le N}|\varphi(U^i)|\Delta t^2\int_0^{t_n}\|S_{tt}(t)\|_{L^2(\mathcal{M})}\mathrm{d}t=C\Delta t^2, \tag{66}$$

where K is a positive constant independent of h and Δt. To estimate the term A_2, we use the Young inequality and the fact that φ is convex; then

$$\begin{aligned}\Delta t\sum_{j=1}^{n}S_t(x,t_{j-1})(\varphi(U^{j-1})-\varphi(u_h^{j-1}))&\le\Delta t\sum_{j=1}^{n}S_t(x,t_{j-1})Z^{j-1}(U^{j-1}-u_h^{j-1})\\&\le C_1\Delta t^2\max_j|Z^{j-1}|^2\sum_{j=1}^{n}|S_t(x,t_{j-1})|^2+C_2\sum_{j=1}^{n}|U^{j-1}-u_h^{j-1}|^2.\end{aligned}$$

Hence,

$$|A_2|\le C_1\Delta t^2\|S_t\|^2_{L^\infty(0,T;L^2(\mathcal{M}))}+C_2\Delta t\sum_{j=1}^{n}\|E(t_{j-1})\|^2_{L^2(\mathcal{M})}. \tag{67}$$

Finally, to bound the term A$_3$ we use Peano's Theorem to estimate the quadrature error. So that, we have that

$$\left|\Delta t\sum_{j=1}^{n}S_t(x,t_{j-1})\varphi(u_h^{j-1})-\int_0^{t_n}\varphi(u_h(t))S_t\mathrm{d}t\right|\le\Delta t\int_0^{t_n}\left|\frac{\partial}{\partial t}(S_t(t)\varphi(u_h(t)))\right|\mathrm{d}t.$$

Then, after applying Lemma 4 (ii) and the facts that $\partial\varphi$ is bounded and $S_{tt}(t)\in L^2(0,T;L^2(\mathcal{M}))$ it follows that there is a bounded constant C independent of Δt and h (but depending on $|\mathcal{M}|$ and t_n) such that

$$|A_3|\le C\Delta t^2. \tag{68}$$

Thus, from (63)–(65) and the estimates (66)–(68) yields

$$|T_5+T_6|\le C\Delta t^2+C\Delta t\sum_{j=1}^{n}\|E(t_{j-1})\|^2_{L^2(\mathcal{M})}+\frac{\gamma_1}{2}\|E(t_n)\|^2_{L^2(\mathcal{M})}, \tag{69}$$

where the constant γ_1 is the same as the one in (64). It remains to estimate the term T_7 in (61). To do so, we set

$$\begin{aligned}\Delta t\int_{\mathcal{M}}\int_0^{t_n}(f(t)u_{ht}(t)-\widehat{f}(t)U_t(t))\mathrm{d}t\mathrm{d}A&=\Delta t\int_{\mathcal{M}}\int_0^{t_n}(f(t)-\widehat{f}(t))u_{ht}(t)\mathrm{d}t\mathrm{d}A\\&\quad+\Delta t\int_{\mathcal{M}}\int_0^{t_n}\widehat{f}(t)(u_{ht}(t)-U_t(t))\mathrm{d}t\mathrm{d}A\equiv(B_1+B_2).\end{aligned} \tag{70}$$

To bound B_1 we apply the Cauchy inequality and use Lemmas 4 and 8 (ii). Hence,

$$|B_1|\le C_1\int_0^{t_n}\|f-\widehat{f}\|^2_{L^2(\mathcal{M})}+C_2\Delta t^2\int_0^{t_n}\|u_{ht}\|^2_{L^2(\mathcal{M})}\le C\Delta t^2. \tag{71}$$

To bound the term B_2 we apply the Young inequality.

$$\begin{aligned}\Delta t\int_{\mathcal{M}}\int_0^{t_n}\widehat{f}(t)(u_{ht}(t)-U_t(t))\mathrm{d}t\mathrm{d}A&=\Delta t\int_{\mathcal{M}}\int_0^{t_n}\widehat{f}(t)E_t(t)\mathrm{d}t\mathrm{d}A=\Delta t\int_{\mathcal{M}}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\widehat{f}(t)E_t(t)\mathrm{d}t\mathrm{d}A\\&=\Delta t\int_{\mathcal{M}}f^nE(t_n)\mathrm{d}A-\Delta t\sum_{j=1}^{n}\int_{\mathcal{M}}\int_{t_{j-1}}^{t_j}E(t_{j-1})(f^j-f^{j-1})\mathrm{d}A\\&\le c_1\Delta t^2\|f\|^2_{L^\infty(0,T;L^2(\mathcal{M}))}+\frac{\gamma_1}{2}\|E(t_n)\|^2_{L^2(\mathcal{M})}\\&\quad+c_2\Delta t\sum_{j=0}^{n-1}\|E(t_j)\|^2_{L^2(\mathcal{M})}+c_3\Delta t^3\sum_{j=1}^{n}\left\|\frac{f^j-f^{j-1}}{\Delta t}\right\|^2_{L^2(\mathcal{M})}.\end{aligned}$$

Since f is Lipschitz continuous, then there exists a constant C such that

$$\left|\Delta t \int_{\mathcal{M}} \int_0^{t_n} \widehat{f}(t)(u_{ht} - U_t)\mathrm{d}t\mathrm{d}A\right| \le C\Delta t^2 + \frac{\gamma_1}{2}\|E(t_n)\|^2_{L^2(\mathcal{M})} + c_2\Delta t \sum_{j=0}^{n-1} \|E(t_j)\|^2_{L^2(\mathcal{M})}, \tag{72}$$

where the constant γ_1 is the same as the one in (64). Thus, from (69)–(72) we have that

$$|T_5 + T_6| + |T_7| \le C\Delta t^2 + C\Delta t \sum_{j=1}^{n} \left\|E(t_{j-1})\right\|^2_{L^2(\mathcal{M})} + \gamma_1 \|E(t_n)\|^2_{L^2(\mathcal{M})}. \tag{73}$$

Thus, from this inequality and (62) it follows the result (46). □

6. Proofs of Theorems 1 and 2

The main idea of the proof of Theorem 1 consists of solving first the problem for a fixed right-hand side term $z \in L^\infty((0,T)\times\mathcal{M})$. Thus, if we denote by u_z such a solution, we shall show, by application of the following fixed point theorem, that for some z the relation $z \in \beta(x, u_z)$ holds

Theorem 6 ([29]). *Let K be a nonempty convex and weakly compact subset in a real Banach space. If $\mathcal{L} : K \to K$ is a function whose graph is weakly × weakly sequentially closed, then $\mathcal{L}$ has at least one fixed point.*

Now we proceed with the proof of Theorem 1 stated in Section 2.

Proof of Theorem 1. As in [15] we denote by H the space $L^2(\mathcal{M})$ equipped with the equivalent inner product

$$\langle \phi, \psi \rangle_H = \int_{\mathcal{M}} \phi\psi c \mathrm{d}A.$$

Let $\widetilde{A}$ be the subdifferential (with the standard inner product) of the convex functional

$$\Phi : \phi \to \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathcal{M}} k\,|\nabla\phi|^p\,\mathrm{d}A & \text{for } \phi \in W^{1,p}(\mathcal{M}), \\ +\infty & \text{otherwise.} \end{cases}$$

It is well known that $dom(\widetilde{A}) \subset W^{1,p}(\mathcal{M})$ is dense in H and that $\widetilde{A}\phi = -\mathrm{div}(k|\nabla\phi|^{p-2}\nabla\phi)$. We define now $\widehat{A}$ by $dom(\widehat{A}) = dom(\widetilde{A})$ and

$$\left\langle \widehat{A}\phi, \psi \right\rangle_H = \left\langle \widetilde{A}\phi, \psi \right\rangle_{L^2(\mathcal{M})}.$$

If we denote by $\overset{*}{\partial}$ the subdifferential with respect to the other inner product, then $\widehat{A}$ coincides with $\overset{*}{\partial}\,\Phi$ (and, by definition, with $\partial\Phi$). We also define the operator

$$\begin{aligned} &\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H \\ &w \longrightarrow \widetilde{A}w + \mathcal{g}(w). \end{aligned}$$

Notice that $\mathcal{A}$ coincides with the subdifferential of the proper lower semicontinuous and convex mapping $\Lambda : D(\Lambda) \subset H \to \mathbb{R}$ defined by

$$\Lambda(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathcal{M}} |\nabla u|^p \mathrm{d}A + \int_{\mathcal{M}} G(u)\mathrm{d}A & u \in D(\Lambda) \\ +\infty \quad u \notin D(\Lambda) \end{cases} \tag{74}$$

where $G(u) = \int_0^u \mathcal{g}(\sigma)\mathrm{d}\sigma$ and

$$D(\Lambda) := \left\{ u \in L^2(\mathcal{M}),\ \nabla u \in L^p(T\mathcal{M}) \text{ and } \int_{\mathcal{M}} G(u)\mathrm{d}A < +\infty \right\},$$

which is dense in H. To apply Theorem 6, we formulate problem (P) as a fixed point problem: u is a solution of (P) if and only if z is a fixed point of the multi-valued operator

$$\mathcal{L} : K \longrightarrow 2^{L^p(0,T;H)}.$$

$\mathcal{L}$ is defined as follows. First, we choose the set K as

$$K = \{z \in L^p(0,T;H) : \quad \|z(t)\|_{L^\infty(\mathcal{M})} \le C_0 \text{ a.e. } t \in (0,T)\}$$

where $C_0 := Q\beta_w \|S\|_{L^\infty((0,T)\times\mathcal{M})}$. We notice that K is a nonempty convex and weakly compact set in $L^p(0,T;H)$. Second, we consider the operator

$$\begin{aligned} &\mathcal{S} : K \longrightarrow C([0,T];H) \\ &z \longrightarrow v \end{aligned}$$

where v is the unique mild solution of the Cauchy problem

$$\begin{cases} c\dfrac{\mathrm{d}v}{\mathrm{d}t}(t) + \mathcal{A}(v) = z & \text{in } H, \\ v(0) = u_0. \end{cases}$$

By virtue of the properties of $\mathcal{A}$ there exists a unique strong solution to this problem (see [6]). Third, we introduce a *selection* operator $\mathcal{F}$ of the graph on the right-hand side of (P) as

$$\begin{aligned} &\mathcal{F} : L^p(0,T;H) \longrightarrow 2^{L^p(0,T;H)} \\ &v \longrightarrow \{h : h \in QS\beta(x,v) + f \quad \text{a.e. } (t,x)\}. \end{aligned}$$

Finally, we define the operator

$$\mathcal{L}(z) = \{h \in L^p(0,T;H) : h \in \mathcal{F}(\mathcal{S}(z))\}.$$

Now, in order to apply Theorem 6 we use the same arguments as in [18] to prove that

(i) for each $v \in L^p((0,T);H)$, $\mathcal{F}(v)$ is a nonempty, convex and closed set of $L^p((0,T);H)$,
(ii) graph $(\mathcal{F})$ is strongly $\times$ weakly sequentially closed in $L^p(0,T;H) \times L^p(0,T;H)$.
(iii) $\mathcal{S}$ is sequentially continuous from $L^p((0,T);H)-$ weak to $C([0,T];H)-$strong.

From (i)–(iii) we conclude that graph($\mathcal{L}$) is weakly $\times$ weakly closed and then, according to Theorem 6, $\mathcal{L}$ has at least a fixed point.

Since $\mathcal{A}$ is T-accretive in $L^2(\mathcal{M})$ and β is a bounded maximal monotone graph, we obtain the following result

Lemma 10. *If $u_0 \in L^\infty(\mathcal{M})$ and $f \in L^\infty((0,T)\times\mathcal{M})$ then $u \in L^\infty((0,T)\times\mathcal{M})$.* $\square$

Our next concern is to prove Theorem 2. To do so we need an auxiliary result on non-degenerate functions showing that, for any $q \geq 1$, the multi-valued function β generates a continuous operator from a subset of $L^\infty(\mathcal{M})$ to $L^q(\mathcal{M})$.

Lemma 11 (*[18]*).

(i) *Let $w, \hat{w} \in L^\infty(\mathcal{M})$ and assume that w satisfies the non-degeneracy property. Then*

$$\begin{aligned} &\forall q \in [1,\infty) \quad \exists \hat{C} > 0 \text{ such that } z, \hat{z} \in L^\infty(\mathcal{M}) \text{ with } z(x) \in \beta(x,w), \quad \hat{z}(x) \in \beta(x,\hat{w}) \text{ a.e. } x \in \mathcal{M} \\ &\quad \|z-\hat{z}\|_q \leq (\beta_w - \beta_i)\min\{\hat{C}\|w-\hat{w}\|_\infty^{1/q}, |\mathcal{M}|^{1/q}\} \quad \square \end{aligned}$$

(ii) *Let $w, \hat{w} \in L^\infty(\mathcal{M})$ and assume that $w, \hat{w}$ satisfy the weak non-degeneracy property. Then*

$$\int_{\mathcal{M}} (z(x)-\hat{z}(x))(w(x)-\hat{w}(x))\mathrm{d}A \leq (\beta_w - \beta_i)C\|w-\hat{w}\|^2_{L^\infty(\mathcal{M})} \tag{75}$$

Proof of Theorem 2. Let $u, \hat{u}$ be bounded weak solutions of (P). We take the difference of the weak formulation (5) of (P) for u and $\hat{u}$ and choose $v = u - \hat{u}$ as a test function. then

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}} c(x)|u(t)-\hat{u}(t)|^2\mathrm{d}A + \int_{\mathcal{M}} (\mathcal{G}(u)-\mathcal{G}(\hat{u}))(u-\hat{u})\mathrm{d}A \\ &\quad + \int_{\mathcal{M}} k(x)\langle|\nabla u(t)|^{p-2}\nabla u(t) - |\nabla\hat{u}(t)|^{p-2}\nabla\hat{u}(t),\ \nabla u(t) - \nabla\hat{u}(t)\rangle\mathrm{d}A \\ &= Q\int_{\mathcal{M}} S(x)(z(x,t)-\hat{z}(x,t))(u(x,t)-\hat{u}(x,t))\mathrm{d}A. \end{aligned} \tag{76}$$

By Lemma 11 and Theorem 1 it is easy to show that if $p > 2$ there exist positive constants C_l, C_0, $\tilde{C}_{1,p,\infty}$ and $\tilde{C}_0$ such that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c(x)(u-\hat{u})\|^2_{L^2(\mathcal{M})} \leq{}& (C_l Q\|S\|_{L^\infty(\mathcal{M})} - \frac{C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{\tilde{C}_{1,p,\infty}})\|u-\hat{u}\|^2_{L^\infty(\mathcal{M})} \\ &+ \tilde{C}_0\|u-\hat{u}\|^2_{L^2(\mathcal{M})}. \end{aligned}$$

Similarly, when $p = 2$ there exist positive constants C_l and $\tilde{C}_{1,2,\sigma}$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c(x)(u-\hat{u})\|^2_{L^2(\mathcal{M})} \le \left(C_l Q\|S\|_{L^\infty(\mathcal{M})} - \frac{|\mathcal{M}|^{\frac{2}{\sigma}}}{\tilde{C}_{1,2,\sigma}}\right)\|u-\hat{u}\|^2_{L^\infty(\mathcal{M})} + \|u-\hat{u}\|^2_{L^2(\mathcal{M})} + \frac{\epsilon}{\tilde{C}_{1,2,\sigma}}.$$

We notice that if $C_l Q\|S\|_{L^\infty(\mathcal{M})} - \frac{C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{\tilde{C}_{1,p,\infty}} < 0$, the application of the Gronwall lemma to the last inequalities yields $\|u-\hat{u}\|^2_{L^2(\mathcal{M})} = 0$, and so the uniqueness of solutions. When $C_l Q\|S\|_{L^\infty(\mathcal{M})} - \frac{C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{\tilde{C}_{1,p,\infty}}$ is not negative, we define a rescaling on $\mathcal{M}$ and obtain the manifold $\mathcal{M}_\delta$ from a new atlas $\{\tilde{W}_\lambda, \tilde{w}_\lambda\}_{\lambda\in\Lambda}$, where $\tilde{w}_\lambda(\tilde{x}) = w_\lambda(\frac{\tilde{x}}{\delta})$ and $\{W_\lambda, w_\lambda\}_{\lambda\in\Lambda}$ an atlas of $\mathcal{M}$. Now, the partition of unity of $\mathcal{M}_\delta$ subordinate to the covering $\tilde{W}_\lambda$ can be defined as $\tilde{\alpha}_\lambda(\tilde{x}) = \alpha_\lambda(\frac{\tilde{x}}{\delta})$ with the new metric verifying $\tilde{g}_{ij} = \delta^2 g_{ij}$. Hence, $|\mathcal{M}_\delta| = \delta^2|\mathcal{M}|$. The formulation of (P) on the manifold $\mathcal{M}_\delta$ is:

$$(P_\delta)\begin{cases} c_\delta(\cdot)\tilde{u}_t - \delta^p \mathrm{div}_{\mathcal{M}_\delta}\left(k_\delta(\cdot)|\nabla_{\mathcal{M}_\delta}\tilde{u}|^{p-2}\nabla_{\mathcal{M}_\delta}\tilde{u}\right) + \mathcal{g}(\cdot,\tilde{u}) \in QS\beta(\cdot,\tilde{u}) + f & \text{in } (0,T)\times\mathcal{M}_\delta \\ \tilde{u}(0,\tilde{x}) = u_0\left(\frac{\tilde{x}}{\delta}\right). \end{cases}$$

If we repeat the last argument for (P_δ), we get for $p > 2$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_\delta(x)u_\delta - \hat{u}_\delta\|^2_{L^2(\mathcal{M}_\delta)} \le \left(C_{l,\delta}Q\|S_\delta\|_{L^\infty(\mathcal{M}_\delta)} - \frac{C_0\delta^p\|u_\delta-\hat{u}_\delta\|^{p-2}_{L^\infty(\mathcal{M}_\delta)}}{\tilde{C}_{1,p,\infty,\delta}}\right)\|u_\delta-\hat{u}_\delta\|^2_{L^\infty(\mathcal{M}_\delta)} + \tilde{C}_0\|u_\delta-\hat{u}_\delta\|^2_{L^2(\mathcal{M}_\delta)}. \quad (77)$$

In the case $p = 2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\delta - \hat{u}_\delta\|^2_{L^2(\mathcal{M}_\delta)} \le \left(C_{l,\delta}Q\|S_\delta\|_{L^\infty(\mathcal{M}_\delta)} - \frac{\delta^2|\mathcal{M}_\delta|^{\frac{2}{\sigma}}}{\tilde{C}_{1,2,\sigma,\delta}}\right)\|u_\delta-\hat{u}_\delta\|^2_{L^\infty(\mathcal{M}_\delta)} \quad (78)$$

$$+ \|u_\delta-\hat{u}_\delta\|^2_{L^2(\mathcal{M}_\delta)} + \frac{\epsilon}{\tilde{C}_{1,2,\sigma,\delta}}. \quad (79)$$

Now, we determine the dependence of the constants $C_{l,\delta}$, $\tilde{C}_{1,p,\infty,\delta}$ and $\tilde{C}_{1,2,\sigma,\delta}$ in terms of δ. For that, we consider the Banach space

$$V_\delta = \{u \in L^2(\mathcal{M}_\delta) : \nabla u \in L^p(T\mathcal{M}_\delta)\}.$$

The constant $C_{l,\delta}$ appears in Lemma 11 when we substitute $\mathcal{M}$ by $\mathcal{M}_\delta$. So that, we have

$$\|z_\delta - \hat{z}_\delta\|_{L^1(\mathcal{M}_\delta)} \le (\beta_w - \beta_i)\tilde{C}_\delta\|u_\delta - \hat{u}_\delta\|_{L^\infty(\mathcal{M}_\delta)},$$

where $\tilde{C}_\delta = \max\{C_\delta, \frac{|\mathcal{M}_\delta|}{\epsilon_0^{p-1}}\} = \delta^2\max\{C, \frac{|\mathcal{M}|}{\epsilon_0^{p-1}}\} = \delta^2\tilde{C}$, C and C_δ are the constants of non-degeneracy for $\mathcal{M}$ and $\mathcal{M}_\delta$ respectively. Then, it follows that

$$C_{l,\delta} = \delta^2 C_l.$$

The constant $\tilde{C}_{1,2,\sigma,\delta}$ verifies

$$\|f\|^2_{L^\sigma(\mathcal{M})} \le \tilde{C}_{1,2,\sigma,\delta}(\|\nabla f\|^2_{L^2(T\mathcal{M}_\delta)} + \|f\|^2_{L^2(\mathcal{M}_\delta)}).$$

$$\|f\|^2_{L^\sigma(\mathcal{M}_\delta)} \le 2\tilde{\mu}^{\frac{2}{\sigma}}k(r,2,\sigma)^2\tilde{\nu}^{-1}\max\{1,\tilde{\mu}\}(1+\sup|\nabla\tilde{\alpha}_\lambda|)^2(\|f\|_2^2 + \|\nabla f\|_2^2).$$

From the relations $\tilde{\nu} = \delta^2\nu$, $\tilde{\mu} = \delta^2\mu$ and $|\tilde{\alpha}_\lambda| = \frac{1}{\delta}|\alpha_\lambda|$ we obtain

$$\tilde{C}_{1,2,\sigma,\delta} = 2\delta^{\frac{4}{\sigma}-2}\mu^{\frac{2}{\sigma}}k(r,p,\sigma)^2\nu^{-1}\max\{1,\delta^2\mu\}\left(1+\sup\frac{1}{\delta}|\nabla\alpha_\lambda|\right)^2.$$

$\tilde{C}_{1,p,\infty,\delta}$ is the constant of the imbedding $V_\delta \subset L^\infty(\mathcal{M}_\delta)$. In particular, if $\delta = 1$ this is the constant $\tilde{C}_{1,p,\infty}$ that for $p > 2$ is given by

$$\tilde{C}_{1,p,\infty} = 2^{p-1}k(p,r)^p\max\{\nu^{\frac{-p}{2}}, \nu^{-1}\mu^{\frac{p}{2}}\}(1+C_{1,2,p}\sup|\nabla\alpha_\lambda|)^p\max\{1,|\mathcal{M}|^{\frac{p-2}{2}}\};$$

hence,

$$\tilde{C}_{1,p,\infty,\delta} = 2^{p-1}k(p,r)^p \max\{\tilde{\nu}^{\frac{-p}{2}}, \tilde{\nu}^{-1}\tilde{\mu}^{\frac{p}{2}}\}(1 + C_{1,2,p,\delta} \sup |\nabla\tilde{\alpha}_\lambda|)^p \max\{1, |\mathcal{M}_\delta|^{\frac{p-2}{2}}\}.$$

By using $\tilde{\nu} = \delta^2\nu$, $\tilde{\mu} = \delta^2\mu$ and $|\nabla\tilde{\alpha}_\lambda| = \frac{1}{\delta}|\nabla\alpha_\lambda|$ we obtain

$$\begin{aligned}\tilde{C}_{1,p,\infty,\delta} &= 2^{p-1}k(p,r)^p \max\{\delta^{-p}\nu^{\frac{-p}{2}}, \delta^{p-2}\nu^{-1}\mu^{\frac{p}{2}}\}(1 + \delta^{\frac{2}{p}-1}\mu^{\frac{1}{p}}k(r,2,p)\nu^{\frac{-1}{2}} \\ &\quad\times \max\{1, \delta\mu^{\frac{1}{2}}\}\left(1 + \sup\frac{1}{\delta}|\nabla\alpha_\lambda|\right)\sup\frac{1}{\delta}|\nabla\alpha_\lambda|)^p \max\{1, \delta^{p-2}|\mathcal{M}|^{\frac{p-2}{2}}\}.\end{aligned}$$

Next, we define $K_{p,\delta}$ by

$$K_{p,\delta} := \begin{cases} C_{l,\delta}Q\|S_\delta\|_{L^\infty(\mathcal{M}_\delta)} - \dfrac{\delta^2|\mathcal{M}_\delta|^{\frac{2}{\sigma}}}{\tilde{C}_{1,2,\sigma,\delta}} & \text{if } p = 2, \\ C_{l,\delta}Q\|S_\delta\|_{L^\infty(\mathcal{M}_\delta)} - \dfrac{\delta^p C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{\tilde{C}_{1,p,\infty,\delta}} & \text{if } p > 2. \end{cases}$$

It is easy to see that $\|S_\delta\|_{L^\infty(\mathcal{M}_\delta)} = \|S\|_{L^\infty(\mathcal{M})}$. By substituting every constant in terms of δ we have that

$$K_{p,\delta} = \begin{cases} \delta^2 C_l Q\|S\|_{L^\infty(\mathcal{M}_\delta)} - \dfrac{\delta^2\delta^{\frac{4}{\sigma}}|\mathcal{M}|^{\frac{2}{\sigma}}}{\delta^{\frac{4}{\sigma}-2}\max\{1,\delta^2\mu\}(1+\frac{1}{\delta}\sup|\nabla\alpha_\lambda|)^2 C_2} & \text{if } p = 2, \\ \delta^2 C_l Q\|S\|_{L^\infty(\mathcal{M})} - \dfrac{\delta^p C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{\max\{\delta^{-p}\nu^{\frac{-p}{2}}, \delta^{p-2}\nu^{-1}\mu^{\frac{p}{2}}\}\tilde{K}_{p,\delta}C_p} & \text{if } p > 2, \end{cases}$$

where

$$\tilde{K}_{p,\delta} = \left(1 + C_{1,2,p,\delta}\sup\frac{1}{\delta}|\nabla\alpha_\lambda|\right)^p \max\{1, \delta^{p-2}|\mathcal{M}|^{\frac{p-2}{2}}\}$$

and C_2 and C_p independent of δ. Then, if $p = 2$

$$\lim_{\delta\to 0} K_{2,\delta} = \lim_{\delta\to 0}\delta^2 C_l Q\|S\|_{L^\infty(\mathcal{M})} - \frac{\delta^4|\mathcal{M}|^{\frac{2}{\sigma}}}{\max\{1,\delta^2\mu\}(1+\frac{1}{\delta}\sup|\nabla\alpha_\lambda|)^2 C_2},$$

and if $p > 2$

$$\lim_{\delta\to 0} K_{p,\delta} = \lim_{\delta\to 0}\delta^2 C_l Q\|S\|_{L^\infty(\mathcal{M})} - \frac{\delta^p C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{\max\{\delta^{-p}\nu^{\frac{-p}{2}}, \delta^{p-2}\nu^{-1}\mu^{\frac{p}{2}}\}\tilde{K}_{p,\delta}C_p}.$$

In both cases the limit is zero and this reduces the proof to the first case.

To prove part (ii) we assume that there exist two solutions u and $\hat{u}$ of (P) verifying the weak non-degeneracy property. Arguing as in (i) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-\hat{u}\|^2_{L^2(\mathcal{M})} + \frac{C_0}{\tilde{C}_{1,p,q}}\|u-\hat{u}\|^p_{L^\infty(\mathcal{M})} \le \int_{\mathcal{M}} QS(z-\hat{z})(u-\hat{u})\mathrm{d}A + \tilde{C}_0\|u-\hat{u}\|^2_{L^2(\mathcal{M})}$$

where $\tilde{C}_{1,p,q} = \tilde{C}_{1,p,\infty}$ if $p > 2$, and equal to $\tilde{C}_{1,2,\sigma}$ if $p = 2$. By Lemma 11, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-\hat{u}\|^2_2 \le \left(CQ\|S\|_{L^\infty(\mathcal{M})} - \frac{C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{C_{1,p}}\right)\|u-\hat{u}\|^2_{L^\infty(\mathcal{M})} + \tilde{C}_0\|u-\hat{u}\|^2_{L^2(\mathcal{M})},$$

with C being the constant of the weak non-degeneracy property. We conclude the uniqueness as in (i) by studying the sign of $CQ\|S\|_{L^\infty(\mathcal{M})} - \frac{C_0\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{C_{1,p,\infty}}$ and by rescaling in the case that such a sign is positive. $\square$

7. Numerical tests

We present numerical results with the term of the radiation energy modelled according to Budyko's formulation, $R_e(u) = Bu + C$, this means that $\mathcal{G}(x,u) = Bu$. In Fig. 1 we show the initial icosahedron that yields the partition D_0 and the partition (mesh) D_4. The numerical tests are carried out on the partition D_6 composed of $N_6 = 81\,920$ triangles and $M = 40\,062$ nodes (or vertices), with an average mesh size $h = 0.02067$ (radians) or equivalently $h \simeq 130$ km.

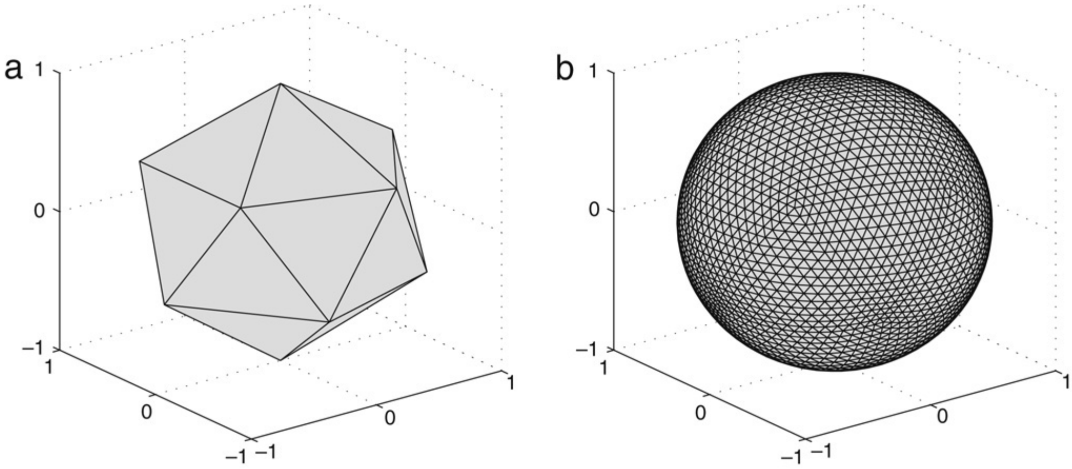

Fig. 1. Initial Icosahedron and mesh after 4 refinements.

As we said in Section 3, we approximate the partition D_k of the 2-sphere $\mathcal{M}$ by the partition D_{hk} composed of triangles in $\mathbb{R}^3$ that generate the polyhedron $\mathcal{M}_h$ of triangular faces. To calculate the numerical solution in the family of finite element spaces $\widehat{V}_h$ we employ the method introduced in [17] after noting that $\nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})$ can be written as

$$\nabla_{\mathcal{M}} u = \nabla u - (\overrightarrow{n}_{\mathcal{M}} \cdot \nabla u)\overrightarrow{n}_{\mathcal{M}},$$

where $\overrightarrow{n}_{\mathcal{M}}$ is the unit outward normal vector on $\mathcal{M}$ and $\nabla u = \left(\frac{\partial u}{\partial x_i}\right)_{i=1,2,3}$ denotes the gradient of u considered as a function of the Cartesian coordinates (x_1, x_2, x_3) referred to the Cartesian coordinate system, the origin of which is at the centre of the sphere. Recalling that $\widehat{u}(x)$, $x \in \mathcal{M}_h$, is a lifting of $u(x^*)$, $x^* \in \mathcal{M}$, then $\nabla_{\mathcal{M}} u$ will be numerically approximated by the approximation to $\nabla_{\mathcal{M}_h} \widehat{u}(x) \in L^p(T\mathcal{M}_h)$ the expression of which is

$$\nabla_{\mathcal{M}_h} \widehat{u}(x) = \nabla \widehat{u}(x) - (\overrightarrow{n}_{\mathcal{M}_h} \cdot \nabla \widehat{u}(x))\overrightarrow{n}_{\mathcal{M}_h} \quad \text{for any } x \in \mathcal{M}_h,$$

where $\overrightarrow{n}_{\mathcal{M}_h}$ denotes the unit outward normal vector on $\mathcal{M}_h$, which is a constant vector on each triangular face Ω_j of $\mathcal{M}_h$, defining thus a piecewise constant approximation to $\overrightarrow{n}_{\mathcal{M}}$. $\widehat{u}(x)$ is approximated by $\widehat{u}_h(x) \in \widehat{V}_h$ satisfying $\widehat{u}_h(P)|_{\Omega_j} \in P_1(\Omega_j)$. The local basis functions $\{\lambda_k(x)\}_{k=1}^3$ are the barycentric coordinates defined by the relations

$$\sum_{k=1}^{3} x_{ki}\lambda_k = x_i, \quad \text{for } i = 1, 2, 3$$
$$\sum_{k=1}^{3} \lambda_k = 1 \quad \forall P \in \Omega_j$$

where x_i are the coordinates of any point $x \in \Omega_j$ and x_{ki} are the coordinates of the vertices of Ω_j. Then, denoting by $\overrightarrow{n}_j$ the unit normal vector on Ω_j we have that for any $x \in \Omega_j$

$$\nabla_{\mathcal{M}_h} \widehat{u}_h(x) = \sum_{k=1}^{3} U_k \nabla\lambda_k - \left(\sum_{l=1}^{3} n_{jl} \sum_{k=1}^{3} U_k \frac{\partial \lambda_k}{\partial x_l}\right)\overrightarrow{n}_j.$$

where $U_k = \widehat{u}_h(x_k)$. We remark that by construction of the family of finite element spaces V_h, U_k are also the values $u_h(x_k)$. Important features that make this formulation attractive for computations are the absence of the so-called "pole problem" and the discretization of the Laplace–Beltrami operator can be managed with the computer codes developed for the Laplace operator in a Cartesian coordinate system. To see this is so, we consider $\int_{\Omega_j} \nabla_{\mathcal{M}_h} \widehat{u}_h \cdot \nabla_{\mathcal{M}_h} \widehat{v}_h \mathrm{d}A_h$ and obtain

$$\int_{\Omega_j} \nabla_{\mathcal{M}_h} \widehat{u}_h \cdot \nabla_{\mathcal{M}_h} \widehat{v}_h \mathrm{d}A_h = VS_jU^{\mathrm{T}}$$

where $V = (V_1, V_2, V_3)$, $U = (U_1, U_2, U_3)$, with V_k and U_k being the values of $\widehat{v}_h$ and $\widehat{u}_h$ at the vertices of Ω_j respectively, and S_j is Ω_j-element symmetric matrix the entries of which are

$$s_{ik} = \int_{\Omega_j} \nabla\lambda_i \cdot (\nabla\lambda_k - (\overrightarrow{n}_j \cdot \nabla\lambda_k)\overrightarrow{n}_j)\, \mathrm{d}A_h, \quad 1 \le i, k \le 3.$$

Note that s_{ik} are the entries of the stiffness matrix corresponding to the Laplace operator minus $\int_{\Omega_j} (\overrightarrow{n}_j \cdot \nabla\lambda_i)(\overrightarrow{n}_j \cdot \nabla\lambda_k)\mathrm{d}A_h$.

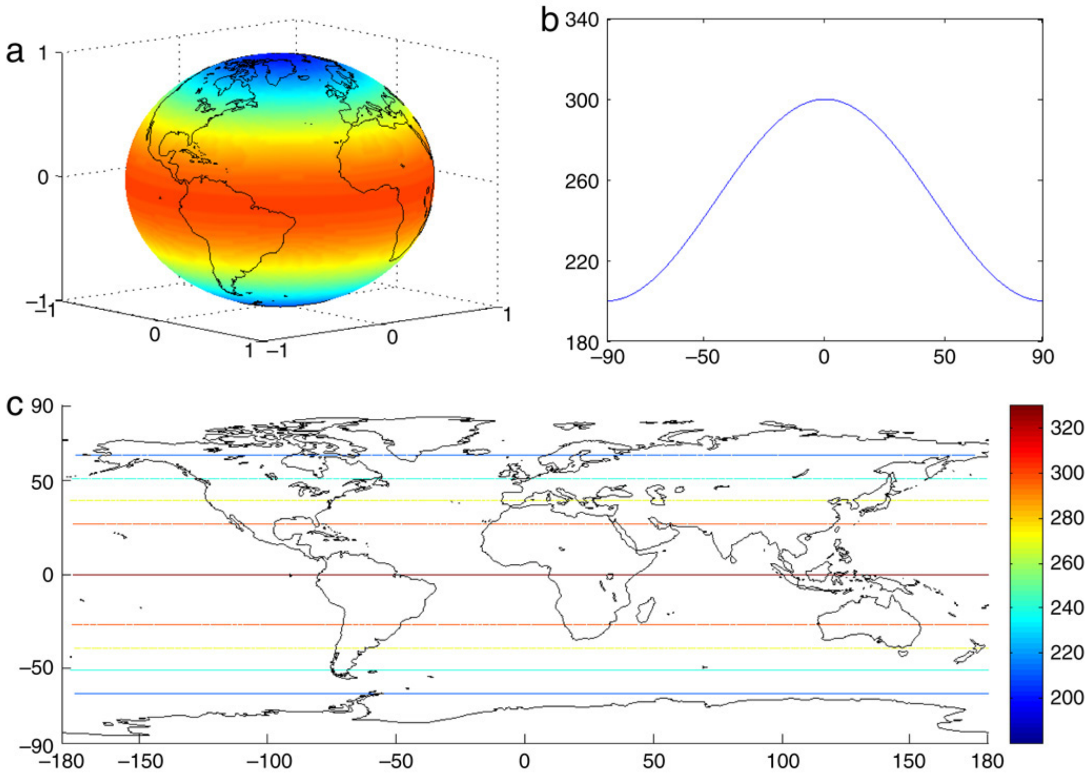

Fig. 2. Distribution of temperature at time $t = 0$.

To calculate the numerical solution it is convenient to work with the non-dimensional formulation of the equations. To this end, we take $T_c = 10^4$ years as the characteristic time scale and the radius a of the Earth as the characteristic space scale. The non-dimensional temperature u is given by the formula

$$u = \frac{u^* - \underline{u_0^*}}{\overline{u_0^*} - \underline{u_0^*}},$$

where $u^*(x, t)$ denotes the sea-level atmospheric temperature expressed in Kelvin degrees, $\underline{u_0^*}$ and $\overline{u_0^*}$ represent the minimum and maximum values of u_0^* (the initial temperature) respectively. In our computations $u_0^* = 300 - 100\cos^2\theta$, $0 \leq \theta \leq \pi$, see Fig. 2, where the right side upper panel shows the distribution of temperature (vertical axis) in Kelvin degrees along the meridian $\varphi = 0^o$, with the horizontal axis being the colatitude ϑ-axis; here the North Pole is located at $\vartheta = 90^o$ and the South Pole at $\vartheta = -90^o$. The lower panel displays the distribution of temperature in the parameter (ϑ, φ)-plane.

The value of the coefficient $c(x)$ is taken as

$$c(x) = \frac{a}{3}\frac{\overline{u_0^*} - \underline{u_0^*}}{T_c}(\overline{\rho c_p})(\rho c)_s,$$

where $\overline{\rho c_p}$ denotes the average product of density times specific heat of the planet Earth, whereas $(\rho c)_s$ is a correction factor to account for the variation of ρc on the Earth surface. We take the following values [21]

$$(\rho c)_s = \begin{cases} 0.8 & \text{for land} \\ 1.2 & \text{for sea} \\ 0.9 & \text{for ice.} \end{cases}$$

The coefficient $k(x)$ is the thermal conductivity given by the formula

$$\frac{\overline{u_0^*} - \underline{u_0^*}}{3a} k_m k_s,$$

where $k_m = 300\ \mathrm{Wm}^{-1}K^{-1}$ denotes the average conductivity of the planet Earth and k_s is a correction factor for the Earth surface that we take as [21]

$$k_s = \left(0.86 + 0.311\cos^2\theta - 0.98\cos^4\theta\right)(1 - 0.73r),$$

with $r = 1$ if $\theta < \frac{\pi}{2} - 0.22\pi$ and zero otherwise. The values of Q, S and the coalbedo as well as the constants B and C of Budyko's model for the radiation energy are borrowed from [21]. Thus

$$QS(t,x) = 340\left(1.24 - 0.72\cos^2\theta\right)$$
$$\beta(x,u) = \begin{cases} 0.8 & \text{if } u^* > 250\ ^\circ\text{K}, \\ 0.25 & \text{otherwise.} \end{cases}$$
$$\mathcal{g}(x,u^*) = 2.03u^*,$$
$$f(t,x) = 2.03 \times 273.16 - 212.$$

To calculate the solution U^n to the fully discrete problem $(P_{h,\Delta t})$ we have to solve a nonlinear problem with two nonlinearities, namely, the one due to the p-Laplacian when $p > 2$,

$$\int_{\mathcal{M}} \left\langle k\left|\nabla U^n\right|^{p-2}\nabla U^n, \nabla v_h\right\rangle \mathrm{d}A,$$

and the other one due to the Heaviside graph on the right-hand side,

$$\int_{\mathcal{M}} QS\beta(x,U^n)v_h \mathrm{d}A.$$

We deal with the p-Laplacian nonlinearity approximating U^n in $|\nabla U^n|^{p-2}$ by the second-order extrapolation formula $2U^{n-1} - U^{n-2}$ when $n > 1$, and by U^{n-1} when $n = 1$; whereas the second nonlinearity is treated by monotone iteration. Thus, at each time step t_n we calculate $U^n \in V_h$ by the following iterative procedure:

For $k = 0$ set $W^0 = 2U^{n-1} - U^{n-2}$ if $n > 1$ or $W^0 = U^{n-1}$ if $n = 1$, then for $k = 1, 2, \ldots$ solve

$$\int_{\mathcal{M}_h} cW^k v_h \mathrm{d}A_h + \Delta t\int_{\mathcal{M}_h}\left\langle k\left|\nabla_{\mathcal{M}_h}W^0\right|^{p-2}\nabla_{\mathcal{M}_h}W^k, \nabla_{\mathcal{M}_h}v_h\right\rangle \mathrm{d}A_h + \Delta t\int_{\mathcal{M}_h}\mathcal{g}(x,W^k)v_h\mathrm{d}A_h$$
$$= \int_{\mathcal{M}_h} cU^n v_h\mathrm{d}A_h + \Delta t\int_{\mathcal{M}_h} QS^n\beta(x,W^{k-1})v_h\mathrm{d}A_h + \Delta t\int_{\mathcal{M}_h} f v_h\mathrm{d}A_h, \quad v_h \in V_h,$$

stop when

$$\frac{\left\|W^k - W^{k-1}\right\|_{L^2(\mathcal{M}_h)}}{\left\|W^0\right\|_{L^2(\mathcal{M}_h)}} \le \mathrm{tol}, \quad \text{or} \quad k = K\mathrm{MAX}.$$

Set

$$U^{n+1} = W^k.$$

In the numerical experiments we take tol $= 10^{-4}$ and $\Delta t = 10^{-2}$ in non-dimensional time units, this corresponds to 100 years of real time. Fig. 3 displays the distribution of temperature at $T = 7000$ years. The remarkable features of this figure as well as those of Fig. 4 are the following: (i) the ice caps get colder and extend towards the equator; (ii) mid and equatorial latitudes get warmer; (iii) existence of narrow free boundaries (in both northern and southern hemispheres) where the temperature experiences a rapid variation; and (iv) there is some degree of asymmetry in the distribution of temperature in midlatitudes due to the distribution of sea and land in the northern and southern hemispheres.

Fig. 4 shows the distribution of temperature at $T = 10^5$ years when the steady state has been reached. The contrast between Figs. 3 and 4 is that the polar region temperatures are colder in Fig. 4 than in Fig. 3, although the geographical extension of the polar regions is almost the same in both figures; on the other hand, the zone between the parallels $\vartheta = \pm 50^o$ are warmer in Fig. 4 than in Fig. 3, particularly in the tropics and the equator. To see the influence of exponent p on the solution, we show in Fig. 5 the distribution of temperature at $T = 10^5$ years (steady state) with $p = 6$. The main difference with respect to Fig. 4 is that the free boundaries in Fig. 5 are wider than that in Fig. 4.

As a final remark, we must say that the main purposes of these experiments are to illustrate the theoretical analysis and to see the viability of the numerical model; so that, no climatic conclusions should be drawn from these experiments. However, the model may be a valuable tool for qualitative climate studies if more realistic initial condition and coefficients are used to simulate climate scenarios.

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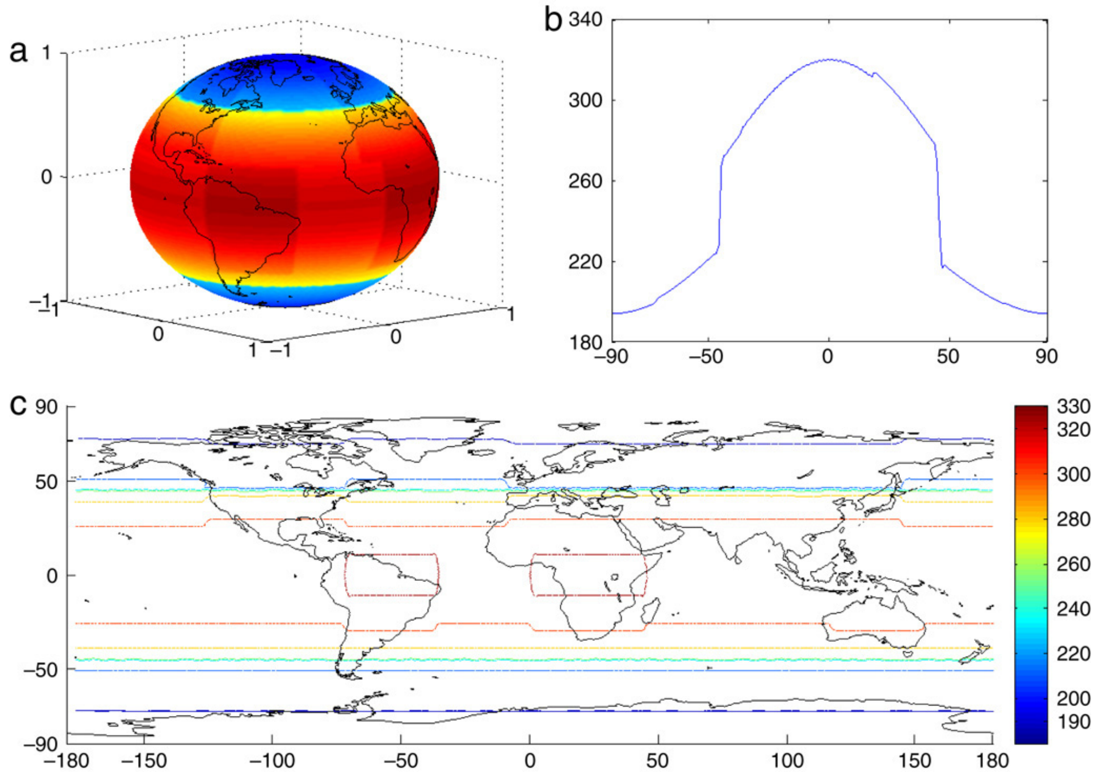

Fig. 3. Distribution of temperature at time $T = 7000$ years with $p = 3$.

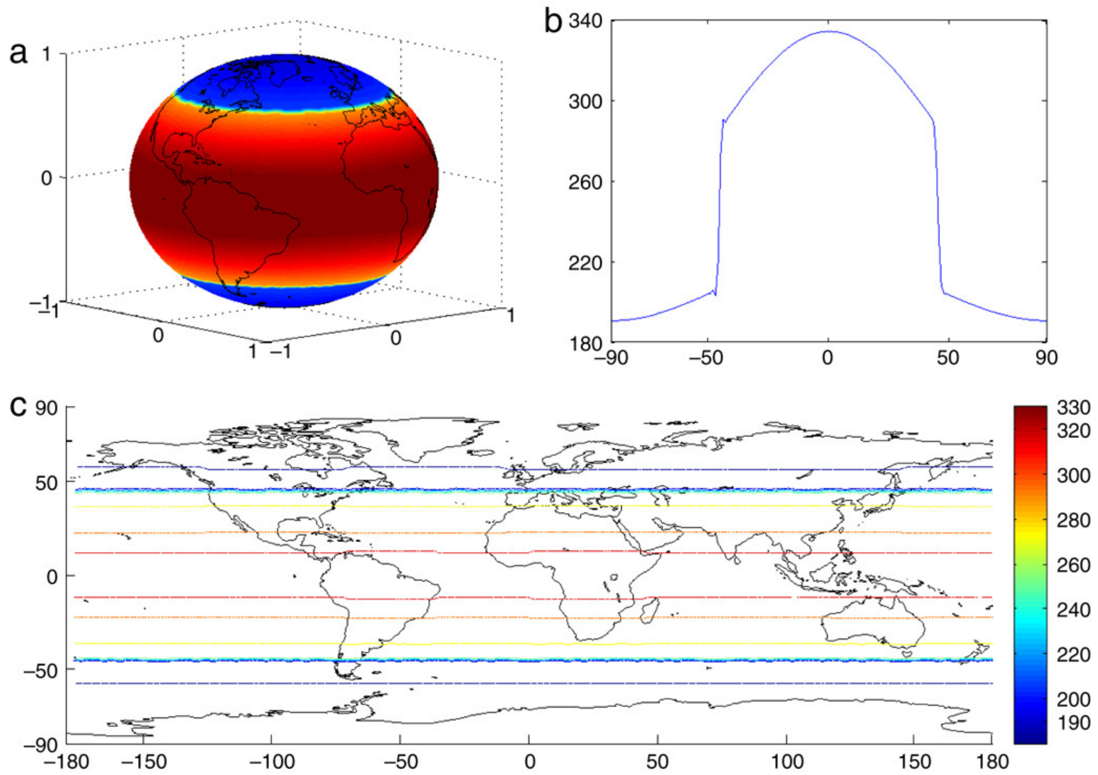

Fig. 4. As Fig. 3 but at $T = 10^5$ years.

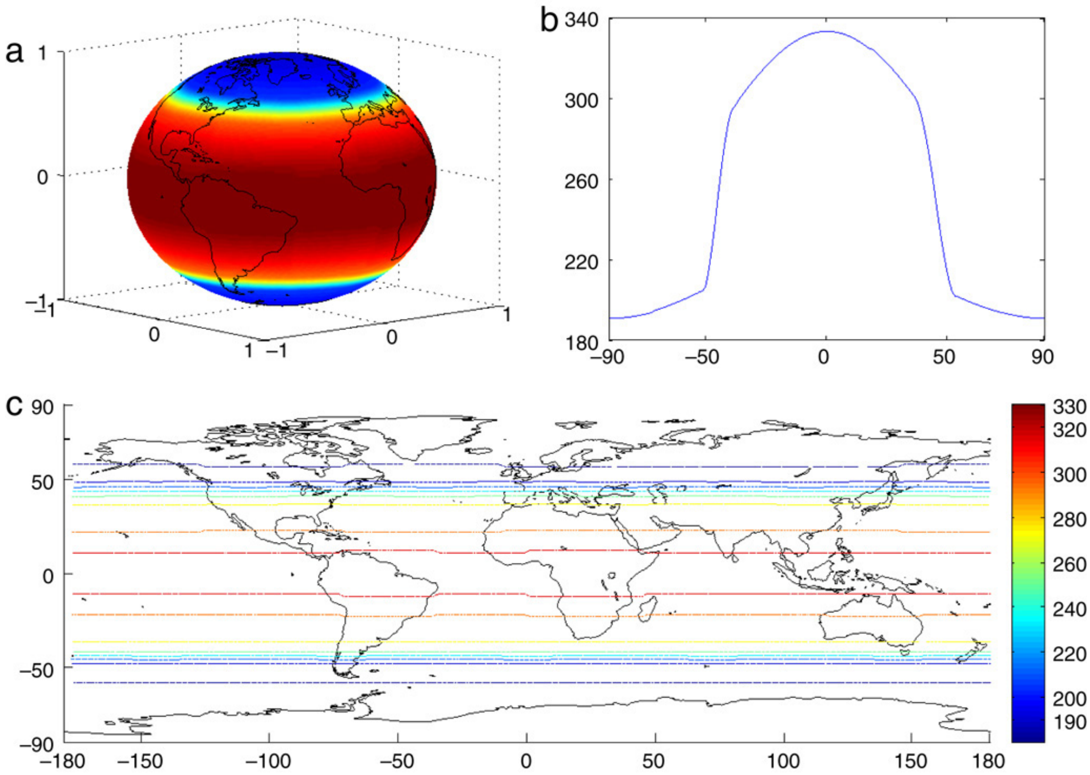

Fig. 5. As Fig. 4 but with $p = 6$.

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