Estimates on the location of the free boundary for the obstacle and Stefan problems by means of some energy methods

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Dedicated to the memory of Jacques-Louis Lions

Abstract. In this paper we use some *energy methods* to study the location (and formation) of the free boundary arising in some unilateral problems, as, for instance, *the obstacle problem* and *the Stefan problem*.

1. Introduction

One of the problems which were formulated and successfully solved during the last half of the past century was the unilateral problems, or variational inequalities, arising in Mechanics, Physics and Economics. After the pioneering works by J.L. Lions and G. Stampacchia [16], [17], this type of problems attracted the attention of Jacques-Louis Lions for many years, who consolidate his researches in the domain in a series of books:[14], [10], [3], [4] and [13].

This type of problems may give rise to a free boundary separating regions of the space where the solution behaves in different ways. This free boundary has an important physical meaning and, besides, it concentrate a very rich information on the solution of the problem (since, usually, it includes the points where the solution loses its regularity).

The main goal of this paper is to extend the, so called, *energy method*, developed since the beginning of the eighties for the study of the free boundaries giving rise by the solutions of nonlinear pdes (see, e.g. the monograph [1] by S.N. Antontsev, J.I. Díaz and S.I. Shmarev to the case of some unilateral problems, as it is the case, for instance, of *the obstacle problem* and *the Stefan problem*.

The qualitative behavior of the coincidence set for the obstacle problem was initiated in 1976 by Brezis and Friedman [6] and Bensoussan and Lions [2] (see also Tartar [20] and Evans and Knerr [11]) by using the maximum principle. Some other general references on the Stefan problem are Friedman [12] and Meirmanov [18]. The energy methods are of special interest in the situations where the traditional methods based on the comparison principles fail. A typical example of such a situation is either a higher-order equation or a system of PDEs. Moreover, even when the comparison principle holds, it may be extremely difficult to construct suitable sub or super-solutions if, for instance, the equation under study contains transport terms and has either variable or unbounded coefficients or the right-hand side.

Here we shall deal with the formation of a free boundary for local solutions of the obstacle problem

$$\frac{\partial \psi(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,D\,u) + B(x,t,u,D\,u) + C(x,t,u) + \beta(u) \ni f(x,t), \qquad (1)$$

where $\beta(u)$ is the maximal monotone graph given by $\beta(u) = \{0\}$ if $u \leq 0$ and $\beta(u) = \phi$ (the empty set) if $u < 0$. The general structural assumptions we shall made are the following

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C_1 |\mathbf{q}|^{p-1}, C_2\,|\mathbf{q}|^p \leq \mathbf{A}(x,t,r,\mathbf{q}) \cdot \mathbf{q}, \qquad (2a)$$

$$|B(x,t,r,\mathbf{q})| \leq C_3\,|r|^{\alpha}\,|\mathbf{q}|^{\beta},\ 0 \leq C(x,t,r)\,r, \qquad (2b)$$

$$C_6\,|r|^{\gamma+1} \leq G(r) \leq C_5\,|r|^{\gamma+1},\ \text{where} \qquad (2c)$$

$$G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau.$$

Here $C_1 - C_6$, p, α, β, σ, γ, k are positive constants which will be specified later on. We shall also consider the Stefan problem

$$\frac{\partial \psi(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,D\,u) + B(x,t,u,D\,u) + C(x,t,u) \ni f(x,t), \qquad (3)$$

where now $\psi(u)$ is the maximal monotone graph $\psi(u) = k_+ u + L$ if $u > 0$, $\psi(u) = k_- u$ if $u < 0$, $\psi(0) = [0, L]$, with k_+, k_- and L positive constants. In both cases, we shall deal with weak solutions satisfying the initial condition

$$u(x,0) = u_0(x) \qquad x \in \Omega. \qquad (4)$$

We point out that this energy method can be applied also to some problems arising in Climatology. For instance, in [8] the existence of the mushy region is proved by giving a partial answer to a question raised by Jacques-Louis Lions (see, e.g., [15]).

2. Statements of the main results

Let us start by considering the obstacle problem.

Definition. A function $u(x,t)$, with $\psi(u) \in C\left([0,T] : L^1_{loc}(\Omega)\right)$, is called weak solution of problem (1), (4) if $u \in L^\infty(0,T;\, L^{\gamma+1}(\Omega')) \cap L^p(0,T;\, W^{1,p}(\Omega'))$, $\overline{\Omega'} \subset \Omega$, $\mathbf{A}(\cdot,\cdot,u,Du)$, $B(\cdot,\cdot,u,Du)$, $C(\cdot,\cdot,u) \in L^1(Q)$; $\liminf_{t\to 0} G(u(\cdot,t)) = G(u_0)$ in $L^1(\Omega)$;

$u(x,t)\geq 0$ and $c(x,t)\in\beta(u(t,x))$ a.e. $(t,x)\in(0,T)\times\Omega$ for some $c\in L^1((0,T)\times\Omega)$, and for every test function $\varphi\in L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)\cap W^{1,2}(0,T;L^\infty(\Omega))$,

$$\int_Q\{\psi(u)\varphi_t-\mathbf{A}\cdot D\varphi-B\,\varphi-C\,\varphi-c\varphi\}\;dxdt\;-\int_\Omega\psi(u)\varphi\,dx\Big|_{t=0}^{t=T}=-\int_Q f\,\varphi\,dx\,dt. \tag{5}$$

In contrast to considerations on the *finite speed of propagation* or the *uniform localization of the support*, we shall use some energy functions defined on domains of a special form. Let us introduce the following notation: given $x_0\in\Omega$ and the nonnegative parameters ϑ and υ, we define the *energy set*

$$P(t)\equiv P(t;\vartheta,\upsilon)=\{(x,s)\in Q:|x-x_0|<\rho(s)\equiv\vartheta(s-t)^\upsilon,s\in(t,T)\}.$$

The shape of $P(t)$, *the local energy set,* is determined by the choice of the parameters ϑ and υ. Here we shall take $\vartheta>0$, $0<\upsilon<1$ and so $P(t)$ becomes a paraboloid (other choices are relevant for the study of different properties: see [1]). We define the *local energy functions*

$$E(P):=\int_{P(t)}|Du(x,\tau)|^p\,dxd\tau,\quad C(P):=\int_{P(t)}|u(x,\tau)|\,dxd\tau$$

$$b(T):=ess\sup_{s\in(t,T)}\int_{|x-x_0|<\vartheta(s-t)^\upsilon}|u(x,s)|^{\gamma+1}\,dx.$$

Although our results have a local nature (for instance, they are independent of the boundary conditions), we shall need some global information on *the global energy function*

$$D(u(\cdot,\cdot)):=ess\sup_{s\in(0,T)}\int_\Omega|u(x,s)|^{\gamma+1}\,dx+\int_Q(|Du|^p+|u|)\,dxdt. \tag{6}$$

For the sake of the exposition, we shall assume the additional condition $\frac{p-1}{p}\leq\gamma\leq p-1$. Our main assumption deals with the forcing term: we assume that there exists $\Theta>0$ and $\rho>0$ such that

$$f(x,t)<-\Theta\text{ on }B_\rho(x_0)\subset\Omega,\text{ a.e. }t\in(0,T). \tag{7}$$

In the presence of the first order term, $B(\cdot,\cdot,u,Du)$, we shall need the extra conditions

$$\begin{cases}\alpha=\gamma-(1+\gamma)\beta/p,\\ C_3<\left(\Theta\frac{p}{p-1}\right)^{(p-\beta)/p}\left(C_2\frac{p}{\beta}\right)^{\beta/p}\text{ if }0<\beta<p,\\ C_3<\Theta\text{ if }\beta=0\text{ (respectively }\Theta<C_2\text{ if }\beta=p).\end{cases} \tag{8}$$

The next result shows how the multivalued term causes the formation of the null-set of the solution, even for positive initial data.

Theorem 1 *There exist some positive constants M, t^*, and $\upsilon\in(0,1)$ such that any weak solution of problem* (1), (4) *with $D(u)\leq M$ satisfies that $u(x,t)\equiv 0$ in $P(t^*:1,\upsilon)$.*

In the case of the Stefan problem a definition of weak solution can be given in similar terms but the integral identity reads now as follows:

$$\int_Q \{\psi_u \varphi_t - \mathbf{A}\cdot D\varphi - B\,\varphi - C\,\varphi\}\; dxdt\; - \int_\Omega \psi(u)\varphi\, dx\Big|_{t=0}^{t=T} = -\int_Q f\,\varphi\, dx\, dt, \qquad (9)$$

for some $\psi_u \in L^1((0,T)\times\Omega)$, $\psi_u(x,t) \in \psi(u(t,x))$ a.e.$(t,x)\in(0,T)\times\Omega$. To simplify the exposition we shall assume now that $\mathbf{A}$ and B are independent of u.

Theorem 2 *Assume that $f(x,t) < -\Theta$ (respect. $f(x,t) > \Theta$) on $B_\rho(x_0)\subset\Omega$, a.e. $t\in(0,T)$. Then there exist some positive constants M, t^*, and $\upsilon\in(0,1)$ such that any weak solution of problem* (3), (4) *with $D(u)\le M$ satisfies that $u(x,t)\le 0$ (respect. $u(x,t)\ge 0$) in $P(t^*:1,\upsilon)$.*

3. Proofs and remarks

The proof of Theorem 1 consists of several parts: Step 1. *The integration-by-parts formula:*

$$\begin{gathered} i_1+i_2+i_3+i_4 = \textstyle\int_{P\cap\{t=T\}} G(u(x,t))dx \\ +\textstyle\int_P \mathbf{A}\cdot Du\, dxd\theta + \int_P B\, udxd\theta + (\int_P C\, udxd\theta - \int_P uf\, dxd\theta) \\ \le \textstyle\int_{\partial_l P} n_x\cdot \mathbf{A}\, u\, d\Gamma d\theta + \int_{\partial_l P} n_\tau G(u(x,t))d\Gamma d\theta \\ +\textstyle\int_{P\cap\{t=0\}} G(u(x,t))dx + := j_1+j_2+j_3, \end{gathered}$$

were $\partial_l P$ denotes the lateral boundary of P i.e. $\partial_l P = \{(x,s):\ |x-x_0| = \vartheta(s-t)^\upsilon,\ s\in(t,T)\}$, $d\Gamma$ is the differential form on the hypersurface $\partial_l P\cap\{t=const\}$, n_x and n_τ are the components of the unit normal vector to $\partial_l P$. This inequality can be proved by taking the cutting function

$$\zeta(x,\theta) := \psi_\varepsilon(|x-x_0|,\theta)\,\xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m(u(x,s))\, ds, \quad h>0,$$

as test function,where T_m is the truncation at the level m,

$$\xi_k(\theta) := \begin{cases} 1 & \textit{if } \theta\in\left[t, T-\frac{1}{k}\right], \\ k(T-\theta) & \textit{for } \theta\in\left[T-\frac{1}{k}, T\right], \\ 0 & \textit{otherwise}, \quad k\in\mathbb{N}, \end{cases} \qquad \psi_\varepsilon(|x-x_0|,\theta) := \begin{cases} 1 & \textit{if } d>\varepsilon, \\ \frac{1}{\varepsilon}d & \textit{if } d<\varepsilon, \\ 0 & \textit{otherwise}, \end{cases}$$

with $d = dist((x,\theta),\partial_l P(t))$ and $\varepsilon>0$. So, $supp\zeta(x,\theta)\equiv P(t)$, $\zeta, \frac{\partial\zeta}{\partial t}\in L^\infty((0,T)\times\Omega)$ and $\frac{\partial\zeta}{\partial x_i}\in L^p((0,T)\times\Omega)$. Using the monotonicity of β and passing to the limits we get the inequality.

Step 2. *A differential inequality for some energy function.* We assume choice P such that it does not touch the initial plane $\{t=0\}$ and $P\subset B_\rho(x_0)\times[0,T]$. Then $i_1+i_2+i_3\le j_1+j_2$. In order to estimate j_1, let us mention that $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_\tau) = \frac{1}{(\vartheta^2\upsilon^2+(\theta-t)^{2(1-\upsilon)})^{1/2}}((\theta-t)^{1-\upsilon}\mathbf{e}_x - \upsilon\mathbf{e}_\tau)$ with $\mathbf{e}_x, \mathbf{e}_\tau$ orthogonal unit vectors to the hyperplane $t=0$ and the axis t, respectively. Then, if we denotes by (ρ,ω), $\rho \geq 0$ and $\omega \in \partial B_1$ the spherical coordinate system in $\mathbb{R}^N$, if $\Phi(\rho,\omega,\theta)$ is the spherical representation of a general function $F(x,t)$, we have

$$I(t) := \int_P F(x,\theta)dxd\theta \equiv \int_t^T d\theta \int_0^{\rho(\theta,t)} \rho^{N-1}d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J|d\omega,$$

where J is the Jacobi matrix and $\rho(\theta,t) = \vartheta(\theta - t)^{\upsilon}$. So,

$$\begin{aligned}\frac{dI(t)}{dt} &= -\int_0^{\rho(\theta,t)} \rho^{N-1}d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J|d\omega\bigg|_{\theta=t} \\ &\quad + \int_t^T \rho_t\rho^{N-1}d\theta \int_{\partial B_1} \Phi(\rho,\omega,t)|J|d\omega = \int_{\partial_l P} \rho_t F(x,\theta)d\Gamma d\theta.\end{aligned} \tag{10}$$

Then, by Hölder's inequality, we get

$$\begin{aligned}\left|\int_{\partial_l P} n_x\cdot \mathbf{A}\, u\, d\Gamma d\theta\right| &\leq M_2 \int_{\partial_l P} |\, n_x||\nabla u|^{p-1}|u|d\Gamma d\theta \\ &\leq M_2 \left(\int_{\partial_l P} |\rho_t||\nabla u|^p d\Gamma d\theta\right)^{(p-1)/p} \left(\int_{\partial_l P} \frac{|\, n_x|^p}{|\rho_t|^{p-1}}|u|^p d\Gamma d\theta\right)^{1/p} \\ &= M_2 \left(-\frac{dE}{dt}\right)^{(p-1)/p} \left(\int_t^T \frac{|\, n_x|^p}{|\rho_t|^{p-1}} \left(\int_{\partial B_{\rho(\theta,t)}} |u|^p d\Gamma\right) d\theta\right)^{1/p}.\end{aligned} \tag{11}$$

To estimate the right-hand side of (11) we use the interpolation inequality ([9]) : if $0 \leq \sigma \leq p-1$, then there exists $L_0 > 0$ such that $\forall\, v \in W^{1,p}(B_\rho)$

$$\|v\|_{p,S_\rho} \leq L_0 \left(\, \|\nabla v\|_{p,B_\rho} + \rho^{\delta}\|v\|_{\sigma+1,B_\rho}\right)^{\tilde{\theta}} \cdot \left(\, \|v\|_{r,B_\rho}\right)^{1-\tilde{\theta}} \tag{12}$$

$r \in [1, 1+\gamma]$, $\quad \tilde{\theta} = \frac{pN - r(N-1)}{(N+1)p - Nr}$, $\quad \delta = -\left(1 + \frac{p-1-\sigma}{p(1+\sigma)}N\right)$. In our case, we shall apply it to the limit case $\sigma = 0$. By Hölder's inequality

$$\left(\int_{B_\rho} |u|^r dx\right)^{1/r} \leq \left(\int_{B_\rho} |u|dx\right)^{1/qr} \cdot \left(\int_{B_\rho} |u|^{\gamma+1}dx\right)^{(q-1)/qr},$$

with $q = \frac{\gamma}{\gamma - r + 1}$.Then

$$
\begin{aligned}
\int_{\partial B_\rho} |u|^p d\Gamma &\le L_0 \left(\left(\int_{B_\rho} |\nabla u|^p + \rho^{\delta p} \left(\int_{B_\rho} |u| \right)^{p/2} \right) \right)^{\tilde{\theta}} \times \left(\int_{B_\rho} |u|^r \right)^{p(1-\tilde{\theta})/r} \\
&\le L_0 \rho^{\delta \tilde{\theta} p} \left(\int_{B_\rho} |\nabla u|^p + \int_{B_\rho} |u| \right)^{\tilde{\theta}} \times \left(\int_{B_\rho} |u| \right)^{p(1-\tilde{\theta})/qr} \left(\int_{B_\rho} |u|^{\gamma+1} \right)^{p(q-1)(1-\tilde{\theta})/qr} \\
&\le K \rho^{\delta \tilde{\theta} p} \left(E_* + C_* \right)^{\tilde{\theta}} C_*^{(1-\tilde{\theta})p/qr} b^{(q-1)(1-\tilde{\theta})p/qr} \\
&\le K \rho^{\delta \tilde{\theta} p} \left(E_* + C_* \right)^{\tilde{\theta} + (1-\tilde{\theta})p/qr} b^{(q-1)(1-\tilde{\theta})p/qr},
\end{aligned}
\tag{13}
$$

where $E_*(t, \rho) := \int_{B_\rho} |\nabla u|^p dx, \quad C_*(t, \rho) := \int_{B_\rho} |u| dx$ and K is a suitable positive constant. Taking $r \in \left[\frac{p(\gamma+1)}{p+\gamma}, \gamma + 1 \right]$ we get that $\mu = \tilde{\theta} + p \frac{1-\tilde{\theta}}{qr} < 1$. Applying once again Hölder's inequality with the exponent μ, we have from (13)

$$
\begin{aligned}
|j_1| \le L \left(-\frac{dE}{dt} \right)^{(p-1)/p} &\times \left(\int_t^T \frac{|\vec{n}_x|^p}{|\rho_t|^{p-1}} K \rho^{\delta \tilde{\theta} p} \left(E_* + C_* \right)^{\mu} b^{(q-1)(1-\tilde{\theta})p/qr} d\tau \right)^{1/p} \\
\le L \left(-\frac{dE}{dt} \right)^{(p-1)/p} &b^{(q-1)(1-\tilde{\theta})/qr} \\
&\times \left(\int_t^T \left(E_* + C_* \right) d\tau \right)^{\frac{\mu}{p}} \left(\int_t^T \left(\frac{|\vec{n}_x|^p}{|\rho_t|^{p-1}} \rho^{\delta \tilde{\theta} p}(\tau) \right)^{\frac{1}{1-\mu}} d\tau \right)^{\frac{1-\mu}{p}} \\
\le L\sigma(t) \left(-\frac{d\left(E + C\right)}{dt} \right)^{(p-1)/p} &b^{(q-1)(1-\tilde{\theta})/qr} \left(E + C \right)^{\frac{\tilde{\theta}}{p} + \frac{1-\tilde{\theta}}{qr}},
\end{aligned}
\tag{14}
$$

for a suitable positive constant L. To obtain (14) we have assumed that

$$
\sigma(t) := \left(\int_t^T \left(\frac{1}{|\rho_t|^{p-1}} \rho^{\delta \tilde{\theta} p}(\tau) \right)^{\frac{1}{1-\mu}} d\tau \right)^{\frac{1-\mu}{p}} < \infty
$$

which is fulfilled if we choose $\nu \in (0, 1)$ sufficiently small because the condition of convergence of the integral $\sigma(t)$ has the form $(1-\nu)(p-1) + \nu\delta\tilde{\theta}p > -(1-\tilde{\theta}) \left(1 - \frac{p}{qr} \right)$. So, we have obtained an estimate of the following type

$$
|j_1| \le L_1 \Lambda(t) D(u)^{(q-1)(1-\tilde{\theta})/qr - \lambda} \left(E + C + b \right)^{1-\omega+\lambda} \left(-\frac{d(E + C)}{dt} \right)^{(p-1)/p}, \tag{15}
$$

where L_1 is a universal positive constant, $D(u)$ is *the total energy* of the solution under investigation, $\lambda \in [0, (q-1)(1-\tilde{\theta})/qr]$ and $\omega := 1 - \frac{\tilde{\theta}}{p} - \frac{1-\tilde{\theta}}{qr} \in \left(1 - \frac{1}{p}, 1 \right)$. This allows us to choose λ so that $\frac{p(\omega-\lambda)}{p-1} \in (0,1)$. Let us estimate j_2. Using the expression for n_τ, we have $|j_2| \le C_5 \int_{\partial_l P} |u|^{1+\gamma} d\Gamma d\theta$. We apply then the interpolation inequality (for the limit case $\sigma = 0$)

$$\|v\|_{\gamma+1,\partial B_\rho} \le L_0 \left(\|\nabla v\|_{p,B_\rho} + \rho^\delta \|v\|_{\sigma+1,B_\rho}\right)^s \cdot \|v\|_{r,B_\rho}^{1-s} \quad \forall\, v \in W^{1,p}(B_\rho) \tag{16}$$

with a universal positive constant $L_0 > 0$ and exponents $s = \frac{(\gamma+1)N - r(N-1)}{(N+r)p - Nr} \frac{p}{\gamma+1}$, $r \in [1+\sigma, 1+\gamma]$. Again

$$\begin{aligned} \int_{\partial B_\rho} |u|^{\gamma+1} dx \le L^{1+\gamma} K^{s(\gamma+1)/\tilde\theta p} \left(\int_{B_\rho} |\nabla u|^p dx + \int_{B_\rho} |u|^{\sigma+1} dx \right)^{s(\gamma+1)/p} \\ \times \left[\left(\int_{B_\rho} |u|^{\sigma+1} dx \right)^{1/qr} \left(\int_{B_\rho} |u|^{\gamma+1} dx \right)^{(q-1)/qr} \right]^{(1-s)(\gamma+1)} . \end{aligned} \tag{17}$$

Here K is the same as before. Let $\eta = \frac{s(\gamma+1)}{p} + \frac{(1-s)(\gamma+1)}{qr} < 1$, $\pi = \frac{(q-1)(1-s)(\gamma+1)}{qr}$, $\eta + \pi \ge 1$. Then,

$$\begin{aligned} |j_2| &= \left| \int_t^T d\tau \int_{\partial B_{\rho(\tau)}} |u|^{\gamma+1} d\Gamma \right| \\ &\le L\,(b(T))^\pi \left(\int_t^T K^{s(\gamma+1)/\tilde\theta p} (E_* + C_*)^\eta |n_\tau| d\tau \right) \\ &\le L\,(E + C + b(T,\Omega))\,(b(T,\Omega))^\kappa \left(\int_t^T \left(K^{s(\gamma+1)/\tilde\theta p} \right)^\varepsilon d\tau \right)^{1/\varepsilon}, \end{aligned} \tag{18}$$

for some $L = L(C_5, L_0)$ and exponents $\kappa := \eta + \pi - 1, \quad \varepsilon = 1/(1-\eta)$. Then. we have

$$C_5 \int_{P \cap \{t=T\}} |u|^{1+\gamma} dx + E + C\Theta \le i_1 + i_2 + i_3, \tag{19}$$

$$|i_4| \le \varepsilon C_3 \frac{p-\beta}{p} C(\rho,t) + \frac{\beta C_3}{p C_2} \varepsilon^{-(p-\beta)/\beta} E(\rho,t), \tag{20}$$

$$K \left(\int_{P \cap \{t=T\}} |u|^{1+\gamma} dx + E + C \right) \le i_1 + i_2 + i_3 + i_4, \tag{21}$$

for different positive constants K. Now, assuming $T - t$ and $D(u)$ so small that

$$L\,(b(T,\Omega))^\kappa \left(\int_t^T \left(K^{s(\gamma+1)/\tilde\theta p} \right)^\varepsilon d\tau \right)^{1/\varepsilon} < \frac{K}{2},$$

we arrive to the inequality

$$E + C + b(T,\Omega) \leq L_1 \Lambda(t) D(u)^{(q-1)(1-\tilde{\theta})/qr-\lambda} \times (E + C + b(T,\Omega))^{1-\omega+\lambda} \left(-\frac{d(E+C)}{dt} \right)^{(p-1)/p}, \tag{22}$$

whence the desired differential inequality for the energy function $Y(t) := E + C$

$$Y^{(\omega-\lambda)p/(p-1)}(t) \leq c(t) \left(-Y(t) \right)', \tag{23}$$

where

$$c(t) = \left(L_1 \left(D(u) \right)^{(q-1)(1-\tilde{\theta})/qr-\lambda} \sigma(t) \right)^{p/(p-1)}, \quad L_1 = const > 0.$$

Notice that $c(t) \to 0$ as $t \to T$. Moreover, the exponent $(\omega - \lambda)\frac{p}{p-1}$ belongs to the interval $(0,1)$ which leads to the result (see [1]). ■

In order to prove Theorem 2, let us start by the case $f(x,t) > \Theta$. It is easy to see that a similar *integration-by-parts formula* can be obtained for $v = -u_- = -\text{mín}(u,0)$ (it suffices to replace $T_m(u(x,s))$ by $T_m(u_-(x,s)))$. In that case $G(u) = \frac{k_-}{2}u^2$. So, $\gamma = 1$ and the arguments of Theorem 1 apply. For the study of the case $f(x,t) < -\Theta$ it is convenient to introduce $v = (\psi(u) - L)_+$ and then multiply the equation by a regular localized approximation of the function $T_m(v(x,s))$. Then, using that $\psi(u)_t sign_0^+ v = v_t sign_0^+ v$ (in a weak sense) and that $C(x,u) sign_0^+ v \geq 0$ (here $sign_0^+ v = 1$ if $v > 0$, and 0 if $v \leq 0$) we get the *integration-by-parts formula* for v with $G(v) = \frac{k_+}{2}v^2$. The arguments of the proof of Theorem 1 lead to the estimate $v = 0$ in $P(t^* : 1, v)$. Finally, the conclusion follows from the fact that $v = 0$ if and only if $u_+ = 0$, $.u_+ = \text{máx}(u,0)$. ■

Remark 1 *The present techniques can be applied to the study of other properties such as the finite speed of propagation, the shrinking of the support, the waiting time or the study of locally vanishing solutions of the associate stationary problems. For some illustration of those properties, global consequences, applications to Fluid Mechanics and a detailed bibliography we send the reader to the monograph [1].*

Remark 2 *The assumptions on $B(x,u,Du)$ can be improved. For instance, the case of $B(x,u,Du) = \mathbf{w}(x,t) \cdot Du$ is considered in Calvo, Díaz, Durany and Schiavi [7].*

Remark 3 *The study of the case of perturbations $C(x,u)$ growing with negative exponents is the main goal of the work [5]. In that case, the applied interpolation inequality is obtained from [19].*

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