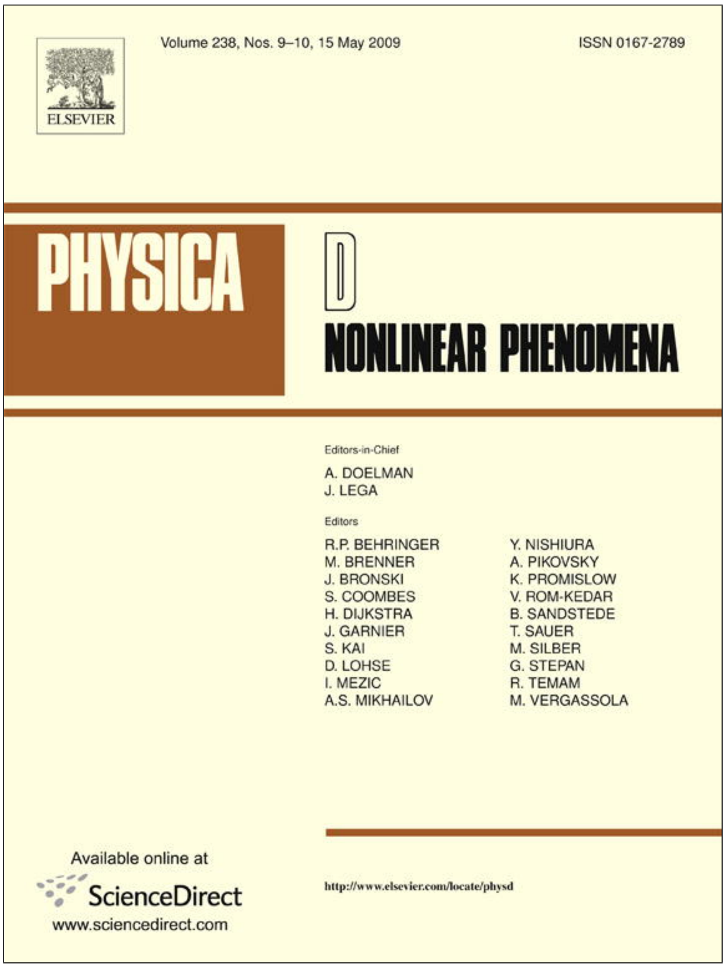
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# On the asymptotic behaviour of solutions of a stochastic energy balance climate model

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ABSTRACT

We prove the existence of a random global attractor for the multivalued random dynamical system associated to a nonlinear multivalued parabolic equation with a stochastic term of amplitude of the order of $\varepsilon$. The equation was initially suggested by North and Cahalan (following a previous deterministic model proposed by M.I. Budyko), for the modeling of some non-deterministic variability (as, for instance, the cyclones which can be treated as a fast varying component and are represented as a white-noise process) in the context of energy balance climate models. We also prove the convergence (in some sense) of the global attractors, when $\varepsilon \to 0$, i.e., the convergence to the global attractor for the associated deterministic case ($\varepsilon = 0$).



## 1. Introduction

This paper deals with the study of the asymptotic behaviour, for large time, of the following stochastic PDE, initially proposed by North and Cahalan [1] (following a previous deterministic model proposed by Budyko [2]), modeling some non-deterministic variability (as, for instance, the cyclones which can be treated as a fast varying component and are represented as a white-noise process) in the context of energy balance climate models:

$$\begin{cases} u_t - u_{xx} + Bu \in QS(x)\beta(u) \\ \quad + h(x) + \varepsilon\phi \dfrac{dW_t}{dt}, \quad (x,t) \in (-1,1) \times \mathbb{R}^+, \\ u_x(-1,t) = u_x(1,t) = 0, \quad t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad x \in (-1,1), \end{cases} \tag{1}$$

where $B, Q, \varepsilon$ are positive constants, $\phi \in H^2(-1,1)$ with $\phi_x(-1) = \phi_x(1) = 0, S, h \in L^\infty(-1,1), u_0 \in L^2(-1,1)$, and

$$\beta \text{ is a bounded maximal monotone graph of } \mathbb{R}^2, \tag{2}$$

i.e., there exist $m, M \in \mathbb{R}$ such that

$$m \le z \le M, \quad \text{for all } z \in \beta(s), s \in \mathbb{R},$$

$$0 < S_0 \le S(x) \le S_1, \text{ a.e. } \quad x \in (-1,1). \tag{3}$$

Here $W_t$ is a two-sided, i.e. $t \in \mathbb{R}$, real Wiener process endowed with the usual Wiener probability space $(\Omega, \mathcal{F}, P)$ (see a detailed definition of this space in Section 2.2). Other motivations are presented in [3] (see the entire proceedings book [4]).

As in the deterministic case (see, e.g., [5–7]), the unknown $u(x,t,\omega)$ represents the average temperature of the Earth surface, where $Q$ is the so-called solar constant which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, the function $S(x)$ is known as the insolation function given by the distribution of incident solar radiation at the top of the atmosphere. When the averaging time is of the order of one year or longer, function $S(x)$ satisfies (3) (in shorter periods we must assume that $S_0 = 0$). The term $\beta$ represents the so-called *coalbedo function* that takes values between 0 and 1. It represents the ratio between the absorbed solar energy and the incident solar energy at the point $x$ on the Earth surface. Obviously, $\beta(u(x,t,\omega))$ depends on the nature of the Earth surface. For instance, it is well known that on ice sheets $\beta(u(x,t,\omega))$ is much smaller than that on the ocean surface because the white color of the ice sheets reflects a large portion of the incident solar energy, whereas the ocean, due to its dark color and high heat capacity, is able to absorb a larger amount of the incident solar energy. As mentioned before, the non-deterministic term was first suggested in [1] and later considered by other authors (see, e.g., [8]).

We recall that, as in the deterministic case, the distribution of temperature $u(x,t,\omega)$ is expressed pointwise after a standard average process, where the spatial variable $x$ is given by $x =$

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$\sin\theta$ where $\theta$ is the latitude. Notice that, for simplicity, we are replacing the natural degenerate diffusion term $((1-x^2)u_x)_x$ by the usual 1$d$-Laplacian operator and that the absence of boundary conditions for the degenerate diffusion is corrected by adding Neumann type boundary conditions since in the degenerate model the meridional heat flux $(1-x^2)u_x$ vanishes at the poles $x = \pm 1$ (this simplification was already considered by many authors: see, e.g., [9,10] and their references). The expression $Bu - h(x)$ represents the average amount of energy radiated to the space. Here, following Budyko [2], the expression corresponds to a linearization of the more general Stefan–Boltzmann nonlinear law.

We point out that we shall not deal here with the existence of weak solutions, since, even under more general assumptions on the stochastic term, it follows from many previous papers as, for instance, Gyöngy and Pardoux [11] or by easy modifications of Bensoussan and Temam [12].

We note that, in virtue of the studies made for the deterministic case [5–7], it is not reasonable to expect the uniqueness of solutions of (1) even if it is a parabolic type problem. Due to that, we shall work here with the associated *multivalued random dynamical system* (MRDS) generated by its solutions. Our main result concerns the asymptotic behaviour of solutions when $t \to +\infty$.

**Theorem 1.1.** *Assume conditions (2) and (3). Then the multivalued random dynamical system associated to (1) has a global random attractor $\mathcal{A}_\varepsilon$.*

For the sake of the completeness in the exposition, we shall recall some basic definitions and results on multivalued random dynamical systems, and its attractors, in a subsection below.

Continuing, and improving, the results initiated in [8], we also consider the question about the convergence (in some sense) of the global attractors $\mathcal{A}_\varepsilon$ when $\varepsilon \to 0$, i.e., the convergence to the global attractor for the associated deterministic case ($\varepsilon = 0$). We prove, in the next section, the following result:

**Theorem 1.2.** *Assume conditions (2) and (3). Then,*

$$\lim_{\varepsilon \searrow 0} dist(\mathcal{A}_\varepsilon(\omega), \mathcal{A}) = 0, \quad \text{for all } \omega \in \Omega,$$

*with $\mathcal{A}$ the global attractor associated to (1) with $\varepsilon = 0$.*

We point out that the study of the $\omega$-limit set for the deterministic case ($\varepsilon = 0$) was carried out in [13] and that the structure of the global attractor may be very complicated (essentially depending on the parameter $Q$) including, a finite number of stationary solutions (see, e.g., [13,14]) or even a set of infinite solutions (see [15,10]).

As a matter of fact, our results can be suitably extended to more general cases: for instance, we can assume an $x$-dependence in the coalbedo multivalued function $\beta(x, u(x, t, \omega))$, we can extend, in a suitable way, our conclusions to the more general case in which $h = h(x, t)$ and $QS = Q(t)S(x, t)$, and the extension to quasilinear diffusion operators (as considered in [5–7]) is also possible. Nevertheless, we shall not present such extensions here to avoid technical details. Some numerical experiences for the deterministic case were given in [7,16]. For the stochastic case, see, for instance, [1,17].

## 2. On the asymptotic behaviour of solutions

### 2.1. Attractors for stochastic partial differential inclusions

Let $(X, d_X)$ be a complete and separable metric space with the Borel $\sigma$-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$ a measure preserving group of transformations in $\Omega$ such that the map $(t, \omega) \mapsto \theta_t\omega$ is measurable and satisfies

$$\theta_{t+s} = \theta_t \circ \theta_s = \theta_s \circ \theta_t; \qquad \theta_0 = Id.$$

The parameter $t$ takes values in $\mathbb{R}$ endowed with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$. Denote by $P(X)$ ($C(X)$) the set of non-empty (non-empty closed) subsets of $X$.

**Remark 2.1.** Throughout this paper all assertions about $\omega$ are assumed to hold on a $\theta_t$ invariant set of full measure.

**Definition 2.2.** A set-valued map $G : \mathbb{R}^+ \times \Omega \times X \to C(X)$ is called a multivalued random dynamical system (MRDS) if it is measurable, i.e., if given $x \in X$ the map

$$(t, \omega, y) \in \mathbb{R}^+ \times \Omega \times X \mapsto dist(x, G(t, \omega, y)) \tag{4}$$

is measurable, where $dist(x, A) = \inf_{a \in A} d_X(x, a)$, for $A \subset X$, and satisfies:

(i) $G(0, \omega, \cdot) = Id$ on $X$;
(ii) $G(t+s, \omega, x) = G(t, \theta_s\omega, G(s, \omega)x)$ (cocycle property) $\forall t, s \in \mathbb{R}^+, x \in X, \omega \in \Omega$.

**Remark 2.3.** Property (4) is equivalent to say that for every open set $O$ the inverse image $\{(t, \omega, y) : G(t, \omega, y) \cap O \neq \varnothing\}$ is measurable (see [18, Proposition 2.1.4] or [19, p. 67]).

Denote an $\epsilon$-neighborhood of a set $A$ by $\mathcal{O}_\epsilon(A) = \{y \in X : dist(y, A) < \epsilon\}$.

**Definition 2.4.** $G(t, \omega, \cdot)$ is said to be *upper semicontinuous* if for all $t \in \mathbb{R}^+$ and all $\omega \in \Omega$, given $x \in X$ and a neighborhood of $G(t, \omega, x)$, $\mathcal{O}(G(t, \omega, x))$, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then

$$G(t, \omega, y) \subset \mathcal{O}(G(t, \omega, x)).$$

On the other hand, $G(t, \omega, \cdot)$ is called *lower semicontinuous* if for all $t \in \mathbb{R}^+$ and all $\omega \in \Omega$, given $x_n \to x$ ($n \to +\infty$) and $y \in G(t, \omega, x)$, there exists $y_n \in G(t, \omega, x_n)$ such that $y_n \to y$.

It is said to be continuous if it is upper and lower semicontinuous.

**Definition 2.5.** $G(t, \omega, \cdot)$ is said to be $\epsilon$-upper semicontinuous if in the definition of upper semicontinuity we replace the neighborhood $\mathcal{O}$ by an $\epsilon$-neighborhood $\mathcal{O}_\epsilon$.

It is clear that any upper semicontinuous map is $\epsilon$-upper semicontinuous. The converse is true if $G$ has compact values [20, p. 45].

We now introduce the generalization of the concept of random attractor (Crauel and Flandoli [21]) to the case of a multivalued random dynamical system and recall here a general result for the existence and uniqueness of attractors. Firstly we need some definitions.

**Definition 2.6.** A closed random set $D$ is a map $D : \Omega \to C(X)$ which is measurable.

A closed random set $D(\omega)$ is said to be negatively (resp. strictly) invariant for the MRDS if

$$D(\theta_t\omega) \subset G(t, \omega, D(\omega)) \text{ (resp. } D(\theta_t\omega) = G(t, \omega, D(\omega))) \quad \forall\, t \in \mathbb{R}^+,\ \omega \in \Omega.$$

Suppose the following conditions for the MRDS $G$:

(H1) There exists an absorbing random compact set $K(\omega)$, that is, for every bounded set $D \subset X$, there exists $t_D(\omega)$ such that for all $t \geq t_D(\omega)$

$$G(t, \theta_{-t}\omega, D) \subset K(\omega), \quad \forall \omega \in \Omega. \tag{5}$$

(H2) $G(t, \omega, \cdot) : X \to C(X)$ is upper semicontinuous, for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Define the $\omega$-limit set $\Lambda(D, \omega) = \Lambda_D(\omega)$ of a bounded set $D \subset X$ as

$$\Lambda_D(\omega) = \cap_{T \geq 0} \overline{\cup_{t \geq T} G(t, \theta_{-t}\omega, D)}. \tag{6}$$

We recall first some auxiliary results, proved in [22, p. 808].

**Lemma 2.7.** *$\Lambda_D(\omega)$ is the set of limits of all converging sequences $\{x_n\}_{n\geq 1}$, where $x_n$ belongs to $G(t_n, \theta_{-t_n}\omega, D)$ with $t_n \nearrow +\infty$.*

**Proposition 2.8.** *Assume conditions* (H1) *and* (H2) *hold. Then:*

(i) *$\Lambda_D(\omega) \subset K(\omega)$ is nonvoid and compact.*
(ii) *$\Lambda_D(\omega)$ is negatively invariant, that is, $G(t, \omega, \Lambda_D(\omega)) \supseteq \Lambda_D(\theta_t\omega)$ for all $t \in \mathbb{R}^+$, $\omega \in \Omega$. If $G(t, \omega)$ is lower semicontinuous, then $\Lambda_D(\omega)$ is strictly invariant.*
(iii) *$\Lambda_D(\omega)$ attracts D, that is, for any $\omega \in \Omega$ we have*

$$\lim_{t\to+\infty} dist(G(t, \theta_{-t}\omega, D), \Lambda_D(\omega)) = 0.$$

**Definition 2.9.** A closed random set $\omega \mapsto \mathcal{A}(\omega)$ is said to be a global random attractor of the MRDS $G$ if:

(i) $G(t, \omega)\mathcal{A}(\omega) \supseteq \mathcal{A}(\theta_t\omega)$, for all $t \geq 0$, $\omega \in \Omega$ (that is, it is negatively invariant);
(ii) for all $D \subset X$ bounded,

$$\lim_{t\to+\infty} \text{dist}(G(t, \theta_{-t}\omega, D), \mathcal{A}(\omega)) = 0;$$

(iii) $\mathcal{A}(\omega)$ is compact for any $\omega \in \Omega$.

We also recall the following theorem on the existence and upper semicontinuity of random attractors for MRDS, proved in [22,23].

**Theorem 2.10.** *Let assumptions $(H1)-(H2)$ hold, the map $(t, \omega) \mapsto \overline{G(t, \omega, D)}$ be measurable for all deterministic bounded sets $D \subset X$, and the map $x \in X \mapsto G(t, \omega, x)$ have compact values. Then*

$$\mathcal{A}(\omega) := \overline{\bigcup_{\substack{D\subset X\\ \text{bounded}}} \Lambda_D(\omega)} \tag{7}$$

*is a global random attractor for G (measurable with respect to $\mathcal{F}$). It is unique and the minimal closed attracting set.*

*Moreover, if the map $x \longmapsto G(t, \omega, x)$ is lower semicontinuous for each fixed $(t, \omega)$, then the global random attractor $\mathcal{A}(\omega)$ is strictly invariant, i.e., $G(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$, for all $t \geq 0$, $\omega \in \Omega$.*

From Theorem 2.10, Kapustyan [24] proved the following result:

**Theorem 2.11.** *Let $\Omega$ be a metrizable topological space, X be a separable Banach space with norm $\|\cdot\|$, $\mathcal{F}$ be the Borel $\sigma$-algebra, and let G satisfy conditions* (i), (ii) *of Definition 2.2. Assume the following conditions:*

(H1b) *There exists a measurable mapping $r : \Omega \to \mathbb{R}^+$ such that for all $\omega \in \Omega$ and for $R > 0$, there is a $T = T(R, \omega) > 1$ such that $\|G(t-1, \theta_{-t}\omega, B_R)\|_+ \leq r(\omega)$ for all $t \geq T$, where $\|A\|_+ = \sup_{y\in A}\|y\|$, for $A \subset X$, and $B_R$ is a closed ball of radius $R > 0$ centered at 0;*
(H2b) *If $x_n \to x_0$ weakly, $t_n \to t_0 > 0$, $\omega_n \to \omega_0$, and $y_n \in G(t_n, \omega_n, x_n)$, then up to a subsequence $y_n \to y_0 \in G(t_0, \omega_0, x_0)$.*

*Then G generates a MRDS and the set $\mathcal{A}(\omega) := \overline{\bigcup_{R>0}\Lambda_{B_R}(\omega)}$ is a global random attractor. It is unique and the minimal closed random attracting set.*

Let $G_\varepsilon : \mathbb{R}^+ \times \Omega \times X \to C(X)$, $\varepsilon \in (0, 1]$, be a parametrized family of MRDS and let $G_0 : \mathbb{R}^+ \times X \to C(X)$ be a deterministic multivalued semiflow (see [25] for the theory of global attractors in this case). We recall that the set $\mathcal{A}$ is a global attractor for $G_0$ if $\mathcal{A} \subset G(t, \mathcal{A})$, for all $t \geq 0$, and $dist(G_0(t, B), \mathcal{A}) \to 0$, as $t \to \infty$, for any bounded set $B$.

The following result improves Theorem 17 in [26].

**Theorem 2.12.** *Let $G_0$ have a compact global attractor $\mathcal{A}$ and let $G_0$ have compact values. Suppose that the map $x \mapsto G_0(t, x)$ is upper semicontinuous for any $t \geq 0$. Assume also that every $G_\varepsilon$ satisfies conditions $(H1)$–$(H2)$ and the following:*

$(G1)$ *For all $\omega \in \Omega$, $t \in \mathbb{R}^+$ it holds*

$$dist(G_\varepsilon(t, \omega, B), G_0(t, B)) \to 0, \text{ as } \varepsilon \searrow 0,$$

*for any compact set B.*

$(G2)$ *There exists a compact set $K \subset X$ such that*

$$\lim_{\varepsilon\searrow 0} dist(K_\varepsilon(\omega), K) = 0, \text{ for all } \omega \in \Omega,$$

*where $K_\varepsilon(\omega)$ is the absorbing set from condition $(H1)$.*

*Then $\lim_{\varepsilon\to 0^+} dist(\mathcal{A}_\varepsilon(\omega), \mathcal{A})$ for all $\omega \in \Omega$.*

**Proof 2.13.** We argue by contradiction. Suppose that there exist $\delta > 0$, $\varepsilon_j \searrow 0$ and $\omega \in \Omega$, such that

$$dist(\mathcal{A}_{\varepsilon_j}(\omega), \mathcal{A}) > \delta.$$

Since $\mathcal{A}_{\varepsilon_j}(\omega)$ are compact sets, there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$, $x_j \in \mathcal{A}_{\varepsilon_j}(\omega)$, such that

$$d_X(x_j, x) > \delta, \quad \text{for all } j \in \mathbb{N}, \text{ for all } x \in \mathcal{A}. \tag{8}$$

We shall prove that there exists a subsequence of $\{x_j\}$ converging to a point of the global attractor $\mathcal{A}$, which contradicts (8).

Indeed, we can write, by the semi-invariance of $\mathcal{A}_{\varepsilon_j}(\omega)$ that

$$x_j \in G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n), \quad \text{with } y_j^n \in \mathcal{A}_{\varepsilon_j}(\theta_{-n}\omega), \quad \text{for all } n \in \mathbb{N}.$$

From $(G2)$ we deduce, as $\mathcal{A}_{\varepsilon_j}(\omega) \subset K_{\varepsilon_j}(\omega)$, that

$$\lim_{j\to\infty} dist(\mathcal{A}_{\varepsilon_j}(\theta_{-n}\omega), K) = 0, \quad \text{for all } n,$$

so that $\lim_{j\to\infty} dist(y_j^n, K) = 0$. Thus, we conclude that there exists a subsequence of $\{y_j^n\}$ converging to a point $y_0^n \in K$.

Now we shall prove that

$$\lim_{j\to+\infty} dist(G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n), \quad G_0(n, K)) = 0. \tag{9}$$

Denote $K_{\bar{j},n} = \overline{\cup_{j\geq\bar{j}} y_j^n}$, which is a compact set. Then by the upper semicontinuity of $G(n, \cdot)$ for any $\nu > 0$ there exists $\bar{J}(\nu, n)$ such that

$$dist(G_0(n, K_{\bar{j},n}), G_0(n, K)) \leq dist(G_0(n, K_{\bar{j},n}), G_0(n, y_0^n)) < \frac{\nu}{2}.$$

Hence

$$\begin{aligned} &dist(G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n), G_0(n, K)) \\ &\quad \leq dist(G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n), G_0(n, K_{\bar{j},n})) \\ &\qquad + dist(G_0(n, K_{\bar{j},n}), G_0(n, K)) \\ &\quad \leq dist(G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n), G_0(n, K_{\bar{j},n})) + \frac{\nu}{2}. \end{aligned} \tag{10}$$

For the first term on the right-hand side of (10) by $(G1)$ for arbitrary $\nu > 0$ there is $J(\nu, n) \geq \bar{J}(\nu, n)$ such that

$$dist(G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n), G_0(n, K_{\bar{j},n})) < \frac{\nu}{2}, \quad \text{if } j \geq J.$$

Therefore it holds (9) and, as $x_j \in G_{\varepsilon_j}(n, \theta_{-n}\omega, y_j^n)$, it is straightforward that

$$\lim_{j\to+\infty} dist(x_j, G_0(n, K)) = 0.$$

Since $G_0$ has compact values, $G_0(n, \cdot)$ is upper semicontinuous and $K$ is compact, the set $G_0(n, K)$ is compact. Indeed, let $y_k \in G_0(n, u_k)$, $u_k \in K$. Then, up to a subsequence, $u_k \to u_0$ and $dist(y_k, G_0(n, u_0)) \to 0$ as $k \to \infty$. Hence, $y_k \to y_0 \in G_0(n, u_0)$.

Then we can ensure that there exists $x_0 \in G_0(n, K)$, for all $n \in \mathbb{N}$, such that $x_j \to x_0$ as $j \to \infty$ (accurate to a subsequence). It follows from the definition of the omega limit set (see [25]) that $x_0 \in \Lambda_K \subset \mathcal{A}$, which contradicts (8). □

*2.2. Existence of random attractors for the climate model*

Let $H$ be a separable Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$.

In Kapustyan [24] it was proved the existence of random attractors for the following abstract stochastic evolution equation

$$\begin{cases} du(t) \in (-\partial\varphi(u(t)) + F(u(t)))dt + \varepsilon\phi \dfrac{dW_t}{dt}, \\ u(x, 0) = u_0, \end{cases} \tag{11}$$

where $\partial\varphi$ is the subdifferential of a proper, convex lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$. Consider the following hypotheses for $F$ and $\varphi$:

(F1) $F : H \to C_v(H)$, where $C_v(H)$ is the set of non-empty bounded closed convex subsets of $H$.
(F2) There exist $D_1, D_2 \geq 0$ such that $\|y\| \leq D_1 + D_2\|v\|$, for all $y \in F(v)$, $v \in H$.
(F3) $F$ is $\epsilon$-upper semicontinuous on $H$.
(F4) There exist $\delta > 0$ and $M > 0$ such that

$$(y, u) \leq -\delta\|u\|^2 + M,$$

for all $u \in D(A)$ and $y \in F(u)$.
(F5) The level sets $M_R = \{u \in D(\varphi) : \|u\| \leq R, \varphi(u) \leq R\}$ are compact in $H$ for any $R > 0$.

In [24] the results on existence and continuity of random attractors are proved by using these conditions, but changing (F3) by the stronger one that $F$ is upper semicontinuous. In order to give a proof of the first theorem of the Introduction, our aim is to put (1) in the abstract form (11) and to improve the result in [24] by proving the existence of the random attractor under merely conditions (F1)–(F5).

Note also that Kapustyan and Valero [27] consider the existence of attractors for the problem

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} \in \Delta u + \widetilde{f}(u) + \widetilde{h}, & \in \mathcal{C} \times (0, T), \\ u = 0 \quad \text{on } \partial\mathcal{C}, \\ u(x, 0) = u_0, \end{cases} \tag{12}$$

where $\mathcal{C} \subset \mathbb{R}^N$ is an open bounded set with smooth boundary, $\widetilde{h} \in L^2(\mathcal{C})$ and $\widetilde{f} : \mathbb{R} \to 2^{\mathbb{R}}$ satisfies:

($\widetilde{f}$1) $\widetilde{f} : \mathbb{R} \to C_v(\mathbb{R})$ ($C_v(\mathbb{R})$ denotes the set of non-empty, closed, bounded and convex subsets of $\mathbb{R}$).
($\widetilde{f}$2) There exists $d_1, d_2 \geq 0$ such that $|y| \leq d_1 + d_2|s|$, for all $y \in \widetilde{f}(s)$, $s \in \mathbb{R}$.
($\widetilde{f}$3) $\widetilde{f}$ is $\epsilon$-upper semicontinuous on $\mathbb{R}$.
($\widetilde{f}$4) There exists $\delta > 0$ and $M > 0$ such that

$$ys \leq (\lambda_1 - \delta)|s|^2 + M,$$

for all $s \in \mathbb{R}$ and $y \in \widetilde{f}(s)$, being $\lambda_1$ the first eigenvalue of $-\Delta$ in $H_0^1(\mathcal{C})$.

Let $H = L^2(\mathcal{C})$. Define the multivalued map $G : H \to 2^H$ as follows:

$$G(y) = \{\xi + \widetilde{h} : \xi \in H, \ \xi(x) \in \widetilde{f}(u(x)), \quad \text{a.e. } x \in \mathcal{C}\}.$$

Then, by Lemma 6.28 in [28] $G(\cdot)$ satisfies (F1)–(F3). In this case condition (F4) is satisfied but changing $\delta$ by $\lambda_1 - \delta$.

**Remark 2.14.** The fact that the map $F$ is $\epsilon$-upper semicontinuous was stated at first in [27, Proposition 2.5]. However, there the proof contained a misleading argument, and a new proof of this fact (more precisely, two different ones) was given in [28, Lemma 6.28].

Thus, if we now define $f : (-1, 1) \times \mathbb{R} \to 2^{\mathbb{R}}$ as

$$f(x, r) = QS(x)\beta(r) - Br, \quad \text{for a.a. } x \in (-1, 1), \ r \in \mathbb{R},$$

we want to generalize the result on existence of random attractors for this a bit more general multivalued map $f$. From now on $H = L^2(-1, 1)$.

By the hypotheses on $\beta$, we know that

($\beta$1) $\beta : \mathbb{R} \to C_v(\mathbb{R})$.
($\beta$2) There exists $d_1 \geq 0$ such that $|y| \leq d_1$, for all $y \in \beta(r)$, $r \in \mathbb{R}$.
($\beta$3) $\beta$ is $\epsilon$-upper semicontinuous on $\mathbb{R}$ (hence, upper semicontinuous as $\beta(s)$ is compact).

Let us prove analogous properties as ($\widetilde{f}$1)–($\widetilde{f}$3) for $f(x, t)$. Also, we prove a modified version of ($\widetilde{f}$4), in which $\lambda_1 = 0$. We have:

(f1) $f : (-1, 1) \times \mathbb{R} \to C_v(\mathbb{R})$ is obvious, since if $(x_0, r) \in (-1, 1) \times \mathbb{R}$, it follows by ($\beta$1) that $QS(x_0)\beta(r) - Br \in C_v(\mathbb{R})$.
(f2) For $(x_0, r) \in (-1, 1) \times \mathbb{R}$, take $y \in QS(x_0)\beta(r) - Br$. Then, there exists $\tilde{y} \in \beta(r)$ such that $y = QS(x_0)\tilde{y} - Br$. Now, by ($\beta$2), there exists $d_1 \geq 0$ such that $|\tilde{y}| \leq d_1$, for all $\tilde{y} \in \beta(r)$, $r \in \mathbb{R}$. Thus,

$$QS(x_0)|\tilde{y}| \leq QS(x_0)d_1,$$

and since

$$|y| \leq QS(x_0)|\tilde{y}| + B|r|,$$

we get that

$$\sup_{y \in f(x_0, r)} |y| \leq QS(x_0)d_1 + B|r| \leq QS_1 d_1 + B|r|,$$

being the last inequality uniform for almost all $x_0 \in (-1, 1)$.
(f3) Given $(x_0, r_0) \in (-1, 1) \times \mathbb{R}$, and $\epsilon > 0$, there exists $\delta > 0$ such that, if $|r - r_0| < \delta$, then $\beta(r) \subset \mathcal{O}_\epsilon(\beta(r_0))$. Thus, $QS(x_0)\beta(r) \subset \mathcal{O}_\epsilon(QS(x_0)\beta(r_0))$, and $QS(x_0)\beta(r) - Br \subset \mathcal{O}_\epsilon(QS(x_0)\beta(r_0) - Br_0)$. Indeed, if we take $\tilde{y} \in \beta(r)$ such that $y = QS(x_0)\tilde{y} - Br$, given $\epsilon > 0$, there exists $\delta < \frac{\epsilon}{2B}$ such that, if $|r - r_0| < \delta$, then $|\tilde{y} - z| < \frac{\epsilon}{2QS(x_0)}$, for some $z \in \beta(r_0)$. Thus,

$$|(QS(x_0)\tilde{y} - Br) - (QS(x_0)z - Br_0)| \leq QS(x_0)|\tilde{y} - z| + |B(r - r_0)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so that $r \mapsto f(x, r)$ is $\epsilon$-upper semicontinuous a.a. $x \in (-1, 1)$.
(f4) For $y = QS(x_0)\tilde{y} - Br$, $\tilde{y} \in \beta(r)$, we have

$$(QS(x_0)\tilde{y} - Br)r \leq QS(x_0)d_1|r| - B|r|^2,$$

so that

$$\sup_{x_0 \in (-1,1)} \sup_{y \in f(x_0, r)} yr \leq -\frac{B}{2}|r|^2 + \frac{QS_1 d_1}{2B}.$$

Define the multivalued map $F : H \to 2^H$ as follows:

$$F(u) = \{y + h : y \in H, \ y(x) \in f(x, u(x)), \text{ for a.a. } x \in (-1, 1)\}.$$

**Proposition 2.15.** *The map F satisfies* (F1)–(F4).

**Proof 2.16.** The map $\beta$ has compact values and it is $\epsilon$-upper semicontinuous, so that it is upper semicontinuous. Therefore, Proposition 2.5 in [18] implies that for arbitrary closed set $C$ the set $E = \{r \in \mathbb{R} : \beta(r) \cap C \neq \emptyset\}$ is closed, so it is a borelian set. Therefore for any $u \in H$ the set

$$\{x \in (-1, 1) : \beta(u(x)) \cap C \neq \emptyset\} = \{x \in (-1, 1) : u(x) \in E\}$$

is measurable. Thus from Theorem 1.35 in [18] we obtain the measurability of the set-valued map $x \mapsto \beta(u(x))$. Hence, $\beta(u(\cdot))$ has a measurable selection $g(\cdot)$ (see [29, Theorem 8.1.3] or

[18, Theorem 2.1]). It follows that $\xi(x) = QS(x)g(x) - Bu(x)$ is a measurable selection of $f(x, u(x))$, and from (f2) we have $\xi \in H$. Then $F(u)$ is non-empty. In view of (f2) we have that

$$\|\xi + h\| \leq \sqrt{\int_\Omega (K_1 + K_2|u(x)|)^2 dx} + \|h\| \leq D_1 + D_2\|y\|,$$

for any $u \in H, \xi + h \in F(u)$, so that (F2) holds and $F(u)$ is bounded. In a similar way as in [25, Lemma 11] we obtain that $F(u)$ is closed and convex. Hence, $F : H \to C_v(H)$ and (F1) holds.

Now we shall prove that $F$ is $\epsilon$-upper semicontinuous on $H$. If this is not the case, then there exist $u \in H$, an $\epsilon$-neighborhood $\mathcal{O}_\epsilon(F(u))$ and sequences $u_n, y_n + h \in F(u_n)$ such that $u_n \to u$ and $y_n + h \notin \mathcal{O}_\epsilon(F(u))$ for all $n$, that is

$$dist(y_n + h, F(u)) \geq \epsilon. \tag{13}$$

Note that

$$y_n(x) = QS(x)\xi_n(x) - Bu_n(x),$$

where $\xi_n(x) \in \beta(u_n(x))$, for a.a. $x \in (-1, 1)$. Also from $S(x) \geq S_0 > 0$ and

$$\xi_n(x) = \frac{y_n(x) + Bu_n(x)}{QS(x)}$$

it follows that $\xi_n \in H$. We have seen that $\beta$ satisfies $(\widetilde{f}1)$–$(\widetilde{f}3)$. Since $\beta$ has compact values in $\mathbb{R}$ and the map $x \mapsto \beta(u(x))$ is measurable (see the first part of the proof), for any $n$ Corollary 8.2.13 in [29] implies the existence of a measurable selection $\eta_n(x) \in \beta(u(x))$ such that

$$dist(\xi_n(x), \beta(u(x))) = |\xi_n(x) - \eta_n(x)| \quad \text{for a.a. } x.$$

As $u_n \to u$ in $H$, passing to a subsequence we have $u_n(x) \to u(x)$ for a.a. $x$. Hence, the $\epsilon$-upper semicontinuity of $\beta$ gives

$$|\xi_n(x) - \eta_n(x)| \leq dist(\beta(u_n(x)), \beta(u(x))) \to 0,$$

as $n \to \infty$, for a.a. $x$. By $(\beta 2)$ we get

$$|\xi_n(x) - \eta_n(x)| \leq 2d_1 \quad \text{for a.a. } x,$$

so the Lebesgue theorem implies $\|\xi_n - \eta_n\| \to 0$, and then if we denote $p_n(x) = QS(x)\eta_n(x) - Bu(x)$, we have

$$\|y_n - p_n\| \to 0, \quad \text{as } n \to \infty.$$

Observe that $p_n + h \in F(u)$. Therefore,

$$dist(y_n + h, F(u)) \leq \|y_n - p_n\| \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction with (13). Thus (F3) is proved.

Finally, we obtain (F4). Indeed, if $y + h \in F(u)$, in view of (f4) we get

$$(y + h, u) \leq \int_{-1}^{1} \left(-\frac{B}{2}|u(x)|^2 + \frac{QS_1 d_1}{2B}\right) dx + \|h\| \|u\| \leq -\frac{B}{4}\|u\|^2 + \frac{QS_1 d_1}{B} + \frac{\|h\|^2}{B}.$$

This concludes the proof. □

If we define the function

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_{-1}^{1} |\nabla u|^2 dx, & \text{if } u \in H^1(-1, 1), \\ +\infty, & \text{otherwise}, \end{cases}$$

then it is well known ([30], see also [31, p. 63]) that $\varphi$ is proper, convex and lower semicontinuous and that $\partial\varphi = -\Delta$ with domain $D(\partial\varphi) = \{u \in H^2(-1, 1) : u_x(-1) = u_x(1) = 0\}$. Also, $D(\varphi) = \{u \in H : \varphi(u) < +\infty\} = H^1(-1, 1)$. Note also that in view of [30] we have $\overline{D(\varphi)} = \overline{D(\partial\varphi)} = H$. Denote $V = H^1(-1, 1) = D(\varphi)$ and let $V^*$ be its conjugate.

As a consequence of the compact embedding $H^1(-1, 1) \subset L^2(-1, 1)$ we have:

**Proposition 2.17.** *The function $\varphi$ satisfies* (F5).

Thus, we can consider (1) in the abstract form (11).

**Remark 2.18.** As it was proved in [5], the case of a degenerate diffusion $((1 - x^2)u_x)_x$ also generates a subdifferential $\partial\varphi$. In fact it remains being true for the case of quasilinear diffusion operators (see [5]).

Now, let us consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \{\omega = w(\cdot) \in C(\mathbb{R}, \mathbb{R}) : w(0) = 0\},$$

equipped with the Borel $\sigma$-algebra $\mathcal{F}$, the Wiener measure $\mathbb{P}$, and the usual uniform convergence on bounded sets of $\mathbb{R}$. Let $\zeta(t) = \phi w(t)$. Each $\omega \in \Omega$ generates a map $\zeta(\cdot) = \phi w(\cdot) \in C(\mathbb{R}, H)$ such that $\zeta(0) = 0$.

Write $A = -\partial\varphi$. For each $\omega \in \Omega$ we make the change of variable $v(t) = u(t) - \varepsilon\zeta(t)$. As a consequence of well-known results (see, e.g. [32]), inclusion (11) formally turns into

$$\begin{cases} \frac{dv}{dt} \in Av(t) + F(v(t) + \varepsilon\zeta(t)) + \varepsilon A\zeta(t), \\ v(0) = v_0 = u_0. \end{cases} \tag{14}$$

Now we have to show that (14) has at least one solution in some sense. Although some existence results can be obtained by obvious modifications of the results of the papers [5–7], we shall carry out this in a different way. Let us define the map $\widetilde{F} : [0, T] \times \Omega \times H \to 2^H$ by $\widetilde{F}(t, \omega, u) = F(u + \varepsilon\zeta(t)) + \varepsilon A\zeta(t)$.

**Definition 2.19.** The process $v : \Omega \times [0, T] \to H$ is called a strong solution of (14) if for each $\omega \in \Omega$:

1. $v(\omega, \cdot) \in C([0, T], H)$;
2. $v(\omega, \cdot)$ is absolutely continuous on any compact subset of $(0, T)$ and almost everywhere (a.e.) differentiable on $(0, T)$;
3. $v(\omega, \cdot)$ satisfies

$$\frac{dv}{dt} = Av(t) + f(t), \quad \text{for a.a. } t \in (0, T), v(0) = v_0, \tag{15}$$

where $f(\cdot) \in L^1(0, T; H)$ and $f(t) \in \widetilde{F}(t, \omega, v(\omega, t))$ for a.a. $t \in (0, T)$ on $H$.

In what follows, we shall omit the variable $\omega$ when no confusion is possible.

Let us now show that $\widetilde{F}$ possesses good properties. First, it is clear that it has at most linear growth. Indeed, by (F2) we obtain

$$\begin{aligned} \|y\| &\leq D_1 + D_2\|u\| + D_2\varepsilon\|\phi\| |w(t)| \\ &\quad + \varepsilon\|A\phi\| |w(t)| \leq \widetilde{D}_1 + D_2\|u\|, \end{aligned} \tag{16}$$

for any $y \in \widetilde{F}(t, \omega, u)$ and $t \in [0, T]$, where $\widetilde{D}_1$ depends on $\omega$ and $T$. Also, it follows from (F1) and (F3) that $\widetilde{F}(t, \omega, u) \in C_v(H)$ and that $u \to \widetilde{F}(t, \omega, u)$ is $\epsilon$-upper semicontinuous. Finally, we shall show that for any $u \in H, \omega \in \Omega$, the map $t \mapsto \widetilde{F}(t, \omega, u)$ has a measurable selection.

**Proposition 2.20.** *The map $t \mapsto \widetilde{F}(t, \omega, u)$ has a measurable selection for all $\omega \in \Omega, u \in H$.*

**Proof 2.21.** First, we consider the map $G$ defined above with $\widetilde{f}(s) = \beta(s)$ and $\widetilde{h}(x) = 0$. For a continuous function $v : [0, T] \to H$ we define the multivalued function $\widetilde{G} : [0, T] \to P(H)$ by

$$t \mapsto \widetilde{G}(t) = G(v(t)).$$

Note that this map has non-empty values by the arguments given in the proof of Proposition 2.15. We shall prove that this map has a measurable selection.

By ($\beta$1)–($\beta$3) we know that $\beta$ is upper semicontinuous. Then by Theorem 1 in [20, p. 84] we have that for any $\sigma > 0$ there exists a locally Lipschitzian map $\beta_\sigma : \mathbb{R} \to \mathbb{R}$ such that its range is contained in the convex hull of the range of $\beta$ and $Graph(\beta_\sigma) \subset Graph(\beta) + \sigma B_1$, where $B_1$ is a closed ball of radius 1 and centered at 0 in $\mathbb{R}^2$. As by ($\beta_2$) we have $|z| \le d_1$, for all $s \in \mathbb{R}$, $z \in \beta(s)$, we obtain also that $|\beta_\sigma(s)| \le d_1$, for any $s \in \mathbb{R}$.

Let $G_\sigma : H \to H$ be given by

$$G_\sigma(u) = \{y \in H : y(x) = \beta_\sigma(u(x)) \text{ for a.a. } x \in (-1, 1)\}.$$

This map is continuous. Indeed, let $u_n \to u$ in $H$. If $y_n = G_\sigma(u_n)$ does not converge to $y = G_\sigma(u)$, then there exist $\delta > 0$ and a subsequence (denoted again $y_n$) such that $\|y_n - y\| \ge \delta$ for any $n$. Passing to a subsequence we have that $y_n(x) \to y(x)$ for a.a. $x \in (-1, 1)$. Then $|y_n(x)| \le d_1$ and the Lebesgue theorem imply that $y_n \to y$, which is a contradiction. Then the map $z_\sigma : [0, T] \to H$ defined by $z_\sigma(t) = G_\sigma(v(t))$ is also continuous and then measurable.

It is well known that a real maximal monotone map can have at most a countable number of discontinuity points. Let $s_0$ be such a point. By $Graph(\beta_\sigma) \subset Graph(\beta) + \sigma B_1$ there exists $(y_\sigma, s_\sigma)$, $y_\sigma \in \beta(s_\sigma)$ such that

$$|\beta_\sigma(s_0) - y_\sigma| + |s_0 - s_\sigma| \le \sigma.$$

Since $\beta$ is $\sigma$-upper semicontinuous and $\beta(s_0)$ is a compact interval, we have $dist(y_\sigma, \beta(s_0)) = |y_\sigma - d_\sigma| \to 0$ as $\sigma \to 0$, for some $d_\sigma \in \beta(s_0)$. Then passing to a subsequence $\sigma_n$ we have that $d_{\sigma_n} \to d_{s_0} \in \beta(s_0)$ and

$$\left|\beta_{\sigma_n}(s_0) - d_{s_0}\right| \le \left|\beta_{\sigma_n}(s_0) - y_{\sigma_n}\right| + \left|y_{\sigma_n} - d_{\sigma_n}\right| + \left|d_{\sigma_n} - d_{s_0}\right| \to 0.$$

We can repeat this for each point of discontinuity and then by a diagonal procedure we can choose a common subsequence for all the points of discontinuity. Clearly, if $s$ is a point of continuity, then $\beta_{\sigma_n}(s) \to d_s$ no matter which sequence we take.

Let $S$ be the set of points of continuity of $\beta$. We define the selection $\overline{\beta}$ of $\beta$ given by

$$\overline{\beta}(s) = \begin{cases} \beta(s), & \text{if } s \in S, \\ d_s, & \text{if } s \notin S, \end{cases}$$

where $d_s$ is chosen for every $s \notin S$ from the above arguments. Hence, $\beta_{\sigma_n}(s) \to \overline{\beta}(s)$ for any $s$.

We state that the map defined by $z(t)(x) = \overline{\beta}(v(t)(x))$, for a.a. $x \in (-1, 1)$, maps every $t$ into $H$ and is measurable. Fix $t \in [0, T]$. We prove first that $z_{\sigma_n}(t) \to z(t)$ in $H$. This follows from the Lebesgue theorem as

$$z_{\sigma_n}(t)(x) = \beta_{\sigma_n}(v(t)(x)) \to \overline{\beta}(v(t)(x)),$$
$$\left|\beta_{\sigma_n}(v(t)(x))\right| \le d_1, \text{ for a.a. } x.$$

Then $z(t) \in H$, for all $t$, and $z(\cdot)$ it is the pointwise limit of a sequence of measurable functions, so that it is measurable. Clearly, $z(\cdot)$ is a selection of $\widetilde{G}$.

If we put $v(t) = u + \varepsilon\zeta(t)$, it follows that

$$r(t) = QSz(t) - B(u + \varepsilon\zeta(t)) + \varepsilon A\zeta(t) + h$$

is a measurable selection of $t \mapsto \widetilde{F}(t, \omega, u)$. □

**Remark 2.22.** When $F$ is upper semicontinuous (as in [24]) the map $t \mapsto \widetilde{F}(t, \omega, x)$ is also upper semicontinuous, and then it follows directly the existence of a measurable selection by the well-known theorems. However, we cannot obtain this property for our problem.

Also, it is well known (see [30] or [31]) that $A = -\partial\varphi$ is an $m$-dissipative operator.

Therefore, by [33] or [34], for each $v_0 \in H$ there exists an integral solution $v(\cdot)$ of (14), which means that $v(0) = v_0$ and for some $f(\cdot) \in L^1(0, T; H)$ such that $f(t) \in \widetilde{F}(t, \omega, v(\omega, t))$, for a.a. $t \in (0, T)$, the following inequality holds:

$$\|v(t) - \xi\|^2 \le \|v(s) - \xi\|^2 + 2\int_s^t (f(\tau) + A\xi, v(\tau) - \xi)\,d\tau, \quad t \ge s, \qquad (17)$$

for all $\xi \in D(A)$, where $(\cdot, \cdot)$ denotes the scalar product in $H$.

Note that (16) implies that $f(\cdot) \in L^2(0, T; H)$. Then it follows from [30] (see also [31, p. 189]) that problem (15) has a unique strong solution. Since every strong solution is an integral one and the integral solution of (15) is unique if $f(\cdot) \in L^1(0, T; H)$ [30] (see also [31]), it follows that our solution $v(\cdot)$ is a strong one (note that a different argument to arrive to this conclusion was given in [5]).

It is clear that the theorem on existence is also true if we take an arbitrary interval $[\tau, T]$. Let $v_i(\cdot)$, $v_1(0) = v_0$, $v_2(T) = v_1(T)$, be strong solutions of (14) defined on $[0, T]$ and $[T, 2T]$, respectively. Then putting

$$v(t) = \begin{cases} v_1(t), & \text{if } t \in [0, T], \\ v_2(t), & \text{if } t \in (T, 2T], \end{cases}$$

and

$$f(t) = \begin{cases} f_1(t), & \text{if } t \in [0, T], \\ f_2(t), & \text{if } t \in (T, 2T], \end{cases}$$

where $f_i(\cdot)$ are the corresponding selection of $\widetilde{F}(t, \omega, v_i(t))$, we can easily see that $v(\cdot)$ satisfies (17) on $[0, 2T]$, so that it is an integral solution. Arguing as above we obtain that $v(\cdot)$ is in fact a strong solution defined on $[0, 2T]$. In this way we can extend every strong solution on $[0, +\infty)$.

Let $\mathcal{D}(v_0, \omega)$ be the set of all strong solution of (14) (which are defined for all $t \ge 0$). Then we define the set-valued map $G : \mathbb{R}^+ \times \Omega \times H \to P(H)$ by

$$G(t, \omega, v_0) = \{v(t) + \varepsilon\zeta(t) : v(\omega, \cdot) \in \mathcal{D}(v_0, \omega)\}.$$

It can be proved exactly in the same way as in [22, Proposition 4] that $G$ satisfies the cocycle property (see Definition 2.2).

Let $\theta_s : \Omega \to \Omega$ be defined by $\theta_s\omega = w(s + \cdot) - w(s) \in \Omega$. Then the function $\widetilde{\zeta}$ corresponding to $\theta_s\omega$ is defined by $\widetilde{\zeta}(\tau) = \zeta(s + \tau) - \zeta(s) = \phi(w(s + \tau) - w(s))$.

Now we state and prove Theorem 1.1.

**Theorem 2.23.** *Assume the conditions (2) and (3). Then the multivalued random dynamical system $G$ associated to (1) has a global random attractor $\mathcal{A}_\varepsilon$.*

**Proof 2.24.** Condition (H1b) in Theorem 2.11 is proved in a similar way as in [22, Proposition 11]. We note that for any $y \in G(t, \theta_s\omega, u_0)$, $y = v(t) + \varepsilon\zeta(t + s) - \varepsilon\zeta(s)$, with $v(\omega, \cdot) \in \mathcal{D}(u_0, \omega)$. After the change of variable $z(t) = v(t) - \varepsilon\zeta(s)$, we obtain that $y = z(t) + \varepsilon\zeta(t + s)$, being $z(\cdot)$ the integral solution (in fact, a strong one) of the problem

$$\begin{cases} \dfrac{dz}{dt} = Az(t) + g(t), \\ z(0) = u_0 - \varepsilon\zeta(s), \end{cases} \qquad (18)$$

where $g(t) \in F(z(t) + \varepsilon\zeta(t + s)) + \varepsilon A\zeta(t + s)$, a.e. in $(0, T)$. For $s = -t_0$ multiplying (18) by $z(t)$ and using (F4) we get

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 \le -\delta(z(t) + \varepsilon\zeta(t - t_0), z(t) + \varepsilon\zeta(t - t_0)) + \varepsilon\|F(z(t) + \varepsilon\zeta(t - t_0))\|_+ \|\zeta(t - t_0)\| + \varepsilon\|A\zeta(t - t_0)\|\,\|z(t)\| + M.$$

Inequality (F2) and Young's inequality then imply

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 \le -\frac{\delta}{2}\|z(t)\|^2 + \varepsilon^2 M_2 \times \left(\|\zeta(t-t_0)\|^2 + \|A\zeta(t-t_0)\|^2\right) + M_3.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 \le -\delta\|z(t)\|^2 + \varepsilon^2 p(t-t_0,\omega) + 2M_3, \tag{19}$$

where $p(t-t_0,\omega) = 2M_2\left(\|\zeta(t-t_0)\|^2 + \|A\zeta(t-t_0)\|^2\right)$. Multiplying (19) by $\exp(\delta t)$ and integrating over $(0,-1+t_0)$ we obtain

$$\|z(-1+t_0)\|^2 \le \mathrm{e}^{-\delta(-1+t_0)}\|u_0\|^2 + \varepsilon^2 \mathrm{e}^{-\delta(-1+t_0)}\|\zeta(-t_0)\|^2 + \int_{-\infty}^{-1} \mathrm{e}^{-\delta(-1-\tau)}\left(\varepsilon^2 p(\tau,\omega) + 2M_3\right)\mathrm{d}\tau.$$

We take $R^2 = 1 + \frac{2M_3}{\delta}$ and

$$r_{1\varepsilon}^2(\theta_{-1}\omega) = R^2 + \varepsilon^2 \sup_{t_0\le -1} \mathrm{e}^{-\delta(-1+t_0)}\|\zeta(-t_0)\|^2 + \varepsilon^2\int_{-\infty}^{-1}\mathrm{e}^{-\delta(-1+t_0)} p(\tau,\omega)\,\mathrm{d}\tau, \tag{20}$$

$$r_\varepsilon(\theta_{-1}\omega) = r_{1\varepsilon}(\theta_{-1}\omega) + \varepsilon\|\zeta(-1)\|. \tag{21}$$

The radius $r_\varepsilon(\theta_{-1}\omega)$ is $\mathbb{P}$-a.s. finite, because $\|\zeta(-t_0)\|^2$ and $p(\tau,\omega)$ have at most polynomial growth for $\mathbb{P}$-almost all $\omega \in \Omega$.

Since $y = z(-1+t_0) + \varepsilon\zeta(-1)$, for $u_0$ in a bounded set $B$ we choose $T(\omega,B) \ge 1$ such that

$$\|y\| \le \|z(-1+t_0)\| + \varepsilon\|\zeta(-1)\| \le r_\varepsilon(\theta_{-1}\omega), \quad \text{if } t_0 \ge T(\omega,B),$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$ and any $y \in G\left(-1+t_0, \theta_{-t_0}\omega, u_0\right)$, $u_0 \in B$. Hence, (H1b) holds.

Let us prove (H2b). Let $y_n \in G(t_n,\omega_n,x_n)$ and $x_n \to x_0$ weakly, $t_n \to t_0$, $t_0 > 0$, $\omega_n \to \omega_0$. Then $y_n = v_n(t_n) + \zeta_n(t)$, where $v_n(\omega_n,\cdot) \in \mathcal{D}(x_n,\omega_n)$, and $\zeta_n \to \zeta_0$ in $C([0,T];H)$. Take $T > t_0$. Multiplying (15) by $v_n$ and using (16) and $\omega_n \to \omega_0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_n\|^2 \le K_1 + K_2\|v_n\|^2,$$

where $K_i$ are constant not depending on $n$, so that Gronwall's lemma and (16) imply

$$\|v_n(t)\| \le C_1,\ \|f_n(t)\| \le C_2, \quad \forall t \in [0,T]. \tag{22}$$

Then arguing as in [22, p. 825] we have also that

$$\int_0^T \varphi(v_n(t))\,\mathrm{d}t \le C_3,\ \varphi(v_n(t)) \le C_4(\delta), \quad \text{if } t \in [\delta,T], \tag{23}$$

$$\int_0^T t\left\|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right\|^2 \mathrm{d}t \le C_5. \tag{24}$$

It follows from (23) and equality (15) that $\frac{\mathrm{d}v_n}{\mathrm{d}t}$ is bounded in $L^2(0,T;V^*)$. Then by the compactness theorem [35] we obtain (up to a subsequence) that

$$v_n \to v \text{ weakly star in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;V),$$
$$\frac{\mathrm{d}v_n}{\mathrm{d}t} \to \frac{\mathrm{d}v}{\mathrm{d}t} \text{ weakly in } L^2\left(0,T;V^*\right),$$
$$v_n \to v \text{ in } L^2(0,T;H),$$
$$v_n(t) \to v(t) \text{ for a.a. } t \in (0,T),$$
$$f_n \to f \text{ weakly in } L^2(0,T;H). \tag{25}$$

Also, by (F5), (22)–(24) and the Ascoli–Arzelà theorem we obtain

$$v_n \to v \text{ in } C([\delta,T];H) \quad \text{for any } \delta > 0. \tag{26}$$

Hence,

$$v_n(t_n) \to v(t_0) \quad \text{in } H. \tag{27}$$

By the compact embedding $H \subset V^*$, (22) and the boundedness of $\frac{dv_n}{dt}$ in $L^2(0,T;V^*)$ it follows using again the Ascoli-Arzelà theorem that $v_n \to v$ in $C([0,T];V^*)$. Therefore, by a standard argument

$$v_n(t) \to v(t) \text{ weakly in } H \text{ for any } t \in [0,T]. \tag{28}$$

Hence, $v(0) = x_0$.

The maps $v_n$ satisfy inequality (17) replacing $f$ by $f_n$. By the convergences in (25)-(26) we have that $v$ satisfies (17) for any $0 < s \le t$. However, $v \in L^2(0,T;V)$, $\frac{dv}{dt} \in L^2(0,T;V^*)$ imply $v \in C([0,T];H)$ [36, p. 261], so (17) holds for any $0 \le s \le t$, as well.

Therefore, in order to show that $v$ is a strong solution of (14) it is necessary to check that $f(t) \in F(v(t)+\varepsilon\zeta_0(t)) + \varepsilon A\zeta_0(t)$, for a.a. $t \in (0,T)$. By condition (F3) for each $\gamma > 0$ there exists $N_0(\gamma)$ such that

$$f_n(t) \in F(v_n(t)+\varepsilon\zeta_n(t)) + \varepsilon A\zeta_n(t) \subset F(v(t)+\varepsilon\zeta_0(t)) + \varepsilon A\zeta_0(t) + \gamma B_1,$$

for $n \ge N_0$, where $B_1$ is a closed ball in $H$ of radius 1 centered at 0. Since the right-hand side of the last expression is a convex set, we have

$$\overline{co}\cup_{n\ge N_0} f_n(t) \subset F(v(t)+\varepsilon\zeta_0(t)) + \varepsilon A\zeta_0(t) + \gamma B_1.$$

But $f_n \to f$ weakly in $L^2(0,T;H)$ implies $f(t) \in \cap_{N\ge N_0}\overline{co}\cup_{n\ge N} f_n(t)$, for a.a. $t$, which is a consequence of Mazur's theorem. Hence, as $\gamma > 0$ is arbitrary, we obtain $f(t) \in F(v(t)+\varepsilon\zeta_0(t)) + \varepsilon A\zeta_0(t)$.

Hence, $v \in \mathcal{D}(x_0,\omega)$ and by (27) we have

$$y_n \to y = v(t_0) + \varepsilon\zeta(t_0) \in G(t_0,\omega_0,x_0).$$

Hence, condition (H2b) is proved.

Now, the result follows from Theorem 2.11. □

By using Theorem 2.12 we can also prove the upper semicontinuity as $\varepsilon \to 0$ of the random attractor to the associated global attractor for the associated (deterministic) partial differential inclusion. Denote by $G_\varepsilon$ the multivalued random dynamical system associated to (1) for each $\varepsilon > 0$. Let $G_0$ be the multivalued deterministic semiflow associated to (1) when $\varepsilon = 0$, which has a global compact attractor. This fact follows from Propositions 2.15 and 2.17 using the results in [27, Theorem 2.3] or [28, Theorem 6.33].

**Theorem 2.25.** *Assume conditions* (2) *and* (3)*. Then,*

$$\lim_{\varepsilon\searrow 0} dist(\mathcal{A}_\varepsilon(\omega),\mathcal{A}) = 0, \quad \text{for all } \omega \in \Omega,$$

*where $\mathcal{A}$ is the global attractor associated to* (1) *with $\varepsilon = 0$.*

**Proof 2.26.** Let us prove that for any bounded weakly closed set $B$ and any $t > 0$ the following holds:

$$dist\left(G_\varepsilon(t,\omega,B), G_0(t,B)\right) \underset{\varepsilon\to 0}{\to} 0, \quad \text{for all } \omega \in \Omega. \tag{29}$$

Assume the opposite, that is, there exist $\delta > 0$, $y_n \in G_{\varepsilon_n}(t,\omega,x_n)$, $x_n \in B$, $\varepsilon_n \to 0$ such that

$$dist(y_n, G_0(t,B)) \ge \delta.$$

Repeating the same arguments of the previous theorem we find up to a subsequence that $y_n \to y_0 \in G_0(t,x_0)$, $x_0 \in B$, which is a contradiction. Hence, as every compact set $B$ is bounded and weakly closed, we obtain that (G1) in Theorem 2.12 holds.

Further, in the proof of Theorem 2.23 we have established that for every $G_\varepsilon$ there exists a random radius given by (21) such that the random closed ball $B_{r_\varepsilon(\theta_{-1}\omega)}$ is absorbing, i.e., for any bounded set $B$ there exists $T(B,\omega)$ such that

$$\|y\| \le r_\varepsilon(\theta_{-1}\omega), \quad \forall y \in G_\varepsilon(-1+s, \theta_{-s}\omega, B), s \ge T.$$

Then it follows from (H2b) that the set $K_\varepsilon(\omega) = \overline{G_\varepsilon\left(1, \theta_{-1}\omega, B_{r_\varepsilon(\theta_{-1}\omega)}\right)}$ is compact, so as, by the cocycle property,

$$G_\varepsilon(s, \theta_{-s}\omega, B) = G_\varepsilon(1, \theta_{-1}\omega, G_\varepsilon(-1+s, \theta_{-s}\omega, B)),$$

we have that $G_\varepsilon$ satisfies (H1).

Let $B_{R+1}$ be a closed ball of radius $R+1$ centered at 0 and define $K = \overline{G_0(1, B_{R+1})}$, which is a compact set because the operator $G_0(1,\cdot)$ is compact, as shown in the proof Theorem 2.3 in [27] (see also [28, Lemma 6.17]). Then $B_{r_\varepsilon(\theta_{-1}\omega)} \subset B_{R+1}$ for $\varepsilon$ small enough and in view of (29) we have

$$\begin{aligned} dist(K_\varepsilon(\omega), K) &\le dist\left(G_\varepsilon\left(1, \theta_{-1}\omega, B_{r_\varepsilon(\theta_{-1}\omega)}\right), G_0(1, B_{R+1})\right) \\ &\le dist\left(G_\varepsilon(1, \theta_{-1}\omega, B_{R+1}), G_0(1, B_{R+1})\right) \to 0, \text{ as } \varepsilon \to 0. \end{aligned}$$

Hence, (G2) holds.

Condition (H2) is a consequence of (H2b), which was proved in the previous theorem. Also, the map $G_0$ has compact values and $x \mapsto G_0(t,x)$ is an upper semicontinuous map (see the proof of Theorem 2.3 in [27] or the proof of Theorem 6.33 in [28]).

Finally, we apply Theorem 2.12. □

**Remark 2.27.** We have considered the asymptotic behaviour of solutions generated by the random inclusion (14). It would be interesting to study the relationship of these solutions with the solutions of the original stochastic equation as done in [37] for a differential inclusion of other type. Moreover, it would be interesting to get similar results when $\phi$ does not satisfy $\phi \in H^2(-1,1)$ with $\phi_x(-1) = \phi_x(1) = 0$ (note that in the case of degenerate diffusion, $((1-x^2)u_x)_x$, this leads to consider $\phi$ such that $\phi \notin H^2(-1,1)$).

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