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Lagrangian approach to the study of level sets II: A quasilinear equation in climatology

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Abstract

We study the dynamics and regularity of the level sets in solutions of the semilinear parabolic equation

$$u_t - \Delta_p u + f \in a\mathbb{H}(u-\mu) \quad \text{in } Q = \Omega \times (0,T],\ p \in (1,\infty),$$

where $\Omega \subset \mathbb{R}^n$ is a ring-shaped domain, $\Delta_p u$ is the p-Laplace operator, a and μ are given positive constants, and $\mathbb{H}(\cdot)$ is the Heaviside maximal monotone graph: $\mathbb{H}(s) = 1$ if $s > 0$, $\mathbb{H}(0) = [0,1]$, $\mathbb{H}(s) = 0$ if $s < 0$. The mathematical models of this type arise in climatology, the case $p = 3$ was proposed and justified by P. Stone in 1972. We establish the conditions on the initial data which guarantee that the level sets $\Gamma_\mu(t) = \{\mathbf{x}\colon u(\mathbf{x},t) = \mu\}$ are hypersurfaces, study the regularity of $\Gamma_\mu(t)$ and derive the differential equation that governs the dynamics of $\Gamma_\mu(t)$. The analysis is based on the introduction of a system of Lagrangian coordinates that transforms the moving surface $\Gamma_\mu(t)$ into a stationary one.

Keywords: Parabolic p-Laplacian; Lagrangian coordinates; Climatic energy balance models; Free boundary problem

1. Introduction

1.1. Statement of the problem

In this paper we deal with the problem

$$\begin{cases} u_t - \Delta_p u + f(x,t) \in a\mathbb{H}(u-\mu) & \text{in } D_T = \Omega \times (0,T], \\ u = \phi & \text{on } S_T = \partial\Omega \times (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $\mu > 0$ and $a > 0$ are prescribed constants, $\mathbb{H}(\cdot)$ is the Heaviside maximal monotone graph in $\mathbb{R}^2$ given by

$$\mathbb{H}(s) = \begin{cases} 1 & \text{if } s > 0, \\ [0,1] & \text{if } s = 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Δ_p is the p-Laplace operator

$$\Delta_p v \equiv \operatorname{div}\bigl(|\nabla v|^{p-2}\nabla v\bigr), \quad p \in (1, \infty).$$

It is assumed throughout the paper that $\Omega \in \mathbb{R}^n$, $n \geqslant 1$, is a ring-shaped domain with the exterior boundary $\partial_e \Omega$ and the interior boundary $\partial_i \Omega$, $\partial_i \Omega \cap \partial_e \Omega = \emptyset$. The function $f(x,t)$ is given and belongs at least $L^{p'}(0,T; L^{p'}(\Omega))$ with $p' = \frac{p}{p-1}$.

Our interest in this problem is motivated by its application in Climatology. Problem (1.1) arises from the mathematical formulation of the Energy Balance Model proposed by M. Budyko in 1969 [1]. The model is obtained from the energy balance equation for the earth surface

$$E = R_a - R_e + D,$$

where E is the accumulation of the total energy, R_a is the co-albedo absorbed energy (represented here by the discontinuous function, i.e., the maximal monotone graph), R_e is the emitted energy and D is the diffusion represented by the second order p-Laplace diffusion operator. If $p = 2$, then the diffusion is described by the linear Laplace operator, as was originally proposed by M. Budyko. In 1972 P. H. Stone [14] proposed to choose for D the nonlinear diffusion operator with $p = 3$. This is the case considered in the present paper: the diffusion operator D is the quasilinear p-Laplacian under the general condition $p \in (1, \infty)$. More information on the physical backgrounds of this model and the further references can be found in [4–6].

The main aim of the paper is to describe the level set Γ_μ which separates the regions

$$D_T^+ = \bigl\{(\mathbf{x},t) \in D_T\colon\ u(\mathbf{x},t) > \mu\bigr\} \quad \text{and} \quad D_T^- = \bigl\{(\mathbf{x},t) \in D_T\colon\ u(\mathbf{x},t) < \mu\bigr\}.$$

The solution of problem (1.1) is understood as follows.

Definition 1.1. A function $u : D_T \mapsto \mathbb{R}$ is said to be a continuous weak solution of problem (1.1) if

(1) $u \in C^0(\overline{D}_T) \cap L^p(0,T;\ W^1_p(\Omega))$ and satisfies the initial and boundary conditions by continuity,
(2) there exists a function $h_u \in L^\infty(D_T)$, such that

$$\begin{cases} h_u : D_T \mapsto [0,1], \\ h_u(\mathbf{x},t) \in \mathbb{H}\bigl(u(\mathbf{x},t) - \mu\bigr) \quad \text{for a.e. } (\mathbf{x},t) \in D_T, \end{cases} \tag{1.2}$$

(3) for every test-function $\eta \in L^p(0,T;\ W_0^{1,p}(\Omega))$, such that $\eta_t \in L^2(D_T)$, the following identity holds:

$$\int_{D_T} \bigl[\eta_t u - \nabla\eta \cdot |\nabla u|^{p-2}\nabla u - \eta f + a\eta h_u\bigr]\, d\mathbf{x}\, dt = \int_\Omega u\eta\, d\mathbf{x}\bigg|_{t=0}^{t=T}. \tag{1.3}$$

According to this definition, problem (1.1) is understood as the problem of finding the functions u and h_u such that

$$\begin{cases} u_t - \Delta_p u + f(x,t) = a h_u \in a\mathbb{H}(u - \mu) \quad \text{in } D_T, \\ u = \phi \quad \text{on } S_T, \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{in } \Omega. \end{cases} \tag{1.4}$$

Throughout the paper we assume that the data of problem (1.4) are subject to the following general restrictions:

$$\begin{cases} \partial\Omega_e, \partial\Omega_i \in C^2, \\ u_0 \in W^{2,q}(\Omega) \quad \text{with some } q > n+2, \qquad |\nabla u_0| \geqslant \epsilon > 0 \quad \text{in } \overline{\Omega}, \\ \phi > \mu \quad \text{on } \partial\Omega_e \times [0,T], \qquad \phi < \mu \quad \text{on } \partial\Omega_i \times [0,T], \\ \phi(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{on } \partial\Omega_e \text{ and } \partial\Omega_i, \\ f(\mathbf{x},t) \in C^\beta\bigl((0,T); L^q(\Omega)\bigr) \quad \text{with some } \beta \in (0,1). \end{cases} \tag{1.5}$$

Further conditions will be specified in the formulation of the main results. We will use the notation

$$\Gamma_\mu = \{(\mathbf{x}, t) \in D_T\colon\ u(\mathbf{x}, t) = \mu\}, \qquad \Gamma_\mu(t) = \Gamma_\mu \cap \{t = \text{const}\}$$

and assume that

$$\gamma = \Gamma_\mu(0) \text{ is a } (n-1)\text{-dimensional surface of the class } C^2. \tag{1.6}$$

We prove that under these conditions the set Γ_μ is a $(n+1)$-dimensional hypersurface (for short times), establish certain regularity properties of Γ_μ, and show that the dynamics of $\Gamma_\mu(t)$ is defined by a differential equation which can be interpreted as a generalization of the Darcy law in filtration theory.

1.2. Previous work. Motivation

The questions of dynamics and regularity of the level set Γ_μ in solutions of Eq. (1.4) were studied thus far only in the linear case $p = 2$—see [7,15] for the problem in the one-dimensional setting and [6] for the case of arbitrary space dimension. The present work continues the study initiated in [6] for the special case $p = 2$, $f \equiv 0$ and extends it to the quasilinear equation ($p \neq 2$) with a nonzero forcing term.

Under the foregoing conditions on the data, existence of a weak solution to problem (1.1) can be proved by the methods developed in [4,5]. The discontinuous term $\mathbb{H}(u - \mu)$ is approximated by a sequence of smooth functions, and the solution of problem (1.1) is then obtained as the limit of the sequence of solutions of the regularized problems. The basics, an excellent insight into this method and a number of relevant results can be found in the monograph [12]. Unfortunately, this approach provides no information about the regularity and qualitative properties of the surface Γ_μ.

For smooth solutions the differential equation of the level surface Γ_μ can be derived in the standard way. Calculating the total derivative of u along Γ_μ we find that

$$0 = du|_{\Gamma_\mu} = (u_t + \mathbf{x}_t \cdot \nabla u)|_{\Gamma_\mu}\, dt. \tag{1.7}$$

Since the normal $\mathbf{n}$ to $\Gamma_\mu(t)$ has the form $\nabla u/|\nabla u|$, Eq. (1.7) formally leads to the differential equation of motion of Γ_μ: if $|\nabla_x u(x_0, t_0)| \neq 0$ at a point $(x_0, t_0) \in \Gamma_\mu$, then at this point the normal velocity of Γ_μ is calculated by the formula

$$\mathbf{x}_t \cdot \mathbf{n} = -\frac{u_t}{|\nabla u|}\bigg|_{\Gamma_\mu}.$$

However, the regularity of the searched solution is insufficient to justify such an equation. Indeed: the nonlinear forcing term in Eq. (1.1) belongs to $L^\infty(D_T)$ and the standard parabolic theory does not guarantee differentiability of the solution across Γ_μ.

The level surface Γ_μ can be regarded as a moving (free) boundary where the nonlinear forcing term has a discontinuity jump. The study of regularity of solutions of the free boundary problems is often based on a suitable change of variables that transforms (locally) the moving boundary into a part of vertical plane—see, for example, [2,3,8] and the references therein. In this approach, the study of the free boundary properties reduces to the study of behavior of the solution to a nonlinear PDE near a known time -independent boundary of the problem domain. The method we use in the present paper is also based on a special (nonlocal) coordinate transformation that renders the free boundary Γ_μ a time-independent surface. We prove the equivalence between the original problem and the new one, and then treat the latter as an independent mathematical problem. An advantage of our method is that the differential equation of the free boundary is included into the formulation of the new problem and does not need any further justification. The detailed description of the coordinate transformation is given in Section 2.

1.3. Main results

Let us a choose a domain $\omega(0) \subset \Omega$ such that $\partial\omega(0) \in C^2$, $\gamma \subset \omega(0)$, and there exist ring-shaped domains ω_0^+ and ω_0^- satisfying the conditions

$$\begin{cases} \omega(0) = \omega_0^+ \cup \omega_0^-, \qquad \gamma = \overline{\omega}_0^+ \cap \overline{\omega}_0^-, \\ \overline{\omega}_0^+ \cap \partial\Omega_e = \emptyset, \qquad \overline{\omega}_0^- \cap \partial\Omega_i = \emptyset, \\ \partial\omega_0^\pm \in C^2. \end{cases} \tag{1.8}$$

Theorem 1.1. *Let conditions* (1.5), (1.6), (1.8) *be fulfilled. Assume that* $a > 0$ *and* $0 \leqslant f < a$ *in* $\overline{D}_T$. *Then there exists* T' *such that*

(1) *problem* (1.1) *has a weak solution* $u \in W^{2,1}_q(D_{T'})$,
(2) *for this solution the surface* Γ_μ *is parametrized by the bijective mapping*

$$\gamma \ni \mathbf{y} \mapsto \mathbf{x}(\mathbf{y}, t) = \mathbf{y} + \nabla_{\mathbf{y}} U \in \Gamma_\mu(t) \tag{1.9}$$

where $U(\mathbf{y}, t)$ *is a function defined on* $\omega(0) \times (0, T')$ *and*

$$D^\beta_{\mathbf{y}} U \in W^{2,1}_q\big(\omega^\pm_0 \times (0, T')\big) \quad \text{for } |\beta| = \textstyle\sum_i \beta_i \leqslant 2.$$

Moreover,

$$\begin{cases} \overline{\omega(0)} \ni \mathbf{y} \mapsto \mathbf{x}(\mathbf{y}, t) = \mathbf{y} + \nabla_{\mathbf{y}} U \in \overline{\omega(t)}, \\ \displaystyle\int\limits_{\omega(t)} u(\mathbf{x}, t)\, d\mathbf{x} = \int\limits_{\omega(0)} u_0(\mathbf{x})\, d\mathbf{x} \quad \forall t \in [0, T']. \end{cases} \tag{1.10}$$

Corollary 1.1. *An immediate byproduct of Theorem* 1.1 *is that for the constructed solution the set* Γ_μ *is an* n*-dimensional hypersurface of the class* $C^{1+\alpha,(1+\alpha)/2}$ *with some* $\alpha \in (0, 1)$, *and that for every* $t \in (0, T']$ *the set* $\Gamma_\mu(t)$ *is a* $(n-1)$*-dimensional hypersurface of the class* $C^{2+\sigma}$ *with* $\sigma \in (0, 1)$.

Remark 1.1. If we drop the regularity assumption (1.6), then, by virtue of conditions (1.5) and the embedding theorems in Sobolev spaces, we would have $\gamma \in C^{1+\alpha}$ with certain $\alpha(q, n) \in (0, 1)$. In this case formulas (1.9) yield $\Gamma_\mu(t) \in C^{1+\alpha}$ for every $t \in (0, T']$.

Remark 1.2. In this paper we do not specially discuss the question of uniqueness of solution to problem (1.1). Under the conditions of Theorem 1.1 the uniqueness follows from the results of [4].

Theorem 1.2. *The points of the surface* $\Gamma_\mu(t)$ *move with the velocity*

$$\mathbf{v}(\mathbf{x}, t) = -\frac{1}{u} |\nabla u|^{p-2} \nabla u + \nabla p,$$

where $p(\mathbf{x}, t)$ *is the solution of the elliptic problem*

$$\begin{cases} \operatorname{div}(u \nabla p) + f - a\chi_{\{u>\mu\}} = 0 \quad \text{in } \omega(t),\ t \in [0, T'], \\ p(\mathbf{x}, t) = 0 \quad \text{on } \partial\omega(t) \end{cases} \tag{1.11}$$

and the moving domain $\omega(t)$ *is given in* (1.10). *The points of the surface* $\Gamma_\mu(0)$ *start moving with the velocity*

$$\mathbf{v}_0(\mathbf{x}) = -\frac{1}{u_0} |\nabla u_0|^{p-2} \nabla u_0 + \nabla p_0,$$

where $p_0 \in W^2_q(\omega(0))$ *is the solution of the elliptic problem*

$$\operatorname{div}(u_0 \nabla p_0) + f(\mathbf{x}, 0) - a\chi_{\{u_0>\mu\}} = 0 \quad \text{in } \omega(0), \qquad p_0 = 0 \quad \text{on } \partial\omega(0).$$

Moreover,

$$\big|\mathbf{v}\big(\mathbf{x}(\mathbf{y}, t), t\big) - \mathbf{v}_0(\mathbf{y})\big|^{(\sigma)}_{\omega(0)} \leqslant C t^\nu \quad \text{with } \nu \in (0, 1). \tag{1.12}$$

The next result refers to case $p = 2$ (linear diffusion).

Theorem 1.3. *Let in the conditions of Theorem* 1.1 $p = 2$ *and* $f \equiv 0$. *Given an arbitrary* $m \in \mathbb{N}$, *there exists* $T = T(m)$ *such that for every fixed* $\mathbf{y} \in \omega(0)$ *the function* $\mathbf{X}(\mathbf{y}, t) = \mathbf{y} + \nabla_{\mathbf{y}} U$ (*the trajectory*) *satisfies the estimate*

$$\sum_{k=0}^{m} \frac{1}{k! M^k} \sum_{|\beta| \leqslant 2} \big\| t^k D^k_t D^\beta_{\mathbf{y}} U(\mathbf{y}, t) \big\|_{W^{2,1}_q(\omega^\pm_0 \times (0, T(m)))} < \infty$$

with a constant M *depending on* n, q, m, γ *and* $\partial\omega^\pm_0$, *but independent of* U.

Remark 1.3. It is proved in [7] that in the case $p = 2$, $n = 1$, and under similar conditions on the initial data, the level curve Γ_μ is represented in the form $x = \zeta(t)$ with $\zeta(t) \in C^\infty(0, T]$. The proof given in [7] is specific for the one-dimensional case and is not applicable to the solutions of multidimensional problem.

1.4. Organization of the paper

The study of the level set Γ_μ is based on the introduction of the system of Lagrangian coordinates frequently used in continuum mechanics. The evolution equation (1.4) is formally considered as the of mass balance equation in the motion of a fictitious fluid, and this motion is then given a counterpart description in the plane of Lagrangian coordinates. The convenience of this method is that on the plane of Lagrangian coordinates the level set Γ_μ is known beforehand.

The introduction of Lagrangian coordinates and the equivalence between the original free-boundary problem and the problem formulated in Lagrangian coordinates are given in Section 2. This part of presentation mostly follows paper [6] where the same method was applied to the study of problem (1.1) with the linear diffusion operator and $f \equiv 0$.

Sections 3-4 are devoted to the study of auxiliary problems formulated in Lagrangian coordinates. In Section 5 we prove solvability of the problem formulated in Lagrangian coordinates and check the equivalence between the original free-boundary problem and its Lagrangian counterpart. The proofs of Theorems 1.1, 1.2 are given in Section 6 and turn out to be simple byproducts of the results obtained for the problem in Lagrangian formulation. The proof of Theorem 1.3 is given in Section 7.

2. Lagrangian coordinates

2.1. The Euler and Lagrangian descriptions of motions of a fluid

Let us consider a fluid occupying a region $\omega(t) \subset \mathbb{R}^n$ assuming that the following conditions are fulfilled:

- the mass of every moving volume $\sigma(t) \subseteq \omega(t)$, constituted by the same particles, does not change with time,
- the boundary $\partial\omega(t)$ of $\omega(t)$ is constituted by the same particles for every $t > 0$ and the velocity of $\partial\omega(t)$ in the normal direction coincides with the normal velocity of the particle constituting $\partial\omega(t)$,
- the continuous velocity field $\mathbf{v}(\mathbf{x}, t)$ is given.

There are two methods of description of such a motion. The first one is the Euler method in which the characteristics of motion are considered as function of the time t and the position of each particle in a coordinate system independent of the fluid. Let $u(\mathbf{x}, t)$ be the density of the fluid. The mathematical description of the fluid motion includes

(a) the mass conservation law

$$\int_{\sigma(t)} u(\mathbf{x}, t)\, d\mathbf{x} = \int_{\sigma(0)} u_0(\mathbf{x})\, d\mathbf{x},$$

where $\sigma(t)$ is an arbitrary fluid volume which evolves with time but is constituted by the same particles at every $t > 0$; in the differential form this law reads

$$u_t + \operatorname{div}(u\mathbf{v}) = 0;$$

(b) the initial and boundary conditions: the initial distribution of density $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ in $\omega(0)$ and the normal velocity of $\partial\omega(t)$ are given.

In this approach the unknowns are the density $u(\mathbf{x},t)$ and the domain $\omega(t)$. They are defined from the conditions

$$\begin{cases} u_t + \operatorname{div}(u\mathbf{v}) = 0 \quad \text{in } \mathcal{C} = \bigcup_{\{t>0\}} \omega(t), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{in } \omega(0), \\ \int\limits_{\omega(t)} u(\mathbf{x},t)\,d\mathbf{x} = \int\limits_{\omega(0)} u_0(\mathbf{x})\,d\mathbf{x} \quad \forall t > 0. \end{cases} \tag{2.1}$$

The solution of this problem is understood in the weak sense.

Definition 2.1. A domain $\mathcal{C} = \bigcup_{t>0} \omega(t)$ and a function $u(\mathbf{x},t) \in C^0(\bar{\mathcal{C}})$ are called a solution of problem (2.1) is for every test-function $\eta \in C^1(\bar{\mathcal{C}})$

$$\int\limits_{\mathcal{C}} (u\eta_t + u\nabla\eta \cdot \mathbf{v})\,d\mathbf{x}\,dt = \int\limits_{\omega(t)} \eta u\,d\mathbf{x}\bigg|_{t=0}^{t=T}. \tag{2.2}$$

In the approach of Lagrange the characteristics of motion are considered as functions of time and the initial position of each particle. In this method the unknowns are: the density $u[\mathbf{X}(\mathbf{y},t),t]$ and the position $\mathbf{X}(\mathbf{y},t) \in \omega(t)$ of the particle which was initially located at the point $\mathbf{y} \in \omega(0)$. The flow is described by the following relations:

(a) the mass conservation law

$$u\big[\mathbf{X}(\mathbf{y},t),t\big]|\mathbf{J}| = u_0(\mathbf{y}) \quad \text{for } (\mathbf{y},t) \in \omega(0) \times [0,T], \tag{2.3}$$

where $\mathbf{J}$ is the Jacobi matrix of the mapping

$$\overline{\omega(0)} \ni \mathbf{y} \mapsto \mathbf{x} = \mathbf{X}(\mathbf{y},t) \in \overline{\omega(t)}, \tag{2.4}$$

(b) the equation of trajectories

$$\begin{cases} \mathbf{X}_t(\mathbf{y},t) = \mathbf{v}\big[\mathbf{X}(\mathbf{y},t),t\big] \quad \text{in } \overline{\omega(0)} \times [0,T], \\ \mathbf{X}(\mathbf{y},0) = \mathbf{y} \quad \text{in } \overline{\omega(0)}. \end{cases} \tag{2.5}$$

Definition 2.2. A vector-valued function $\mathbf{X}(\mathbf{y},t)$ and a scalar function $u(\mathbf{X},t)$ are said to be a weak solution of system (2.3), (2.5) if

$$\mathbf{X}_t, \mathbf{v}[\mathbf{X},t] \in \big(W^{1,0}_q(Q_T)\big)^n \quad \text{with some } q > 1, \qquad u\big[\mathbf{X}(\mathbf{y},t),t\big] \in C^0(\overline{Q}_T),$$

for every test-function $\Phi \in W^{1,0}_{q'}(Q_T)$

$$\int\limits_{Q_T} \nabla\Phi \cdot \big(\mathbf{X}_t - \mathbf{v}[\mathbf{X},t]\big)\,d\mathbf{x}\,dt = 0, \tag{2.6}$$

and (2.3) holds at every point of Q_T.

Theorem 2.1. *Let* $(\mathbf{X},u)$ *is a weak solution of problem* (2.3), (2.5) *in the sense of Definition* 2.2. *If there exist positive constants* λ_1, λ_2 *such that* $\lambda_1 < |\mathbf{J}| < \lambda_2$ *in* Q_T *and if the mapping* (2.4) *is bijective, then the function* $u(\mathbf{x},t)$ *defined by the formulas*

$$u(\mathbf{x},t) = u_0(\mathbf{y})\big|\mathbf{J}^{-1}\big|, \quad \mathbf{x} = \mathbf{X}(\mathbf{y},t),\ (\mathbf{y},t) \in \overline{Q}_T, \tag{2.7}$$

is a solution of problem (2.1) *in the sense of Definition* 2.1.

Proof. Let us check that the function $u(\mathbf{x},t)$ satisfies identity (2.2). Let $\psi(\mathbf{y},t)$ be a suitable test-function. Passing to the Lagrangian coordinates $\mathbf{y}$ and applying (2.3), we have:

$$\int_{\omega(t)} \psi u \, d\mathbf{y}\Big|_{t=0}^{t=T} = \int_0^T \frac{d}{dt}\biggl(\int_{\omega(t)} \psi(\mathbf{x},t)u(\mathbf{x},t)\,d\mathbf{x}\biggr)dt$$
$$= \int_0^T \frac{d}{dt}\biggl(\int_{\omega(0)} \psi\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]u\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]|\mathbf{J}|\,d\mathbf{y}\biggr)dt$$
$$= \int_0^T\biggl(\int_{\omega(0)} \biggl(\frac{d}{dt}\psi\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]\biggr)u\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]|\mathbf{J}| + \psi\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]\frac{d}{dt}\bigl(u\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]|\mathbf{J}|\bigr)\,d\mathbf{y}\biggr)dt$$
$$= \int_{Q_T} u_0 \frac{d}{dt}\bigl(\psi\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]\bigr)\,d\mathbf{x}\,dt.$$

By virtue of (2.6)

$$\int_{Q_T} u_0(\mathbf{y})\bigl[\psi_t\bigl[\mathbf{X}(\mathbf{y},t),t\bigr] + u_0(\mathbf{y})\nabla_{\mathbf{x}}\psi\bigl[\mathbf{X}(\mathbf{y},t),t\bigr]\cdot\mathbf{X}_t(\mathbf{y},t)\bigr]\,d\mathbf{y}\,dt$$
$$= \int_{Q_T} u_0(\psi_t + \nabla_{\mathbf{x}}\psi\cdot\mathbf{v})\,d\mathbf{y}\,dt$$
$$= \int_{Q_T} u(\psi_t + \nabla_{\mathbf{x}}\psi\cdot\mathbf{v})|\mathbf{J}|\,d\mathbf{y}\,dt$$
$$= \int_{\mathcal{C}} (u\psi_t + u\nabla_{\mathbf{x}}\psi\cdot\mathbf{v})\,d\mathbf{x}\,dt.$$

The initial condition for $u(\mathbf{x},t)$ is fulfilled by definition. By (2.3), for every $t>0$

$$\int_{\omega(t)} u(\mathbf{x},t)\,d\mathbf{x} = \int_{\omega(0)} u|\mathbf{J}|\,d\mathbf{y} = \int_{\omega(0)} u_0(\mathbf{y})\,d\mathbf{y}. \qquad \square$$

Corollary 2.1. *The assertion of Theorem* 2.1 *remains true if condition* (2.6) *is substituted by the following one*: *there exists a nondegenerate symmetric matrix* $\mathbf{A}$ *with differentiable entries* A_{ij} *such that*

$$\operatorname{div}\bigl(\mathbf{A}\cdot\bigl(\mathbf{X}_t - \mathbf{v}[\mathbf{X},t]\bigr)\bigr) = 0 \quad \text{a.e. in } Q_T. \tag{2.8}$$

Proof. For every given $\Psi \in W_q^{1,0}(Q_T)$ there exists a function $\Phi \in W_q^{1,0}(Q_T)$ defined as a weak solution of the co-normal derivative problem

$$\operatorname{div}(\mathbf{A}\cdot\nabla\Phi - \nabla\Psi) = 0 \quad \text{in } \omega(0), \qquad (\mathbf{A}\cdot\nabla\Phi - \nabla\Psi)\cdot\mathbf{n}|_{\partial\omega(0)} = 0.$$

Multiplying (2.8) by Φ and integrating by parts in $\omega(0)$, we obtain (2.6). $\square$

Corollary 2.2. *In the conditions of Corollary* 2.1

$$\bigl(\mathbf{A}\cdot\bigl(\mathbf{X}_t - \mathbf{v}[\mathbf{X},t]\bigr)\bigr)\cdot\mathbf{n} = 0 \quad \text{on } S_T = \partial\omega(0)\times[0,T], \tag{2.9}$$

where $\mathbf{n}$ *is the exterior normal to* $\partial\omega(0)$.

2.2. Local coordinates in the equation without mass conservation

Let us revert to problem (2.1). We want to consider this problem as the mathematical description of motion of a fluid with density $u(\mathbf{x},t)$ and velocity

$$\mathbf{v}(\mathbf{x},t) = -u^{-1}|\nabla u|^{p-2}\nabla u + \nabla p, \tag{2.10}$$

where $p(\mathbf{x},t)$ is the new unknown. Let us denote

$$\begin{aligned}
&\omega^+(t)=\{\mathbf{x}\in\omega(t):\ u(\mathbf{x},t)>\mu\}, &&\omega^-(t)=\{\mathbf{x}\in\omega(t):\ u(\mathbf{x},t)<\mu\},\\
&\omega_0^+=\{\mathbf{x}\in\omega(0):\ u_0(\mathbf{x})>\mu\}, &&\omega_0^-=\{\mathbf{x}\in\omega(0):\ u_0(\mathbf{x})<\mu\},\\
&Q_T^\pm=\omega_0^\pm\times(0,T], &&Q_T=Q_T^+\cup Q_T^-\cup\{\gamma\times(0,T]\},\\
&S_T=\text{the lateral boundary of } Q_T, &&S_T^\pm=\text{the lateral boundaries of } Q_T^\pm.
\end{aligned}$$

We will search for the solutions satisfying several additional conditions:

- for every $t>0$ the set $\Gamma_\mu(t)$ is a $(n-1)$-dimensional manifold separating the regions $\omega^\pm(t)$,
- for every $t>0$ $\Gamma_\mu(t)$ is constituted by the same particles,
- the displacement $\mathbf{X}$ is the potential vector

  $$\mathbf{X}=\mathbf{y}+\nabla_{\mathbf{y}}U,\quad (\mathbf{y},t)\in\overline{Q}_T,$$

  which makes symmetric the Jacobi matrix $\mathbf{J}$, and

  $$\omega^\pm(t)=\{\mathbf{x}:\ \mathbf{x}=\mathbf{X}(\mathbf{y},t),\ \mathbf{y}\in\omega_0^\pm\},\qquad \Gamma_\mu(t)=\{\mathbf{x}:\ \mathbf{x}=\mathbf{X}(\mathbf{y},t),\ \mathbf{y}\in\gamma\},$$

- the density $u(\mathbf{x},t)$ is a solution of problem (2.1) with the $\mathbf{v}$ given by (2.10).

Adding these assumptions to conditions (2.3), (2.5), we arrive at the following problem: to find scalar functions $U(\mathbf{y},t)$, $P(\mathbf{y},t)$, $R(\mathbf{y},t)$ such that $\mathbf{X}=\mathbf{y}+\nabla U$ is a solution of system (2.5) with the velocity field given by (2.10),

$$R(\mathbf{y},t)\equiv u[\mathbf{X}(\mathbf{y},t),t]=\mu\quad\text{for } \mathbf{y}\in\Gamma_\mu(0).$$

The latter condition yields $|\mathbf{J}|=1$ on $\Gamma_\mu(0)$. According to Theorem 2.1, the solution of problem (2.3), (2.5) generates a weak solution of problem (2.1): for every $\psi\in C^0(\bar{\mathcal{C}})\cap C^1(\mathcal{C})$

$$\int\limits_{\mathcal{C}}\bigl(u\psi_t-\nabla\psi\cdot\bigl(|\nabla u|^{p-2}\nabla u-u\,\nabla p\bigr)\bigr)\,d\mathbf{x}\,dt=\int\limits_{\omega(t)}\eta u\,d\mathbf{x}\Big|_{t=0}^{t=T}. \tag{2.11}$$

Let us now claim that the function p is chosen as follows: $p\in W^{1,0}_{p'}(\mathcal{C})$ and for every $\psi\in C^0(\bar{\mathcal{C}})\cap C^1(\mathcal{C})$

$$\int\limits_{\mathcal{C}}\bigl[u\nabla\psi\cdot\nabla p-\psi\bigl(f-a\,\chi_{\omega^+(t)}\bigr)\bigr]\,d\mathbf{x}\,dt=0, \tag{2.12}$$

where $\chi_{\omega^+(t)}$ is the characteristic function of the set $\{u>\mu\}$. If such a problem has a solution, then, gathering (2.11) with (2.12), we conclude that $u(\mathbf{x},t)$ is a solution of the free boundary problem (2.1).

Let us formulate the problem in the plane of Lagrangian coordinates. Formally passing in (2.12) to Lagrangian coordinates we find that

$$\int\limits_{Q_T}\bigl[u_0\bigl((\mathbf{J}^{-1})^2\cdot\nabla\Psi\bigr)\cdot\nabla P-\Psi\bigl(f(\mathbf{y}+\nabla U)-a\chi_{\omega_0^+}\bigr)|\mathbf{J}|\bigr]\,d\mathbf{x}\,dt=0.$$

The boundary conditions for P follow from the trajectory equation as $(\mathbf{y},t)\in S_T$. Integrating by parts in $\omega(0)$ we obtain the equation for defining P:

$$\operatorname{div}\bigl(u_0\bigl(\mathbf{J}^{-1}\bigr)^2\cdot\nabla P\bigr)=\bigl(a\chi_{\omega_0^+}-f(\mathbf{X},t)\bigr)|\mathbf{J}|\quad\text{in } Q_T, \tag{2.13}$$

Gathering now all the above conditions arrive at the following problem: to find the scalar functions U, R, P satisfying the system of equations

$$\begin{cases}\operatorname{div}\bigl(\mathbf{J}\cdot\nabla U_t+R^{-1}\bigl|\mathbf{J}^{-1}\cdot\nabla_{\mathbf{y}}R\bigr|^{p-2}\nabla_{\mathbf{y}}R-\nabla_{\mathbf{y}}P\bigr)=0,\\ \operatorname{div}\bigl(u_0\bigl(\mathbf{J}^{-1}\bigr)^2\nabla P\bigr)=\bigl(-f(\mathbf{y}+\nabla U,t)+a\chi_{\omega_0^+}\bigr)|\mathbf{J}|,\\ R|\mathbf{J}|=u_0\quad\text{in } Q_T^\pm,\end{cases}$$

and the initial and boundary conditions

$$\begin{cases} U=0 \quad \text{on } S_T^{\pm}, \qquad U(\mathbf{y},t)=0 \quad \text{in } \omega(0), \\ \left[R^{-1}\left|\mathbf{J}^{-1}\cdot\nabla_{\mathbf{y}}R\right|^{p-2}\nabla_{\mathbf{y}}R-\nabla_{\mathbf{y}}P\right]\cdot\mathbf{n}=0 \quad \text{on } \gamma\times[0,T], \\ |\mathbf{J}|=1 \quad \text{on } \gamma\times[0,T], \qquad P=0 \quad \text{on } S_T. \end{cases}$$

Recall that according to Corollary 2.2 the trajectory equation is fulfilled on the surface $\Gamma_\mu(t)$, which leads to the condition of zero jump of the normal velocity across $\Gamma_\mu(t)$. The boundary condition for P on $\partial\omega_0^{\pm}$ is automatically fulfilled because of the trajectory equation.

Excluding from this system the function R, we arrive at the following problem: to find the functions $U(\mathbf{y},t)$ (the potential) and $P(\mathbf{y},t)=p(\mathbf{x},t)$ (the artificial pressure) which satisfy the system of two scalar nonlinear equations

$$\begin{cases} \operatorname{div}(\mathbf{J}\nabla U_t+\mathbf{V}-\nabla P)=0, \\ \operatorname{div}\left(\rho_0\left(\mathbf{J}^{-1}\right)^2\nabla P\right)=a\left(f(\mathbf{y}+\nabla U,t)-\chi_{\omega_0^+}\right)|\mathbf{J}| \quad \text{in } Q_T^{\pm} \end{cases} \tag{2.14}$$

with

$$\mathbf{V}=u_0^{-1}|\mathbf{J}|\left|\mathbf{J}^{-1}\nabla\left(u_0\left|\mathbf{J}^{-1}\right|\right)\right|^{p-2}\nabla\left(\mathbf{J}^{-1}\nabla\left(u_0\left|\mathbf{J}^{-1}\right|\right)\right),$$

and the initial and boundary conditions

$$\begin{cases} U=0 \quad \text{on } S_T^{\pm}, \qquad P=0 \quad \text{on } S_T, \\ \left[(\mathbf{V}-\nabla P)\cdot\mathbf{n}\right]_{\gamma\times[0,T]}=0, \\ |\mathbf{J}|=1 \quad \text{on } \gamma\times[0,T]. \end{cases} \tag{2.15}$$

3. Auxiliary nonlinear problem

In this section we consider the auxiliary problem of finding a function U under the assumption that the second unknown, P, is given. This problem splits into two similar problems posed on the cylinders Q_T^+ and Q_T^-. We limit ourselves by considering the problem in Q_T^+, the problem in Q_T^- is studied in the same way.

3.1. Formulation of the problem

Let us fix a function P and consider the auxiliary problem of defining the function U from the conditions

$$\begin{cases} \mathcal{H}_1(U)\equiv\operatorname{div}[\mathbf{J}\nabla U_t+\mathbf{V}-\nabla P]=0 \quad \text{in } Q_T^+, \\ \mathcal{H}_2(U)\equiv\operatorname{Det}[\mathbf{J}]-1=0 \quad \text{on } \gamma\times[0,T], \\ U=0 \quad \text{on } S_T^+, \qquad U(\mathbf{y},0)=0 \quad \text{in } \omega_0^+ \end{cases} \tag{3.1}$$

with

$$\mathbf{V}=u_0^{-1}|\mathbf{J}|\left|\mathbf{J}^{-1}\nabla\left(u_0\left|\mathbf{J}^{-1}\right|\right)\right|^{p-2}\nabla\left(\mathbf{J}^{-1}\nabla\left(u_0\left|\mathbf{J}^{-1}\right|\right)\right).$$

This problem can be formulated as the functional equation

$$\mathcal{H}(U)\equiv\left\{\mathcal{H}_1(U),\mathcal{H}_2(U)\right\}=\mathbf{0}.$$

The existence of a unique solution of problem (3.1) will be proved by means of an abstract version of the modified Newton method [9, Chapter XVIII].

Theorem 3.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and assume that the following conditions hold*:

(1) *the operator $\mathcal{H}:\mathcal{X}\mapsto\mathcal{Y}$ admits a strong* (*Frechét*) *differential $\mathcal{H}'(\cdot)$ in a ball $B_r(0)\subset\mathcal{X}$ of radius $r>0$,*
(2) *the operator $\mathcal{H}'(V):\mathcal{X}\mapsto\mathcal{Y}$ is Lipschitz-continuous in $B_r(0)$,*

$$\left\|\mathcal{H}'(U_1)-\mathcal{H}'(U_2)\right\|\leqslant L\|U_1-U_2\|, \quad L=\text{const},$$

(3) *there exists the inverse operator* $[\mathcal{H}'(0)]^{-1}$ *and*

$$\big\|\big[\mathcal{H}'(0)\big]^{-1}\big\| = M, \qquad \big\|\big[\mathcal{H}'(0)\big]^{-1}\big\langle\mathcal{H}(0)\big\rangle\big\| = \Lambda.$$

Then, if $\lambda = M\Lambda L < 1/4$*, the equation* $\mathcal{H}(U) = 0$ *has a unique solution* U^* *in the ball* $B_{\Lambda t_0}(0)$ *where* t_0 *is the least root of the equation* $\lambda t^2 - t + 1 = 0$*. Moreover, the solution* U^* *is obtained as the limit of the sequence*

$$U_{n+1} = U_n - \big[\mathcal{H}'(0)\big]^{-1}\big\langle\mathcal{H}(U_n)\big\rangle, \quad U_0 = 0. \tag{3.2}$$

It is known that if an operator is weakly differentiable (in the sense of Gateaux), and its Gateaux differential is Lipschitz-continuous, then the operator is strongly differentiable and its weak and strong differentials coincide [9, Chapter XVIII]. Due to condition (2) of Theorem 3.1, we may take for $\mathcal{H}(0)$ the weak differential of $\mathcal{H}$ at the initial state $U_0 = 0$, which is easy to obtain by means of formal linearization. The proof of existence of a solution to the nonlinear problem (3.1) reduces then to the detailed study of the linear problem $\mathcal{H}'(0)\langle U\rangle = (F, \Phi)$.

Let us fix $q > n + 2$ and introduce the Banach spaces

$$\mathcal{Z}^+ = \big\{U\colon\ U \in W^4_q\big(Q^+_T\big),\ U_t \in W^2_q\big(Q^+_T\big),\ U = 0 \text{ on } S^+_T,\ U(\mathbf{y}, 0) = 0 \text{ in } \omega^+_0\big\},$$
$$\mathcal{Y}^+ = \big\{f\colon\ f \in W^2_q\big(Q^+_T\big)\big\},$$
$$\mathcal{X}^+ = \big\{\phi\colon\ \phi \in W^{2,1}_q\big(Q^+_T\big),\ \phi(\mathbf{y}, 0) = 0 \text{ in } \omega^+_0\big\}$$

with the norms

$$\|U\|_{\mathcal{Z}^+} = \|U\|^{(4)}_{q,Q^+_T} + \|U_t\|^{(2)}_{q,Q^+_T}, \qquad \|f\|_{\mathcal{Y}^+} = \|f\|^{(2)}_{q,Q^+_T}, \qquad \|\phi\|_{\mathcal{X}^+} = \|\phi\|_{W^{2,1}_q(Q^+_T)}.$$

3.2. The linear problem

To calculate the Gateaux derivative of $\mathcal{H}$ we use its definition as

$$\mathcal{H}'_i(0)\langle U\rangle = \frac{d\mathcal{H}_i(\epsilon U)}{d\epsilon}\bigg|_{\epsilon=0},$$

where ϵ is a small parameter and

$$\mathcal{H}_1(\epsilon U) = \operatorname{div}\big(\epsilon\big(\mathbf{I} + \epsilon\mathbf{H}(U)\big)\nabla U_t + u_0^{-1}|\mathbf{J}|\big|\mathbf{J}^{-1}\nabla\big(u_0\big|\mathbf{J}^{-1}\big|\big)\big|^{p-2}\nabla\big(\mathbf{J}^{-1}\nabla\big(u_0\big|\mathbf{J}^{-1}\big|\big)\big) - \nabla P\big),$$
$$\mathcal{H}_2(\epsilon U) = \operatorname{Det}\big[\mathbf{I} + \epsilon\mathbf{H}(U)\big] - 1.$$

Obviously,

$$\frac{d}{d\epsilon}\operatorname{div}\big[\epsilon\big(\mathbf{I} + \epsilon\mathbf{H}(U)\big)\nabla U_t\big]\bigg|_{\epsilon=0} = \Delta U_t.$$

For every matrix $\mathbf{B}$ and $\mu = \text{const}$ Newton's formulas hold

$$\operatorname{Det}[\mu\mathbf{I} - \mathbf{B}] = \sum_{k=0}^{n}(-1)^k\alpha_k\mu^{n-k},$$

where $\alpha_0 = 1$, $k\alpha_k = \sum_{i=1}^{k}\alpha_{k-i}\operatorname{trace}(\mathbf{B}^i)$ for $1 \leqslant k \leqslant n$. It follows that $\mathcal{H}'_2(0)\langle U\rangle = \Delta U$. Next,

$$\frac{d}{d\epsilon}\big(\operatorname{Det}\big[\mathbf{I} + \epsilon\mathbf{H}(U)\big]\big)\bigg|_{\epsilon=0} = \big(1 + \epsilon\Delta U + \mathcal{O}\big(\epsilon^2\big)\big)\big|_{\epsilon=0} = \Delta U,$$
$$\begin{aligned}\frac{d}{d\epsilon}\big(\mathbf{I} + \epsilon\mathbf{H}(U)\big)^{-1}\nabla\big(u_0\operatorname{Det}\big[\big(\mathbf{I} + \epsilon\mathbf{H}(U)\big)^{-1}\big]\big)\bigg|_{\epsilon=0} &= \frac{d}{d\epsilon}\big(\nabla u_0 - \epsilon\big(\mathbf{H}(U)\nabla u_0 + \nabla(u_0\Delta U)\big) + \mathcal{O}\big(\epsilon^2\big)\big)\bigg|_{\epsilon=0}\\ &= -\mathbf{H}(U)\nabla u_0 - \nabla(u_0\Delta U)\end{aligned}$$

and

$$\frac{d}{d\epsilon}\left|\left(\mathbf{I}+\epsilon\mathbf{H}(U)\right)^{-1}\nabla\left(u_0\,\mathrm{Det}\left[\left(\mathbf{I}+\epsilon\mathbf{H}(U)\right)^{-1}\right]\right)\right|^{p-2}\Bigg|_{\epsilon=0}$$
$$= -(p-2)|\nabla u_0|^{p-4}\left(\nabla u_0\cdot\left(\mathbf{H}(U)\cdot\nabla u_0+\nabla(u_0\Delta U)\right)\right).$$

Gathering these formulas, we find that

$$\begin{aligned}
\mathcal{H}_2'(0)\langle U\rangle &= \Delta U,\\
\mathcal{H}_1'(0)\langle U\rangle &= \Delta U_t - \operatorname{div}\Bigg[\frac{\Delta U}{u_0}|\nabla u_0|^{p-2}\nabla u_0 + (p-2)\frac{1}{u_0}|\nabla u_0|^{p-4}\left(\nabla u_0\cdot\left(\mathbf{H}(U)\cdot\nabla u_0+\nabla(u_0\Delta U)\right)\right)\cdot\nabla u_0\\
&\qquad + \frac{1}{u_0}|\nabla u_0|^{p-2}\left(\mathbf{H}(U)\nabla u_0+\nabla(u_0\Delta U)\right)\Bigg]\\
&\equiv \Delta U_t - \mathcal{L}\left(\Delta U,\mathbf{H}(U)\right).
\end{aligned}$$

The linear problem $\mathcal{H}'(0)\langle U\rangle = (\Delta f,\phi)$ now reads as follows: it is requested to find a function $U\in\mathcal{Z}^+$ such that

$$\begin{cases}
(\Delta U)_t - \mathcal{L}\left(\Delta U,\mathbf{H}(U)\right) = \Delta f \quad \text{in } Q_T^+,\\
(\Delta\, U-\psi)\big|_{\gamma\times[0,T]} = 0,\\
U(\mathbf{x},0)=0 \quad \text{in } \omega_0^+, \qquad U=0 \quad \text{on } \partial\Omega_e\times[0,T].
\end{cases} \tag{3.3}$$

The existence of a solution is proved by means of the Schauder Fixed Point Principle. Let us introduce the Banach space

$$\mathcal{S}^+ = \left\{V\colon\ V\in W_q^3(Q_T^+),\ V=0 \text{ on } S_T^+,\ V(\mathbf{y},0)=0 \text{ in } \omega_0^+\right\}, \qquad \|V\|_{\mathcal{S}^+} = \|V\|^{(3)}_{q,Q_T^+},$$

and consider the problem

$$\begin{cases}
(\Delta U)_t - \mathcal{L}\left(\Delta U,\mathbf{H}(V)\right) = \tau\Delta f \quad \text{in } Q_T^+,\\
(\Delta U-\tau\psi)\big|_{\gamma\times[0,T]} = 0,\\
U(\mathbf{x},0)=0 \quad \text{in } \omega_0^+, \qquad U=0 \quad \text{on } S_T^+,\ \tau\in[0,1],
\end{cases} \tag{3.4}$$

with an arbitrary function $V\in\mathcal{S}^+$. Let us denote

$$\mathcal{S}_R^+ = \left\{V\in\mathcal{S}^+\colon\ \|V\|_{\mathcal{S}^+}<R\right\}.$$

The solution of problem (3.4) is considered as the solution of the functional equation

$$U=\Phi(V,\tau), \qquad \Phi:\mathcal{S}_R^+\times[0,1]\mapsto\mathcal{S}^+.$$

If the operator Φ has a fixed point for $\tau=1$, this point is a solution of problem (3.3). To prove the existence of a fixed point of the operator $\Phi(\cdot,\tau)$ it suffices to check that (see [11, Chapter 4, Section 10])

(a) the mapping $\Phi(\cdot,\tau):\mathcal{S}_R^+\mapsto\mathcal{S}_R^+$ is continuous and compact,
(b) for every $\tau\in[0,1]$ the fixed points of the operator $U=\Phi(V,\tau)$ satisfy the estimate $\|U\|_{\mathcal{S}^+}\leqslant R'$ for some $R'>0$.

Notice that since problem (3.4) is linear with respect to U, V and τ, so is Φ, which yields $U=\tau\Phi(V)$.

Lemma 3.1. *For every $\tau\in[0,1]$, $V\in\mathcal{S}^+$ and every $f\in\mathcal{Y}^+$, $\psi\in\mathcal{X}^+$ problem* (3.4) *has a solution $U\in\mathcal{Z}^+$ such that*

$$\|U\|_{\mathcal{Z}^+}\leqslant C\left(\tau\|\psi\|_{\mathcal{X}^+}+\tau\|f\|_{\mathcal{Y}^+}+\|V\|_{\mathcal{S}^+}\right) \tag{3.5}$$

with an absolute constant C depending on n, q, the properties of $\partial\omega_0^+$ and γ, $\inf u_0$ and $\sup|\nabla u_0|$, but independent of ψ, f and U.

Proof. Set $W = \Delta U$ and choose W according to the conditions

$$\begin{cases} W_t - \mathcal{L}\big(W, \mathbf{H}(V)\big) = \tau \Delta f \in L^q\big(Q_T^+\big), \\ W - \tau\psi = 0 \quad \text{on } \gamma \times [0, T], \\ W = 0 \quad \text{on } S_T^+, \qquad W(\mathbf{y}, 0) = 0 \quad \text{in } \omega_0^+. \end{cases} \tag{3.6}$$

The equation for W has the form

$$W_t - \operatorname{div}\big(a(\mathbf{y}, \nabla W) + \mathbf{b}(\mathbf{y})W\big) = h \tag{3.7}$$

with

$$\mathbf{a}(\mathbf{y}, \xi) = |\nabla u_0|^{p-2}\big[\xi + (p-2)(\nu \cdot \xi) \cdot \nu\big], \qquad \nu = \frac{\nabla u_0}{|\nabla u_0|},$$

$$\mathbf{b}(\mathbf{y}) = \frac{p-1}{u_0}|\nabla u_0|^{p-2}\nabla u_0,$$

$$h = \tau\Delta f + \operatorname{div}\left[\frac{1}{u_0}|\nabla u_0|^{p-2}\Big((p-2)\big(\nu \cdot \big(\mathbf{H}(V) \cdot \nu\big)\big) \cdot \nabla u_0 + \big(\mathbf{H}(V) \cdot \nabla u_0\big)\Big)\right].$$

Eq. (3.7) is linear with respect to W and V. Under conditions (1.5) on u_0, for every $p > 1$ and $\xi \in \mathbb{R}^n$, $|\xi| \neq 0$,

$$\mathbf{a}(\mathbf{y}, \xi) \cdot \xi = |\nabla u_0|^{p-2}\big[|\xi|^2 + (p-2)|\nu \cdot \xi|^2\big] \geqslant |u_0|^{p-2}|\xi|^2 \begin{cases} 1 & \text{if } p \geqslant 2, \\ p-1 & \text{if } p \in (1, 2). \end{cases}$$

The function $\psi \in W_q^{2,1}(Q_T^+)$ with $q > n+2$ is Hölder-continuous in $\overline{Q}_T^+$ and satisfies the zero-order compatibility condition on the hypersurface γ as $t = 0$. For every $f \in W_q^2(Q_T^+)$ and $V \in W_q^3(Q_T)$ we have $h \in L^q(Q_T^+)$. By the assumption $\partial\omega_0^+$, $\gamma \in C^2$. It follows from the classical parabolic theory (see, e.g., [10, Chapter 4, Section 9]), that for every $f, \psi \in W_q^2(Q_T^+)$ and $V \in W_q^3(Q_T^+)$ problem (3.6) has a unique solution $W \in W_q^{2,1}(Q_T^+)$ which satisfies the estimate

$$\|\Delta U\|_{W_q^{2,1}(Q_T^+)} = \|W\|_{W_q^{2,1}(Q_T^+)} \leqslant C\Big(\|\psi\|_{W_q^{2,1}(Q_T^+)} + \|f\|_{q,Q_T^+}^{(2)} + \|V\|_{q,Q_T^+}^{(3)}\Big). \tag{3.8}$$

The function U is now defined as the solution of the Dirichlet problem for the Poisson equation with the right-hand side $\Delta U \in W_q^2(Q_T^+)$. □

Lemma 3.2. *The operator $\tau\Phi(V) : \mathcal{S}_R^+ \times [0, 1] \mapsto \mathcal{S}^+$ is continuous and compact.*

Proof. Continuity of Φ follows from the linearity of problem (3.4) with respect to U, V, τ, and estimate (3.5). Compactness of the mapping follows from [13]. □

Lemma 3.3. *There exists T^*, depending on the constant C in the conditions of Lemma* 3.1 *and $|\omega_0^+|$, such that for every $\tau \in [0, 1]$ the fixed points of the mapping $U = \tau\Phi(V)$ satisfy the estimate*

$$\|U\|_{\mathcal{S}^+} \leqslant \|U\|_{\mathcal{Z}^+} \leqslant 2C\big(\|f\|_{\mathcal{Y}^+} + \|\psi\|_{\mathcal{X}^+}\big) \equiv R' \tag{3.9}$$

on the time interval $[0, T^*]$.

Proof. Let $U \in \mathcal{S}^+$ be a fixed point of the mapping $U = \tau\Phi(V)$. Applying Hölder's inequality we have that for every $s > q$ and every $v \in W_q^3(Q_T^+)$

$$\|v\|_{q,Q_T^+}^{(3)} \leqslant \big(T|\omega_0^+|\big)^{\frac{1}{s'}}\|v\|_{s,Q_T^+}^{(3)}, \qquad \frac{1}{s'} = \frac{1}{q} - \frac{1}{s}.$$

Since $s > q > n+2$, it follows from Sobolev's embedding theorem that

$$\|U\|_{q,Q_T^+}^{(3)} = \|U\|_{\mathcal{S}^+} \leqslant \big(T|\omega_0^+|\big)^{\frac{1}{s'}}\|U\|_{\mathcal{Z}^+}.$$

The assertion follows now if we substitute this inequality into (3.5) and claim that T is appropriately small: $T < T^*$ with

$$2\left(T^*\left|\omega_0^+\right|\right)^{\frac{1}{s'}} = C$$

and the constant C from (3.5). □

Gathering the above lemmas and applying the Schauder Fixed Point Principle, we conclude that for $\tau = 1$ problem (3.4) has a fixed point $U \in \mathcal{S}^+_{R'}$ with R' given in (3.9). We summarize these conclusions in the following theorem.

Theorem 3.2. *There exists T^*, depending on $|\omega_0^+|$, n, q, $\inf_\Omega u_0$ and $\sup_\Omega |\nabla u_0|$ such that for every $f \in \mathcal{Y}^+$, $\psi \in \mathcal{X}^+$ problem* (3.3) *has at least one solution $U \in \mathcal{Z}^+$ satisfying the estimate*

$$\|U\|_{\mathcal{Z}^+} \leqslant 2C\left(\|f\|_{\mathcal{Y}^+} + \|\phi\|_{\mathcal{X}^+}\right) \tag{3.10}$$

with the constant C from (3.5).

Corollary 3.1. *$M = \|\mathcal{H}^{-1}(0)\| \leqslant 2C$ with the constant C from* (3.10).

Corollary 3.2.

$$\Lambda = \left\|\mathcal{H}^{-1}(0)\langle\mathcal{H}(0)\rangle\right\| \leqslant 2C\left(T^{1/q}\left(\left\|\frac{\Delta_p u_0}{u_0}\right\|_{L^q(\omega_0^+)} + \left\|\frac{|\nabla u_0|^p}{u_0^2}\right\|_{L^q(\omega_0^+)}\right) + \|P\|_{\mathcal{Y}^+}\right)$$

with the constant C from (3.10).

Proof. The estimate follows from (3.10) with

$$\Delta f = \mathcal{H}_1(0) = \operatorname{div}\left(\frac{1}{u_0}|\nabla u_0|^{p-2}\nabla u_0 - \nabla P\right), \quad \psi = 0. \qquad \square$$

3.3. The nonlinear problem

To apply Theorem 3.1 we have to check Lipschitz continuity of the Gateaux derivative of the operator $\mathcal{H}$ defined by

$$\mathcal{H}'(V)\langle U\rangle = \frac{d}{d\epsilon}\mathcal{H}(V+\epsilon U)\bigg|_{\epsilon=0} \in \mathcal{Y}^+ \times \mathcal{X}^+, \quad U, V \in \mathcal{Z}^+,$$

and to ensure the fulfillment of the relations

$$\forall U \in \mathcal{Z}^+, \quad \mathcal{H}_1(U) \in \mathcal{Y}^+, \qquad \mathcal{H}_2(U) \in \mathcal{X}^+. \tag{3.11}$$

Given the functions $U, V \in \mathcal{Z}^+$, we define the matrices

$$\mathbf{B} = \mathbf{I} + \mathbf{H}(V) + \epsilon\mathbf{H}(U) \equiv \left(\mathbf{I} + \mathbf{H}(V)\right)(\mathbf{I} + \epsilon\mathbf{A}), \quad \mathbf{A} = \left(\mathbf{I} + \mathbf{H}(V)\right)^{-1}\mathbf{H}(U),$$

and $\mathbf{B}_0 = \mathbf{B}|_{\epsilon=0}$. By the definition

$$\mathcal{H}_1(V+\epsilon U) = \operatorname{div}\left(\mathbf{B}\nabla(V_t + \epsilon U_t) + u_0^{-1}|\mathbf{B}|\left|\mathbf{B}^{-1}\nabla\left(u_0\left|\mathbf{B}^{-1}\right|\right)\right|^{p-2}\nabla\left(\mathbf{B}^{-1}\nabla\left(u_0\left|\mathbf{B}^{-1}\right|\right)\right) - \nabla P\right),$$
$$\mathcal{H}_2(V+\epsilon U) = |\mathbf{B}| - 1.$$

Using the easily verified formula

$$\mathbf{B} = \mathbf{I} + \mathbf{H}(V) + \epsilon\mathbf{H}(U) = \left(\mathbf{I} + \epsilon\left(\mathbf{I} + \mathbf{H}(V)\right)^{-1}\mathbf{H}(U)\right)\left(\mathbf{I} + \mathbf{H}(V)\right)$$
$$= \left(\mathbf{I} + \mathbf{H}(V)\right)\left(\mathbf{I} + \epsilon A + \mathcal{O}\left(\epsilon^2\right)\right)$$

we find that

$$|\mathbf{B}| = 1 + \epsilon \operatorname{trace}\mathbf{A} + \mathcal{O}\left(\epsilon^2\right), \qquad \left|\mathbf{B}^{-1}\right| = 1 - \epsilon\operatorname{trace}\mathbf{A} + \mathcal{O}\left(\epsilon^2\right).$$

Then

$$\begin{aligned}\mathcal{H}_1'(V)\langle U\rangle = \operatorname{div}\big\{&\mathbf{H}(U)\nabla V_t + \big(\mathbf{I}+\mathbf{H}(V)\big)\nabla U_t - u_0^{-1}\operatorname{trace}\mathbf{A}\big|\mathbf{B}_0^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)\big|^{p-2}\mathbf{B}_0^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)\\ &-(p-2)u_0^{-1}|\mathbf{B}_0|\big|\mathbf{B}_0^{-1}\cdot\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)\big|^{p-4}\\ &\times\big(\mathbf{H}(U)^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)\mathbf{B}_0^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\operatorname{trace}\mathbf{A}\big)\big)\cdot\big(\mathbf{B}_0^{-1}\cdot\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)\big)\\ &-u_0^{-1}|\mathbf{B}_0|\big||\mathbf{B}_0|\big|B_0^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)\big|^{p-2}\\ &\times\big(\mathbf{H}(U)^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\big)+\mathbf{B}_0^{-1}\nabla\big(u_0\big|\mathbf{B}_0^{-1}\big|\operatorname{trace}\mathbf{A}\big)\big)\big\},\\ \mathcal{H}_2'(V)\langle U\rangle = \operatorname{trace}&\,\mathbf{A}.\end{aligned}$$

The elements of the inverse matrix can be expressed through the algebraic adjoints and the determinant, the determinants are polynomials of n-th power. Further, the embedding theorems yield that for $q > n+2$

$$\forall U\in\mathcal{Z}^+,\qquad \sum_{|\gamma|=2,3}\big\langle D_{\mathbf{y}}^{\gamma}U\big\rangle_{Q_T^+}^{(\alpha)} \leqslant C\|U\|_{\mathcal{Z}^+} \tag{3.12}$$

with some $\alpha\in(0,1)$ (see, e.g., [10, Chapter 2, Lemma 3.3]), whence, since $U(\mathbf{y},0)\equiv 0$,

$$\sum_{|\gamma|=2,3}\sup_{Q_T^+}\big|D_{\mathbf{y}}^{\gamma}U\big| \leqslant CT^{\alpha/2}\|U\|_{\mathcal{Z}^+}. \tag{3.13}$$

The last estimate implies the inequality

$$\big|\mathbf{I}+\mathbf{H}(U)\big| \geqslant 1 - C(n)T^{\alpha/2}\|U\|_{\mathcal{Z}^+} > \frac{1}{2}, \tag{3.14}$$

provided that $\|U\|_{\mathcal{Z}^+}\leqslant 1$ and T is sufficiently small. It is now straightforward to check that for every $V_1, V_2\in\mathcal{Z}^+$ with $\|V_i\|_{\mathcal{Z}^+}\leqslant 1/2$

$$\big\|\big(\mathcal{H}_i'(V_1)-\mathcal{H}_i'(V_2)\big)\langle U\rangle\big\| \leqslant L\|V_1-V_2\|_{\mathcal{Z}^+}\|U\|_{\mathcal{Z}^+} \tag{3.15}$$

with $L = L(n,\omega_0^+,n,T)\to 0$ as $T\to 0$. Relations (3.11) follow by the same arguments.

The next theorem is an immediate byproduct of Theorem 3.1.

Theorem 3.3. *Let $P\in W_q^2(Q_T^+)$ with $q>n+2$. Then one may choose T_* so small that $\lambda = ML\Lambda < 1/4$ with the constants Λ, M and L from Corollaries* 3.1, 3.2 *and estimate* (3.15)*, and problem* (3.1) *has a unique solution*

$$U\in\mathcal{B}_r(0) = \big\{W:\ \|W\|_{\mathcal{Z}^+} < r\big\},\qquad r = \frac{\Lambda}{2\lambda}\big(1-\sqrt{1-4\lambda}\big) < 2\Lambda. \tag{3.16}$$

The same assertion is true for problem (3.1) *in the cylinder Q_T^-.*

4. Auxiliary linear elliptic problem

In this section we consider the problem of finding a function P satisfying the following conditions: for every $t\in[0,T^*]$

$$\begin{cases}\mathcal{M}P := \operatorname{div}\big(u_0\big(\mathbf{J}^{-1}\big)^2\nabla P\big) = \big(f(\mathbf{y}+\nabla U,t) - a\,\chi_{\omega_0^+}\big)|\mathbf{J}| & \text{in } \omega_0^{\pm},\\ \big[\big(\mu\big(\mathbf{J}^{-1}\big)^2\nabla P-\Psi\big)\cdot\mathbf{n}\big]_{\gamma} = 0,\\ P = 0 \quad \text{on } \partial\omega_0^{\pm},\end{cases} \tag{4.1}$$

where $U\in\mathcal{Z}^+$ is a given function, $\mathbf{J} = \mathbf{I}+\mathbf{H}(U)$ and

$$\Psi = \mu\big(\mathbf{J}^{-1}\big)^2\big|\mathbf{J}^{-1}\nabla\big(u_0\big|\mathbf{J}^{-1}\big|\big)\big|^{p-2}\big(\mathbf{J}^{-1}\nabla\big(u_0\big|\mathbf{J}^{-1}\big|\big)\big)\in W_q^1\big(Q_T^{\pm}\big).$$

Theorem 4.1. *Let $U \in \mathcal{Z}^{\pm}$. Then T^* can be taken so small that for a.e. $t \in (0, T)$ problem* (4.1) *has a solution $P(\cdot, t) \in W_q^2(\omega_0^{\pm})$, and this solution satisfies the estimate*

$$\|P\|_{q,\omega_0^{\pm}}^{(2)} \leqslant C\big[\big(a + \|f\|_{q,\Omega}\big)\big\|\mathrm{Det}[\mathbf{J}]\big\|_{q,\omega_0^{\pm}} + \|\Psi\|_{W_q^1(\omega_0^{\pm})}\big] \tag{4.2}$$

with an absolute constant C.

Proof. Let us take for P in ω_0^+ the solution of the Dirichlet problem for the linear uniformly elliptic equation

$$\begin{cases} \mathcal{M}P^+ = (f-a)|\mathbf{J}| & \text{in } \omega_0^+, \\ P^+ = 0 & \text{on } \gamma \text{ and } \partial\omega_0^+, \end{cases}$$

and then continue P^+ to ω_0^- by the solution of the problem

$$\begin{cases} \mathcal{M}P^- = f|\mathbf{J}| & \text{in } \omega_0^-, \\ P^- = 0 & \text{on } \partial\omega_0^-, \\ \rho_0\big(\mathbf{J}^{-1}\big)^2\nabla P^- \cdot \mathbf{n} = \rho_0\big(\mathbf{J}^{-1}\big)^2\nabla P^+ \cdot \mathbf{n} - [\Psi \cdot \mathbf{n}]_\gamma. \end{cases}$$

These problems have solutions which satisfy the estimates (see, e.g., [11, Chapter 3, Sections 5–6, 15]): for a.e. $t \in (0, T^*)$

$$\big\|P^+(\cdot,t)\big\|_{W_q^2(\omega_0^+)} \leqslant C\big\|\mathrm{Det}[\mathbf{J}]\big\|_{q,\omega_0^+}\big(a + \|f\|_{q,\Omega}\big),$$
$$\big\|P^-(\cdot,t)\big\|_{W_q^2(\omega_0^-)} \leqslant C\big(\big\|\mathrm{Det}[\mathbf{J}]\big\|_{q,\omega_0^-}\|f\|_{q,\Omega} + \|\Psi\|_{W_q^1(\omega_0^-)}\big). \qquad \square$$

Corollary 4.1. *Under the conditions of Theorem* 4.1

$$\|P\|_{\mathcal{Y}^{\pm}} \leqslant C\big(1 + \|f\|_{q,Q_{T^*}^{\pm}}\big)\big(1 + \|U\|_{\mathcal{Z}^{\pm}}\big).$$

Lemma 4.1. *Under the conditions of Theorem* 4.1 *for every $t \in (0, T^*)$*

$$\big|P(\cdot,t) - P_0\big|_{\omega_0^{\pm}}^{(1+\sigma)} \leqslant Ct^{\delta}, \quad 0 < \sigma < 1 - \frac{n}{q},\ \delta = \min\{\beta, \alpha/2\},$$

where $\alpha \in (0,1)$ is the exponent from (3.13)*, $\beta \in (0,1)$ is taken from* (1.5)*, and P_0 is the solution of the problem*

$$\mathrm{div}(u_0\nabla P_0) = f(\mathbf{y}, 0) - a\chi_{\omega_0^+} \quad \text{in } \omega(0), \qquad P_0 = 0 \quad \text{on } \partial\omega_0^{\pm}. \tag{4.3}$$

Proof. Problem (4.3) has a unique solution $P_0 \in W_q^2(\omega(0)) \cap C^{1+\sigma}(\omega(0))$ with $0 < \sigma < 1 - n/q$. Since $u_0 \in W_q^2(\omega(0))$, this solution automatically satisfies the jump condition $[u_0\nabla(P_0 - \ln u_0)]_\gamma = 0$. Problem (4.1) is linear and its solution continuously depends on the data. Let us fix $t \in (0, T^*)$ and consider the function $P - P_0$ which solves the problem

$$\begin{cases} \mathrm{div}\big(u_0\nabla(P - P_0)\big) = F & \text{in } \omega_0^{\pm}, \\ \big[u_0\nabla(P - P_0)\cdot\mathbf{n}\big]_\gamma = \sigma, \qquad P - P_0 = 0 & \text{on } \partial\omega_0^{\pm} \end{cases}$$

with

$$F = \big(f(\mathbf{y} + \nabla_{\mathbf{y}}U, t) - f(\mathbf{y}, 0)\big) + f(\mathbf{y}, t)\big(|\mathbf{J}| - 1\big) + a\big(1 - |\mathbf{J}|\big)\chi_{\omega_0^+} \in L^q\big(\omega_0^+ \cup \omega_0^-\big),$$
$$\sigma = \mu\big[\big(\big(\mathbf{J}^{-1}\big)^2 - \mathbf{I}\big)\cdot\nabla P\big]_\gamma \cdot \mathbf{n} + [\mu\nabla\ln u_0 - \Psi]_\gamma \cdot \mathbf{n} + \big[\big(\mathbf{I} - \big(\mathbf{J}^{-1}\big)^2\big)\cdot\Psi\big]_\gamma \cdot \mathbf{n}.$$

According to (3.13) and the regularity properties of f we have that

$$\|F\|_{q,\omega_0^{\pm}} \leqslant Ct^{\delta}\big(\|1 + \|U\|_{\mathcal{Z}^{\pm}}\big),$$
$$\|\sigma\|_{\omega_0^{\pm}}^{(1)} \leqslant Ct^{\delta}\big(\|u_0\|_{W_q^2(\omega(0))} + \|P\|_{W_q^2(\omega_0^{\pm})} + \|U\|_{\mathcal{Z}^{\pm}}\big).$$

Following the proof of Theorem 4.1, we now find that

$$\|P(\cdot,t)-P_0\|_{W^2_q(\omega_0^\pm)}\leqslant Ct^\delta$$

and the assertion follows from the embedding theorems in Sobolev spaces. □

Corollary 4.2. *By virtue of* (3.12)*, it follows by the same arguments that for every* $t_1, t_2 \in [0, T^*]$

$$\big|P(\cdot,t_1)-P(\cdot,t_2)\big|^{(1+\sigma)}_{\omega_0^\pm}\leqslant C\big\|P(\cdot,t_1)-P(\cdot,t_2)\big\|_{W^2_q(\omega_0^\pm)}\leqslant C'|t_2-t_1|^\delta.$$

5. Existence of solutions to the problems in Lagrangian and Euler formulations

Theorem 5.1. *There exists* T^* *such that for every* $T\in(0,T^*)$ *problem* (PL) *has a solution* $(U,P)\in\mathcal{Z}^\pm\times\mathcal{Y}^\pm$*, which generates a solution of problem* (2.1) *with the velocity field defined by* (2.10).

Proof. Let us consider the sequences

$$\{U_k\}\in\mathcal{Z}^+\cap\mathcal{Z}^-,\qquad\{P_k\}\in W^2_q\big(Q_T^+\big)\cap W^2_q\big(Q_T^-\big)$$

defined iteratively: $U_0=0$, for every $k\geqslant 1$ U_k is a solution of problem (3.1) with $P=P_{k-1}$, P_k is a solution of problem (4.1) with $U=U_k$. By Theorems 3.3 and 4.1 for all sufficiently small T

$$\|P_k\|_{\mathcal{Y}^\pm}+\|U_k\|_{\mathcal{Z}^\pm}\leqslant\lambda$$

with some absolute constant λ. This estimate together with (3.13) mean that the sequences $\{U_k\}$ and $\{P_k\}$ contain subsequences (which we assume to coincide with the whole of these sequences) such that

$$\begin{aligned}&U_k\to U\quad\text{as }k\to\infty\text{ weakly in }\mathcal{Z}^\pm,\\&D^2_{ij}U_k\to D^2_{ij}U\quad\text{as }k\to\infty\text{ in }C^{\alpha',\alpha'/2}\big(\overline{Q}_T^\pm\big),\\&P_k\to P,\quad\text{as }k\to\infty\text{ weakly in }W^2_q\big(Q_T^\pm\big).\end{aligned}\tag{5.1}$$

By the method of construction, each of the pairs (U_k, P_{k-1}) satisfies (2.6) with $\mathbf{v}$ defined by (2.10), which allows us to pass to the limit as $k\to\infty$. It remains to check the bijection of the mapping $\mathbf{y}\mapsto\mathbf{x}$. According to (3.13) the Jacobian $|\mathbf{J}|$ is bounded away from zero and infinity in $Q_T^\pm$ (for small T) and the mapping $\omega_0^\pm\ni\mathbf{y}\mapsto\mathbf{x}\in\omega^\pm(t)$ is locally invertible in a neighborhood of every interior point of $Q_T^\pm$. It is then sufficient to check that T can be chosen so small that the images of two arbitrary boundary points $\mathbf{y},\mathbf{z}\in\partial\omega_0^\pm$ (or $\mathbf{y},\mathbf{z}\in\gamma$), $\mathbf{y}\neq\mathbf{z}$, do not coincide on the interval $[0,T^*]$. The arguments are the same for the three possibilities. For example, fix two arbitrary points $\mathbf{y},\mathbf{z}\in\partial\omega_0^+$ and denote by $\mathbf{X}(\mathbf{y},t)$ and $\mathbf{X}(\mathbf{z},t)$ their images at the instant t. By the definition $\mathbf{X}(\mathbf{s},t)=\mathbf{s}+\nabla U(\mathbf{s},t)$. Further,

$$\begin{aligned}\big|\mathbf{X}(\mathbf{y},t)-\mathbf{X}(\mathbf{z},t)\big|&=\big|\mathbf{y}-\mathbf{z}+\nabla\big(U(\mathbf{y},t)-U(\mathbf{z},t)\big)\big|\\&\geqslant|\mathbf{y}-\mathbf{z}|-\int\limits_{L(\mathbf{y},\mathbf{z})}\left|\frac{d}{dl}\nabla U(\mathbf{s},t)\right|dS,\end{aligned}$$

where $L(\mathbf{y},\mathbf{z})\subset\omega_0$ is a curve connecting $\mathbf{y}$ and $\mathbf{z}$. For every surface $\partial\omega_0^+\in C^{0,1}$ the curve $L(\mathbf{y},\mathbf{z})$ can be chosen Lipschitz-continuous and there exist finite positive constants K_1, K_2 (depending on the geometry of $\partial\omega_0^+$) such that

$$K_1|\mathbf{y}-\mathbf{z}|\leqslant\int\limits_{L(\mathbf{y},\mathbf{z})}dS\leqslant K_2|\mathbf{y}-\mathbf{z}|.$$

By (3.13)

$$\big|\mathbf{X}(\mathbf{y},t)-\mathbf{X}(\mathbf{z},t)\big|\geqslant|\mathbf{y}-\mathbf{z}|-\sum_{i,j=1}^n\sup_{\omega_0^+}\big|D^2_{ij}U\big|\int\limits_{L(\mathbf{y},\mathbf{z})}dS\geqslant|\mathbf{y}-\mathbf{z}|\big(1-CK_2T^{\alpha/2}\big)$$

with the constant C from Theorem 5.1. Since C is defined through the data of problem (PL), it follows that the trajectories of two arbitrary points separated at the initial instant cannot touch if T^* is chosen appropriately small. □

6. Solution of problem (1.4). Proof of Theorems 1.1–1.2

6.1. According to Theorems 2.1 and 5.1 the pair $(u, \mathcal{C})$ defined by formulas (2.7) is a solution of problem (2.1).

Lemma 6.1. *There exists T' such that $|\nabla_{\mathbf{x}} u| \geqslant \epsilon/2$ in $\mathcal{C}' = \mathcal{C} \cap \{t < T'\}$.*

Proof. By the definition $u(\mathbf{x}, t) = u_0(\mathbf{x})|\mathbf{J}^{-1}|$ in $\mathcal{C}$, whence

$$\nabla_{\mathbf{x}} u = \nabla_{\mathbf{y}} u_0(\mathbf{y}) \cdot \mathbf{J}^{-1}\big|\mathbf{J}^{-1}\big| + u_0(\mathbf{y})\nabla_{\mathbf{y}}\big(\big|\mathbf{J}^{-1}\big|\big) \cdot \mathbf{J}^{-1}$$

and

$$|\nabla_{\mathbf{x}} u| \geqslant \epsilon\big(1 - \mathcal{O}\big(t^{-\alpha/2}\big)\big) - \sup_{\Omega} u_0 \mathcal{O}\big(t^{-\alpha/2}\big) \geqslant \epsilon/2 \quad \text{as } t \to 0. \qquad \square$$

Corollary 6.1. *Let $\mathcal{C}^{\pm}$ be the domains bounded by the surfaces Γ_μ and $\Sigma^{\pm}$. Then $u < \mu$ in $\mathcal{C}^- \cap \{t < T'\}$ and $u > \mu$ in $\mathcal{C}^+ \cap \{t < T'\}$.*

The value of T' can be taken so small that $\gamma \times [0, T'] \subset \mathcal{C}$, and there exist smooth vertical surfaces $\sigma^{\pm} \subset \mathcal{C}^{\pm} \cap \{t < T'\}$. Let $S^{\pm}$ be the cylinders with the lateral boundaries $\{\partial\Omega_i \times (0, T'^{+}\}$, and $\{\partial\Omega_e \times (0, T'^{-}\}$. Consider the problem

$$\begin{cases} \mathcal{N}v \equiv v_t - \Delta_p v + f = 0 & \text{in } S^-, \\ v - u = 0 & \text{on } \sigma^-, \\ v - \phi = 0 & \text{on } \partial\Omega_e \times (0, T'], \\ v(\mathbf{x}, 0) - u_0(\mathbf{x}) = 0, \end{cases} \tag{6.1}$$

where $u(\mathbf{x}, t)$ is the constructed solution of problem (2.1).

Lemma 6.2. *Let the conditions of Theorem* 1.1 *be fulfilled. Then problem* (6.1) *has a solution $v \in W^{2,1}_q(S^+)$ such that*

$$\|v\|_{W^{2,1}_q(S^-)} \leqslant C\big[\|u_0\|_{W^2_q(\Omega)} + \|f\|_{q,S^-} + \|u\|_{W^{2,1}_q(\mathcal{C}^-)} + \|\phi\|_{W_q^{2-\frac{2}{q},1-\frac{1}{q}}(\partial\Omega_e \times (0,T'])}\big].$$

Moreover, if $f \leqslant 0$ in S, then $v < \mu$ in $\overline{S}$.

Proof. The proof is an imitation of the proof of Theorem 3.2. We make use of the modified Newton's method and search a solution of problem (6.1) as the limit of the sequence $\{v_k\}$ with

$$v_{k+1} = v_k - \big(\mathcal{N}'(0)\big)^{-1}\langle \mathcal{N}v_k\rangle, \quad v_0 = u_0.$$

Here $\mathcal{N}'(0)$ is the Frechét differential of $\mathcal{N}$ at the initial function u_0. For every $w \in W^{2,1}_q(S)$ with $q > n + 2$ we have (see (3.13))

$$w_t, \Delta_p w \in L^q(S^-), \quad w \in W_q^{2-\frac{2}{q},1-\frac{1}{q}}(\sigma^-), \quad w \in W_q^{2-\frac{2}{q},1-\frac{1}{q}}\big(\partial\Omega_e \times (0, T']\big).$$

According to Theorem 3.1, the proof of existence of a solution to problem (6.1) reduces to and checking the Lipschitz-continuity of the operator $\mathcal{N}$ linearized at an arbitrary element of $W^{2,1}_q(S^-)$ and solving the problem linearized at the initial function u_0. The latter has the form

$$\begin{cases} w_t - \operatorname{div}\big(|\nabla u_0|^{p-2}\big[\nabla w + (p-2)(\nabla w \cdot \nu) \cdot \nu\big]\big) = \psi \in L^q(S^-), & \nu = \dfrac{\nabla u_0}{|\nabla u_0|}, \\ w(\mathbf{x}, 0) = 0, \qquad w = g \quad \text{on } \sigma^-, \qquad w = h \quad \text{on } \partial\Omega_e \times (0, T']. \end{cases}$$

Since $p > 1$ and $|\nabla u_0| \geqslant \epsilon > 0$ this is a uniformly parabolic equation with the data satisfying the zero-order compatibility conditions. For every $\psi \in L^q(S^-)$, $g \in W_q^{2-2/q,1-1/q}(\sigma^-)$, $h \in W_q^{2-2q,1-1/q}(\partial\Omega_e \times (0, T'])$ with $q > n + 2$ the linearized problem has a unique solution $w \in W^{2,1}_q(S^-)$ which satisfies the estimate

$$\|w\|_{W_q^{2,1}(S^-)} \leqslant C\big[\|u_0\|_{W_q^2(\Omega)} + \|\psi\|_{q,S^-} + \|g\|_{W_q^{2-\frac{2}{q},1-\frac{1}{q}}(\sigma^-)} + \|h\|_{W_q^{2-\frac{2}{q},1-\frac{1}{q}}(\partial\Omega_e\times(0,T'])}\big].$$

This gives the estimate on $\|(\mathcal{N}'(0))^{-1}\|$. Solving the linearized problem with $\psi = \mathcal{N}u_0$, $g = u_0$ and $h = u_0$, we obtain the estimate

$$\big\|\big(\mathcal{N}'(0)\big)^{-1}\big\langle\mathcal{N}(u_0)\big\rangle\big\| \leqslant CT^{1/q}\|u_0\|_{W_q^2(\Omega)}.$$

Checking the Lipschitz-continuity of the operator $\mathcal{N}'(w)$ with $w \in W_q^{2,1}(S^-)$, $q > n+2$, is straightforward (see the proof of Theorem 3.2).

By Corollary 6.1 $u < \mu$ in $\mathcal{C}^- \cap \{t < T'\}$. The inequality $u < \mu$ in $\overline{S}^-$ follows then the maximum principle because $v_t - \Delta_p v \leqslant 0$ in S^-, and $v < \mu$ on the parabolic boundary of S^-. □

The continuation from $\mathcal{C}^+ \cap \{t < T'\}$ to the rest of $D^+_{T'}$ is performed in the same way. We solve the problem

$$\begin{cases} w_t - \Delta_p w = -f + a & \text{in } S^+,\\ w - u = 0 & \text{on } \sigma^+,\\ w - \phi = 0 & \text{on } \partial\Omega_i \times (0, T'],\\ w(\mathbf{x}, 0) - u_0(\mathbf{x}) = 0. \end{cases}$$

The only difference between this case and the already considered one is the claim $a - f \geqslant 0$. Since $w > \mu$ on the parabolic boundary of S^+, this claim yields the inequality $u > \mu$ in $\overline{S}^+$.

Let us define the function

$$\widetilde{u}(\mathbf{x}, t) = \begin{cases} u(\mathbf{x}, t) & \text{in } D_{T'} \setminus (S^+ \cup S^-),\\ v(\mathbf{x}, t) & \text{in } S^-,\\ w(\mathbf{x}, t) & \text{in } S^+. \end{cases}$$

By construction, for every smooth test-function $\eta(\mathbf{x}, t)$

$$\int\limits_{D_T} \big[\eta_t\widetilde{u} - \nabla\eta\cdot|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u} - \eta\, f(x,t) + a\eta\, h_{\widetilde{u}}\big]\,d\mathbf{x}\,dt + \int\limits_{S^\pm} \eta\big[|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u}\big]\cdot\mathbf{n}\,dS = \int\limits_{\Omega} \widetilde{u}\eta\,d\mathbf{x}\Big|_{t=0}^{t=T}.$$

Since $\widetilde{u} \in W_q^{2,1}(S^+) \cap W_q^{2,1}(S^-) \cap W_q^{2,1}(\mathcal{C})$, then $[\nabla\widetilde{u}]_{S^\pm}\cdot\mathbf{n} = 0$ and the assertion of Theorem 1.1 follows.

The conclusion about the regularity of Γ_μ follows from the representation and the inclusions

$$\mathbf{x}|_{\Gamma_\mu} = (\mathbf{y} + \nabla U)|_{\mathbf{y}\in\gamma}, \quad U \in W_q^4\big(Q_T^\pm\big), \quad U_t \in W_q^2\big(Q_T^\pm\big).$$

6.2. Theorem 1.2 is an immediate byproduct of Theorems 1.1 and 5.1 with the function p defined as a solution of the problem

$$\begin{cases} \operatorname{div}(u\nabla p) + a\chi_{\omega^+(t)} = 0 & \text{in } \omega^\pm(t),\\ [\nabla p\cdot\mathbf{n}]|_{\Gamma_\mu(t)} = [\nabla \ln u\cdot\mathbf{n}]|_{\Gamma_\mu(t)},\\ p = 0 & \text{on } \partial\omega^\pm(t). \end{cases} \tag{6.2}$$

By Theorem 1.1, the constructed solution of problem (1.4) $\widetilde{u}(\mathbf{x}, t) \in W_q^{2,1}(D_{T'})$. It follows that $\nabla\widetilde{u} \in C^{\alpha,\alpha/2}(D_{T'})$ whence $[\nabla\widetilde{u}\cdot\mathbf{n}]_{\Gamma_\mu(t)} = 0$, and problems (1.11), (6.2) are equivalent for $\widetilde{u} \in W_q^{2,1}(\mathcal{C})$.

Since p is defined as the solution of problem (1.11), we also have $p \in W_q^2(\omega(t)) \cap C^{1+\sigma}(\omega(t))$ with $\sigma \in (0, 1 - n/q)$. By Corollary 4.2 $\nabla_{\mathbf{x}} p = \nabla_{\mathbf{y}} P \cdot \mathbf{J}^{-1}$ is Hölder continuous with respect to t, whence Hölder continuity of $\mathbf{v}$ in $\overline{\mathcal{C}}$ and relation (1.12) follows.

7. Higher regularity. Proof of Theorem 1.3

Fix an arbitrary $m \in \mathbb{N}$ and define the function spaces

$$\mathcal{Z}_m^{\pm} = \{v\colon\ v^{(k)} \equiv t^k D_t^k v \in \mathcal{Z}^{\pm},\ k = 0, 1, 2, \dots, m\},$$
$$\mathcal{Y}_m^{\pm} = \{f\colon\ f^{(k)} \equiv t^k D_t^k f \in \mathcal{Y}^{\pm},\ k = 0, 1, 2, \dots, m\},$$
$$\mathcal{X}_m^{\pm} = \{\phi\colon\ \phi^{(k)} \equiv t^k D_t^k \phi \in \mathcal{X}^{\pm},\ k = 0, 1, 2, \dots, m\}$$

with the norms

$$\|v\|_{\mathcal{Z}_m^{\pm}} = \sum_{i=0}^{m} \frac{1}{M^i i!} \|v^{(i)}\|_{\mathcal{Z}^{\pm}},$$
$$\|f\|_{\mathcal{Y}_m^{\pm}} = \sum_{i=0}^{m} \frac{1}{M^i i!} \|f^{(i)}\|_{\mathcal{Y}^{\pm}},$$
$$\|\phi\|_{\mathcal{X}_m^{\pm}} = \sum_{i=0}^{m} \frac{1}{M^i i!} \|\phi^{(i)}\|_{\mathcal{X}^{\pm}}.$$

Here M is a constant which depends on n, q and the differential properties of γ and $\partial\omega^{\pm}$ and will be chosen later.

Lemma 7.1. *For every $u, v \in \mathcal{Z}_m^{\pm}$ $\|uv\|_{\mathcal{Z}_m^{\pm}} \leqslant \|u\|_{\mathcal{Z}_m^{\pm}} \|v\|_{\mathcal{Z}_m^{\pm}}$.*

Proof. The assertion immediately follows from the Cauchy permutation formula

$$\begin{aligned}\left(\sum_{i=0}^{m} \frac{|D_t^i v|}{M^i i!}\right) \times \left(\sum_{i=0}^{m} \frac{|D_t^i u|}{M^i i!}\right) &= \sum_{n=0}^{m} \frac{1}{M^n} \sum_{i+j=n} \frac{1}{i!j!} |D_t^i u| |D_t^j v| \\ &\geqslant \sum_{n=0}^{m} \frac{1}{M^n} \left| \sum_{i=0}^{n} \frac{D_t^i u D_t^{n-i} v}{i!(n-i)!} \right| = \sum_{n=0}^{m} \frac{1}{M^n n!} \left| \sum_{i=0}^{n} \binom{i}{n} D_t^i u D_t^{n-i} v \right| \\ &= \sum_{n=0}^{m} \frac{|D_t^n(uv)|}{M^n n!}. \qquad \square\end{aligned}$$

The next assertion is a byproduct of (3.13).

Lemma 7.2. *For every $u \in \mathcal{Z}_m^{\pm}$ with $q > n+2$*

$$\sum_{k=0}^{m} \sum_{|\gamma|=2,3} \sup_{Q_T^+} \left| D_{\mathbf{y}}^{\gamma} \left(t^k D_t^k u\right)\right| \leqslant C T^{\alpha/2} \|u\|_{\tilde{\mathcal{Z}}^{\pm}}.$$

Since $\mathcal{Z}_m^{\pm} \subset \mathcal{Z}^{\pm}$, to prove Theorem 1.3 it suffices to check that the constructed solution $(U, P) \in \mathcal{X}^{\pm} \times \mathcal{Y}^{\pm}$ of problem (2.14)–(2.15) belongs to $\mathcal{Z}_m^{\pm} \times \mathcal{Y}_m^{\pm}$. Let us revise the proofs of Theorems 3.3, 4.1, 5.1.

Lemma 7.3. *Let the conditions of Theorem* 1.3 *be fulfilled and $P \in \mathcal{Y}_m^{\pm}$ with $m \in \mathbb{N}$. Then T can be chosen so small that problem* (3.1) *has a solution $U \in \mathcal{Z}_m^{\pm}$ such that $\|U\|_{\mathcal{Z}_m^{\pm}} \leqslant C(T) \to 0$ as $T \to 0$.*

Following the proof of Theorem 3.3, we study first the linearized problem (3.3). The existence of a solution $U \in \mathcal{Z}^+$ is already proven in Theorem 3.2. Let us assume that for some $s \geqslant 0$ the function

$$W^{(s)} = \frac{t^s}{M^s s!} D_t^s(\Delta U)$$

solves problem (3.6) with the data

$$\Delta f^{(s)} = \frac{1}{M^s s!} t^s D_t^s(\Delta f), \qquad \phi^{(s)} = \frac{1}{M^s s!} t^s D_t^s \phi$$

and satisfies the estimate

$$\|W^{(s)}\|_{\mathcal{Z}^+} \leqslant C\big(\|f^{(s)}\|_{\mathcal{Y}^+} + \|\phi^{(s)}\|_{\mathcal{X}^+}\big).$$

The problem for the function $W^{(s+1)}$ has the form (recall that $p=2$)

$$\begin{cases} W_t^{(s+1)} - \Delta W^{(s+1)} = \dfrac{1}{M} W_t^{(s)} + \tau f^{(s+1)} & \text{in } Q_T^+, \\ W^{(s+1)} - \tau\psi^{(s+1)} = 0 \quad \text{on } \gamma \times [0,T], \\ W^{(s+1)} = 0 \quad \text{on } S_T^+, \qquad W^{(s+1)}(\mathbf{y},0) = 0 & \text{in } \omega_0^+. \end{cases}$$

To pose the initial condition for $W^{(s+1)}$ we use the fact that if $g_t \in L^q(Q_T^+)$, then $tg_t \to 0$ as $t \to 0$. By Lemma 3.1

$$\|W^{(s+1)}\|_{\mathcal{Z}^+} \leqslant \tau C\big(\|f^{(s+1)}\|_{\mathcal{Y}^+} + \|\phi^{(s+1)}\|_{\mathcal{X}^+}\big) + \frac{C}{M}\|W^{(s)}\|_{\mathcal{Z}^+}.$$

Let us claim that $M > 2C$ and then take the sum of these estimates for $s = 0, 1, \ldots, m-1$:

$$\|W\|_{\mathcal{Z}_m^+} = \sum_{s=0}^{m} \|W^{(s)}\|_{\mathcal{Z}^+} \leqslant \tau C\big(\|f\|_{\mathcal{Y}_m^+} + \|\phi\|_{\mathcal{X}_m^+}\big) + \frac{1}{2}\|W\|_{\mathcal{Z}_m^+},$$

whence the estimate

$$\|W\|_{\mathcal{Z}_m^+} \leqslant 2\tau C\big(\|f\|_{\mathcal{Y}_m^+} + \|\phi\|_{\mathcal{X}_m^+}\big).$$

The rest of the proof of Theorem 3.3 does not need any substantial change and consists in applying Lemmas 7.1, 7.2 to estimate the nonlinear terms.

Lemma 7.4. *T^* can be chosen so small that for every $U \in \mathcal{Z}_m^\pm$ and a.e. $t \in (0,T^*)$ problem* (4.1) *has a solution $P(\cdot,t) \in \mathcal{Y}_m^\pm$ which satisfies the estimate*

$$\|P\|_{\mathcal{Y}_m^\pm} \leqslant C\big(1 + \|U\|_{\mathcal{Y}_m^\pm}\big).$$

Derivation of this estimate is similar to the parabolic case. Set $P^{(s)} = \frac{1}{M^s s!} D_t^s P$. This function satisfies the conditions

$$\begin{cases} \operatorname{div}\big(u_0(\mathbf{J}^{-1})^2 \nabla P^{(s)}\big) = -\dfrac{at^s}{M^s s!} D_t^s\big(|\mathbf{J}|\big)\chi_{\omega_0^+} - \operatorname{div}\Phi_s \quad \text{in } \omega_0^\pm, \\ \big[\big(\mu(\mathbf{J}^{-1})^2 \nabla P^{(s)} - \Psi^{(s)}\big) \cdot \mathbf{n}\big]_\gamma = [\Phi_s \cdot \mathbf{n}]_\gamma, \\ P^{(s)} = 0 \quad \text{on } \partial\omega_0^\pm \end{cases} \tag{7.1}$$

with

$$(\Phi_{\mathbf{s}})_r = u_0 \sum_{j=0}^{s-1} \binom{j}{s} \frac{t^{s-j}}{M^{s-j}(s-j)!} \sum_{i=1}^{n} D_t^{s-j}\big((\mathbf{J}^{-1})^2\big)_{ir} D_i P^{(j)} + u_0 \frac{t^s}{M^s s!} \sum_{i=1}^{n} D_t^s\big((\mathbf{J}^{-1})^2\big)_{ir} D_i P, \quad r = 1, \ldots, n.$$

By Lemmas 7.1, 7.2, for every $U \in \mathcal{Z}_m^\pm$

$$\|\operatorname{div}\Phi_s(\cdot,t)\|_{q,\omega_0^\pm} \leqslant 2^s C(t) \sum_{j=0}^{s-1} \|P^{(j)}\|_{W_q^2(\omega_0^\pm)} \quad \text{with } C(t) \to 0 \text{ as } t \to 0.$$

By Theorem 4.1 the solution of problem (7.1) satisfies the estimate

$$\|P^{(s)}\|_{W_q^2(Q_T^\pm)} \leqslant C\big(1 + \|U\|_{\mathcal{Z}_m^\pm} + \|\Psi\|_{W_q^1(Q_T^\pm)}\big) + C(T) \sum_{j=0}^{s-1} \|P^{(j)}\|_{W_q^2(Q_T^\pm)}.$$

Summing these estimates for $s = 1, 2, \dots, m$ and taking T appropriately small we find that

$$\|P\|_{\mathcal{Y}_m^\pm} \leqslant C\big(1 + \|U\|_{\mathcal{Z}_m^\pm}\big).$$

The sequence $\{(U_k,\ P_{k-1})\}$ contains a subsequence that converges (weakly) to a solution $(U,\ P)$ of problem (PL). At the same time, this subsequence is uniformly bounded in the norm of $\mathcal{Z}_m^\pm \times \mathcal{Y}_m^\pm$, which yields the assertion of Theorem 1.3.

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