Construction of the maximal solution of Backus' problem in geodesy and geomagnetism

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ABSTRACT

The (simplified) Backus' Problem (BP) consists in finding a harmonic function u on the domain exterior to the three dimensional unit sphere S, such that u tends to zero at infinity and the norm of the gradient of u takes prescribed values g on S. Except for a change of sign, the solution is not unique in general. However, there is uniqueness of solutions in the class of functions with the additional property that the radial component of the gradient of u on S is nonpositive. This is the geodetically relevant case. If a solution u with this property exists, then u is the maximal solution of the problem (and −u the minimal one). In this paper we propose a method of successive approximations to get this particular solution of BP and prove the convergence for functions g close to a constant function.

Keywords: harmonic functions, fully nonlinear boundary problem, geodesy, geomagnetism

1. INTRODUCTION

Let Ω be the exterior domain to the unit sphere S in $\mathbb{R}^3$. Let $\mathcal{H}(\Omega)$ be the real space of functions which are harmonic in Ω and tend to zero at infinity. We use the notation $\mathcal{H}^k(\bar{\Omega}) = \mathcal{H}(\Omega) \cap C^k(\bar{\Omega})$, where $k \in \{0,1\}$. Here $C^0(\bar{\Omega})$ denotes the set of continuous functions on $\bar{\Omega} = \Omega \cup S$ and $C^1(\bar{\Omega})$ the set of functions having continuous derivatives of order 1 in Ω and with continuous extensions to $\bar{\Omega}$.

For $x \in \bar{\Omega}$ we write $r = |x|$ and $s = x/|x|$. Each function $u \in \mathcal{H}^0(\bar{\Omega})$ can be expanded in outer harmonics

$$u = \sum_{n=0}^{\infty} u_n , \tag{1}$$

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where

$$u_n(x) = r^{-(n+1)} Y_n(s) \tag{2}$$

and, with $\langle \, , \rangle$ denoting the scalar product,

$$Y_n(s) = \frac{2n+1}{4\pi} \int_S u(s') P_n(\langle s, s' \rangle) \, ds' \, . \tag{3}$$

Here P_n is the Legendre polynomial of degree n. Since $P_0 = 1$, we have

$$Y_0 = \frac{1}{4\pi} \int_S u \, ds \, . \tag{4}$$

The series expansion (1) is absolutely and uniformly convergent on each subset $\Omega' \subset \Omega$ with $\operatorname{dist}(\Omega', \Omega) > 0$ (*Freeden and Michel, 2004, Chapter 3*); see also (*Sansò and Venuti, 2005, Section 2*). The function $Y_n(s)$ in Eq.(3) is a spherical harmonic of order *n*, and the series $\sum_{n=0}^{\infty} Y_n(s)$ is the Fourier series of $u|_S$ in terms of the spherical harmonics; this series converges in $L^2(S)$ (cf. *Vladimirov, 1971*).

Let $C_+(S)$ be the set of nonnegative continuous functions on *S*. We define the map $G : \mathcal{H}^1(\bar{\Omega}) \to C_+(S)$ by

$$G(u)(x) = |\nabla u(x)| , \qquad x \in S \, , \tag{5}$$

where ∇u is the gradient of *u*. For a given $g \in C_+(S)$, BP consists in solving the equation $G(u) = g$ (*Backus, 1968)*:

$$\mathrm{BP}: \begin{cases} \Delta u = 0 & \text{in } \Omega \, , \\ |\nabla u| = g & \text{on } S \, , \\ u(x) \to 0 & \text{as } x \to \infty \, . \end{cases} \tag{6}$$

Not considering a change of sign $(G(-u) = G(u))$, the solution of this boundary problem is not unique in general as shown by *Backus (1970)*. However, the solution is unique if *u* is subject to the condition $\partial u / \partial r \le 0$ (or alternatively $\partial u / \partial r \ge 0$), where $\partial u / \partial r$ is the radial component of ∇u on the unit sphere (*Díaz et al., 2006*). Let

$$K = \left\{ u \in \mathcal{H}^1(\bar{\Omega}) : \partial u / \partial r \le 0 \right\} , \tag{7}$$

and let $u \in K$ satisfy $G(u) = g$. Then $|u^*| < u$ for any other solution $u^* \neq -u$ of the equation $G(u^*) = g$ (*Díaz et al., 2006*). In other words, $-u$ and u are the minimal and the maximal solutions of BP. For $g \equiv 1$ the maximal solution is $u = 1/r$. We note that if $u \in K \setminus \{0\}$, then u is a non-constant positive function $(u > 0)$ in $\bar{\Omega}$. Since $G(u) = G(-u)$ we observe that the maximal solution, if it exists, must be positive.

BP has application in geodesy and geomagnetism. Many geodesists have contributed to the study of BP: e.g. *Koch and Pope (1972)*, *Bjerhammar and Svensson (1983)*, *Grafarend (1989)*, *Heck (1989)*, *Sacerdote and Sansò (1989)*, *Holota (1997, 2005)*, *Čunderlík et al. (2008)* and *Čunderlík and Mikula (2010)*. The major achievements in the present dates are local solvability results and a deep knowledge of the linearized problem which is a regular oblique derivative problem in the most general case. In geodesy g is the length of the gravity vector. Since the Earth's gravity points toward the interior, the restriction $u \in K$ is natural. By virtue of Eq.(1), the solution must be of the form

$$u = \frac{c}{r} + \sum_{n=1}^{\infty} u_n , \tag{8}$$

where $c = (4\pi)^{-1} \int_S u \, ds > 0$ is a positive constant.

In contrast to that, in geomagnetism the solution must satisfy Eq.(8) with $c = 0$ (cf. *Campbell, 1997*):

$$\int_S u \, ds = 0 . \tag{9}$$

Hence u changes its sign on S, so that neither u nor $-u$ belong to K and then $\partial u / \partial r$ changes its sign on S too. As an example, Fig. 1 displays the field of a magnetic dipole. It is observed that:

1. ∇u is tangential to the unit sphere along the equator E.
2. $\nabla u|_E$ is orthogonal to E.
3. $\partial u / \partial r$ changes its sign on S through E from plus to minus in the direction of the vector field $\nabla u|_E$.

The magnetic inclination is defined as the angle measured from the horizontal plane to the magnetic field vector where downward is positive. Fig. 2 shows the inclination of the Earth magnetic field. It is clearly seen that ∇u is tangential to the Earth's surface along a closed curve called the *dip equator*.

In Section 2.1 we propose an iterative algorithm to construct the solution u of BP such that $u \in K$, and hence the maximal solution of this problem. This is the relevant solution in geodesy. Our construction is adapted from *Sacerdote and Sansò (1989)*. In Section 2.2 we give two numerical examples and find, in particular, the maximal solution of BP where g is the norm of the gradient of $u^* = z/r^3$. This can be considered as an explicit example

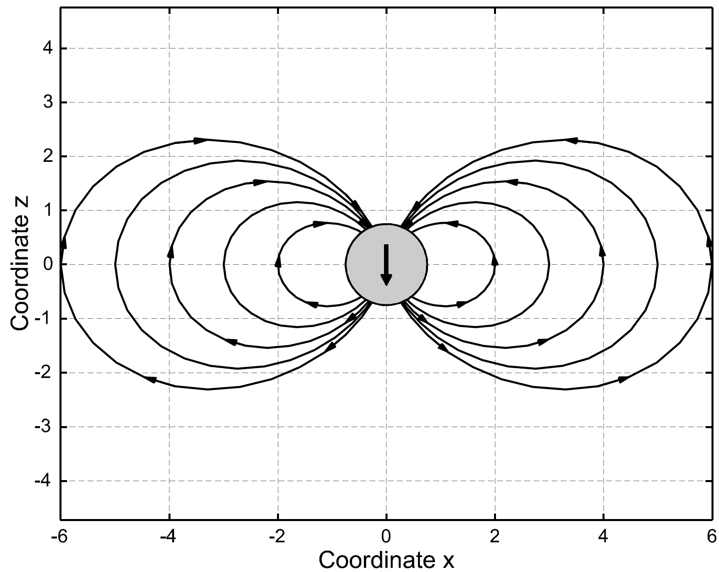

Fig. 1. Magnetic dipole field using the Matlab function lforce2d by A. Abokhodair (*http://www.mathworks.com/matlabcentral/fileexchange*).

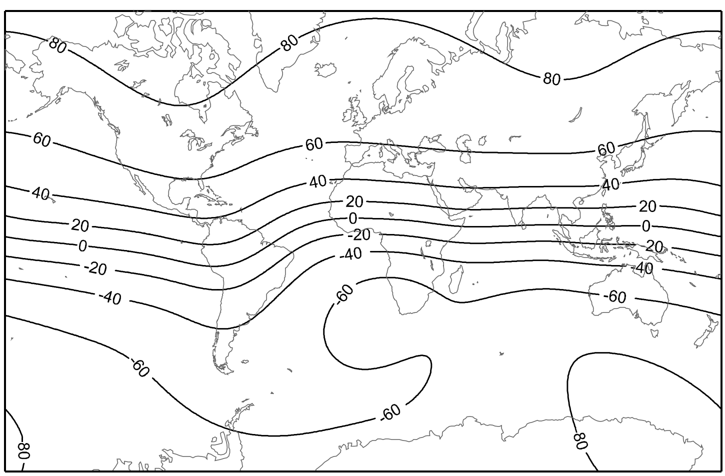

Fig. 2. Inclination of the Earth magnetic field (epoch 2005) from the USA National Geophysical Data Center (NGDC) (*http://www.ngdc.noaa.gov/geomag/data.shtml*).

of the non-uniqueness of solutions of BP complementing the results in *Backus (1970)*. Finally, in Section 3 we prove the convergence of the successive approximations to $u \in K$ for functions g close enough to a constant function in a Hölder space.

2. SUCCESSIVE APPROXIMATIONS FOR CONSTRUCTING THE MAXIMAL SOLUTION OF BACKUS' PROBLEM

2.1. The Sequence of Successive Approximations

To simplify notation, here and subsequently, we use the same letter G for the map $G:\mathcal{H}^1(\bar{\Omega}) \to C_+(S)$ defined by

$$G(u)(x) = |\nabla u(x)|^2, \qquad x \in S\,. \tag{10}$$

Let $u \in K$ satisfy $G(u) = f$, where $f = g^2$. Since $u > 0$, there exists an unknown positive constant μ such that

$$\mu^{1/2} u = \frac{1}{r} + v\,, \tag{11}$$

with $v \in \mathcal{H}^1(\bar{\Omega})$ without outer harmonic of order zero. By Eq.(4), this is equivalent to the property

$$\int_S v\, ds = 0\,. \tag{12}$$

Note that the constant μ in Eq.(11) is such that $\mu^{-1/2} = (4\pi)^{-1} \int_S u\, ds$. In addition, the restriction $\partial u/\partial r \le 0$ on S holds if, and only if, $\partial v/\partial r \le 1$ on S.

Since $G\left(\mu^{1/2} u\right) = \mu f$, from Eq.(11) it is easy to check that the function v must satisfy the following boundary condition on S:

$$2\frac{\partial v}{\partial r} = 1 + G(v) - \mu f\,. \tag{13}$$

We recall the Green second identity, where $u, v \in C^1(\bar{\Omega})$ and regular at infinity,

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_S \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r} \right) ds\,. \tag{14}$$

Taking for u in the identity (14) the function $u = 1/r$, we see that

$$\int_S \left(\frac{\partial v}{\partial r} + v \right) ds = 0 \tag{15}$$

for all $v \in \mathcal{H}^1(\bar{\Omega})$. Consequently, $v \in \mathcal{H}^1(\bar{\Omega})$ satisfies Eq.(12) if, and only if

$$\int_S \frac{\partial v}{\partial r}\, ds = 0 \,. \tag{16}$$

Hence the constant μ in Eq.(13) must be such that

$$\int_S \left[1 + G(v) - \mu f\right] ds = 0 \,, \tag{17}$$

i.e.

$$\mu = \mu(v) = \mu_0 \left[1 + \frac{1}{4\pi} \int_S G(v)\, ds\right], \tag{18}$$

where μ_0 is given by

$$\mu_0^{-1} = f_0 := \frac{1}{4\pi} \int_S f \; ds \,. \tag{19}$$

Therefore, to solve the equation $G_K(u) = f$, where G_K is the restriction of G to K, we must find a solution of the nonlinear problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \,, \\ 2\dfrac{\partial v}{\partial r} = 1 + G(v) - \mu(v) f & \text{on } S \,, \\ v(x) \to 0 & \text{as } x \to \infty \,. \end{cases} \tag{20}$$

with the additional property that $\partial v / \partial r \leq 1$ on S.

To this end, we define a sequence $\{v_n : n = 1, 2, \ldots\} \in \mathcal{H}^1(\bar{\Omega})$ by the following boundary conditions on S

$$2\frac{\partial v_1}{\partial r} = 1 - \mu_0 f \;, \tag{21a}$$

$$2\frac{\partial v_{n+1}}{\partial r} = 1 + G(v_n) - \mu_n f \;, \quad n \geq 1, \tag{21b}$$

where

$$\mu_n := \mu(v_n) = \mu_0 \left[1 + \frac{1}{4\pi} \int_S G(v_n)\, ds\right]. \tag{22}$$

We observe that each v_n satisfies

$$\int_S v_n \; ds = 0 \,. \tag{23}$$

In fact, by Eq.(15) we have

$$-2\int_S v_1\, ds = 2\int_S \frac{\partial v_1}{\partial r} ds = \int_S \left(1-\mu_0 f\right) ds = 0 \,,$$

and, if $n \geq 1$,

$$\begin{aligned} -2\int_S v_{n+1}\, ds &= 2\int_S \frac{\partial v_{n+1}}{\partial r} ds = 4\pi + \int_S G\left(v_n\right) ds - \mu_n \int_S f\; ds \\ &= 4\pi\left[1+\frac{1}{4\pi}\int_S G\left(v_n\right) ds\right] - 4\pi\mu_0^{-1}\mu_n = 0 \end{aligned}$$

by Eq.(22).

In $\bar{\Omega}$ the functions v_n are given by

$$2v_1 = -\frac{1}{r} - \mu_0 F \,, \qquad 2v_{n+1} = -\frac{1}{r} + G_n - \mu_n F \,, \qquad n \geq 1, \tag{24}$$

where F and G_n are the unique solutions of the exterior Neumann problems

$$\Delta F = 0 \;\text{ in } \Omega, \qquad F\left(x\right) \to 0 \;\text{ as }\; x \to \infty \,, \qquad \frac{\partial F}{\partial r} = f \;\text{ on } S, \tag{25}$$

and

$$\Delta G_n = 0 \;\text{ in } \Omega, \qquad G_n\left(x\right) \to 0 \;\text{ as }\; x \to \infty \,, \qquad \frac{\partial G_n}{\partial r} = G\left(v_n\right) \;\text{ on } S, \tag{26}$$

respectively.

By Eq.(11), if the sequences $\left\{v_n\right\}_{n\geq 1} \in \mathcal{H}^1\left(\bar{\Omega}\right)$ and $\left\{\mu_n\right\}_{n\geq 1} \in \mathbb{R}^+$ converge, the former in an adequate functional space to a function $v \in \mathcal{H}^1\left(\bar{\Omega}\right)$ with $\partial v/\partial r \leq 1$ on S, then the successive approximations to construct the maximal solution of BP are

$$u_n = \mu_n^{-1/2}\left(r^{-1} + v_n\right), \qquad n \geq 1. \tag{27}$$

Schematically, the sequence of computations is

$$\begin{gathered} \mu_0 \to v_1 \to \mu_1 \to u_1 \\ \left(v_1, \mu_1\right) \to v_2 \to \mu_2 \to u_2 \\ \left(v_2, \mu_2\right) \to v_3 \to \mu_3 \to u_3 \\ \vdots \\ \left(v_{n-1}, \mu_{n-1}\right) \to v_n \to \mu_n \to u_n \end{gathered}$$

We treat the question of the convergence of the sequence $\{u_n\}$ in Section 3.

Remark 2.1. This construction is motivated by *Sacerdote and Sansò (1989)*. These authors seek a solution of BP in the form

$$u = \frac{A}{r} + \delta u\,, \qquad \delta u \in \mathcal{H}^1\left(\bar{\Omega}\right), \tag{28}$$

where Ω is the exterior of a closed surface S such that $|x| = 1 + \rho(x/r)$ if $x \in S$, with ρ being a known function defined on the unit sphere. Then, they propose the following recursion procedure

$$-\frac{\partial\,\delta u_{n+1}}{\partial r} = -\frac{(1+\rho)^2}{2A}\left|\nabla\,\delta u_n\right|^2 + \frac{(1+\rho)^2}{2A}\left[g^2 - \frac{A^2}{(1+\rho)^4}\right], \quad n \geq 1\,. \tag{29}$$

If S is the unit sphere ($\rho = 0$), then Eq.(29) becomes

$$-\frac{\partial\,\delta u_{n+1}}{\partial r} = -\frac{1}{2A}\left|\nabla\,\delta u_n\right|^2 + \frac{1}{2A}\left(g^2 - A^2\right). \tag{30}$$

It is clear that $\delta u_n = \delta u_n(x, A)$ for $n \geq 2$. The difference between this procedure and ours is that the function δu is not subjected to any condition and the constant A must be chosen in such a way that the sequence $\{\delta u_n(x, A)\}$ converges.

Remark 2.2. We must assume that $f > 0$ on S. In fact, if $f(x_0) = 0$ at some point $x_0 \in S$, then $\partial u(x_0)/\partial r = 0$ (and $\nabla_t u(x_0) = 0$ as well, where $\nabla_t u$ is the tangential component of ∇u on the unit sphere S). Hence, by Eq.(11) the function v has to satisfy the strong condition

$$\frac{\partial v}{\partial r}(x_0) = 1\,. \tag{31}$$

(Note that since

$$2\frac{\partial v}{\partial r} = 1 + \left|\frac{\partial v}{\partial r}\right|^2 + \left|\nabla_t v\right|^2 - \mu f \tag{32}$$

by Eq.(13), $\nabla_t v(x_0) = 0$ if v satisfies Eq.(31).) If we do not take into consideration the constraint (31) the recursive approach described above can diverge. For example, if $f = 5\cos^4\theta + 4\cos^2\theta$, where θ is the colatitude on the sphere, then $f = 0$ along the equator $\theta = \pi/2$.

The maximal solution of the corresponding BP is given by

$$u = \frac{1}{r} + \frac{2}{3}\left(\frac{1}{r}\right)^3 P_2(\cos\theta) \in \mathcal{H}^1\left(\bar{\Omega}\right). \tag{33}$$

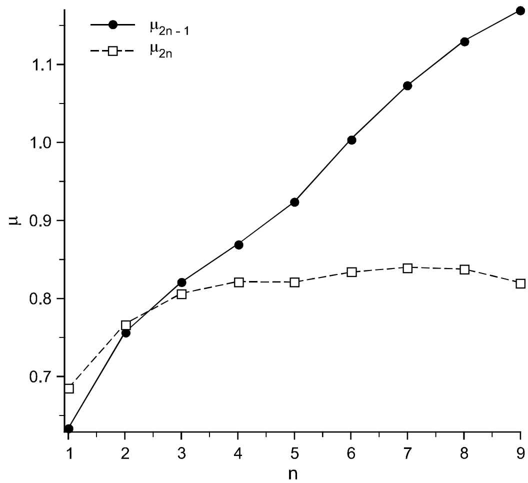

Fig. 3. Sequences μ_{2n-1} and μ_{2n} $(n = 1, \ldots, 9)$ for the Backus' problem associated with $f = 5\cos^4\theta + 4\cos^2\theta$.

In fact, $\partial u/\partial r|_{r=1} = -3\cos^2\theta \leq 0$ which shows that $u \in K$, $\partial u/\partial\theta|_{r=1} = -\sin 2\theta$ and $|\nabla u|^2_{r=1} = (\partial u/\partial r)^2 + (\partial u/\partial\theta)^2 = f$. However, the sequence (μ_n, v_n), computed as we describe in the next section in detail, seems to diverge. This is shown in Fig. 3 where we plot the sets $\{\mu_{2n-1}\}_{n=1,\ldots,9}$ and $\{\mu_{2n}\}_{n=1,\ldots,9}$. Note that in this example $\mu = 1$.

2.2. Numerical Examples

Let $\theta \in [0, \pi]$ denote the colatitude on the unit sphere, and set $t = \cos\theta \in [-1, 1]$. In the examples below, to generate the sequence $\{v_n\}$, we have to solve Neumann problems of the form

$$\Delta u = 0 \text{ in } \Omega, \quad u(x) \to 0 \text{ as } x \to \infty, \quad \frac{\partial u}{\partial r} = \varphi(t) \text{ on } S, \tag{34}$$

where $\varphi(t)$ is an even polynomial of degree $2N$

$$\varphi(t) = \varphi_0 + \sum_{i=1}^{N} \varphi_i t^{2i}, \tag{35}$$

with the property

$$\int_{-1}^{1} \varphi(t)\, \mathrm{d}t = 0. \tag{36}$$

The solution of this boundary-value problem is given by

$$u = \sum_{n=1}^{N} u_n \left(\frac{1}{r}\right)^{2n+1} P_{2n}(t), \tag{37}$$

where

$$u_n = -\frac{4n+1}{2n+1} \sum_{i=n}^{N} a_{in} \varphi_i \tag{38}$$

and

$$a_{in} = \frac{1}{2} \int_{-1}^{1} t^{2i} P_{2n}(t)\,\mathrm{d}t = \frac{(2i)!}{2^{i-n}(i-n)!(2i+2n+1)!!}, \qquad (i = n, n+1, \ldots). \tag{39}$$

Here we have used the Schmied formula (*Weisstein, 2010, Eq.15*). For the properties of Legendre polynomials we also refer to *Weisstein (2010)*.

The function $G(u)$ is an even polynomial in t of degree $4N$. To prove this assertion we first observe that $(\partial u/\partial r)^2 = \varphi^2$ is an even polynomial of degree $4N$. For the tangential component of ∇u we have

$$\frac{\partial u}{\partial \theta} = \sum_{n=1}^{N} u_n \frac{dP_{2n}(t)}{d\theta}. \tag{40}$$

From the identities

$$\frac{dP_n(t)}{d\theta} = \frac{ntP_n(t) - nP_{n-1}(t)}{\sqrt{1-t^2}} \tag{41}$$

and

$$(n+1)P_{n+1}(t) = (2n+1)P_n(t) - nP_{n-1}(t), \tag{42}$$

we get

$$\frac{dP_{2n}(t)}{d\theta} = \frac{2n(2n+1)}{4n+1} \frac{\left[P_{2n+1}(t) - P_{2n-1}(t)\right]}{\sqrt{1-t^2}}. \tag{43}$$

Since $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ for all n, $t = \pm 1$ are roots of the polynomial $P_{2n+1}(t) - P_{2n-1}(t)$ and we can write $P_{2n+1}(t) - P_{2n-1}(t) = (t-1)(t+1)Q_{2n-1}(t)$, where $Q_{2n-1}(t)$ is an odd polynomial of degree $2n - 1$. Hence

$$\frac{\partial u}{\partial \theta} = \sum_{n=1}^{N} c_n \frac{(t-1)(t+1)}{\sqrt{1-t^2}} Q_{2n-1}(t), \tag{44}$$

where

$$c_n = u_n \frac{2n(2n+1)}{4n+1}. \tag{45}$$

From Eq.(44) we have

$$\left(\frac{\partial u}{\partial \theta}\right)^2 = \sum_{n,m=1}^{N} c_n c_m \left(1-t^2\right) Q_{2n-1}(t) Q_{2m-1}(t), \tag{46}$$

which is an even polynomial of degree 4*N*.

To compute the sequence $\{\mu_n\}$ we need to evaluate the integral on *S* of the square of the modulus of the gradient of harmonic functions like Eq.(37). For this purpose we use the identity (*Hörmander, 1976, Eq.1.2.4*)

$$\int_S |\nabla u|^2 \, ds = \int_S \frac{\partial u}{\partial r}\left(2\frac{\partial u}{\partial r} + u\right) ds, \tag{47}$$

where $u \in \mathcal{H}^1\left(\bar{\Omega}\right)$. Since on *S*

$$\frac{\partial u}{\partial r} = -\sum_{n=1}^{N} (2n+1) u_n P_{2n}(t) \tag{48}$$

and

$$2\frac{\partial u}{\partial r} + u = -\sum_{n=1}^{N} (4n+1) u_n P_{2n}(t), \tag{49}$$

using the property

$$\int_{-1}^{1} P_n(t) P_m(t) \, \mathrm{d}t = \frac{2}{2n+1} \delta_{mn}, \tag{50}$$

where δ_{mn} is the Kronecker delta, we arrive at the following expression for the integral of $|\nabla u|^2$:

$$\int_S |\nabla u|^2 \, ds = 4\pi \sum_{n=1}^{N} (2n+1) u_n^2 . \tag{51}$$

Example 2.1. (Unknown maximal solution). To illustrate our approach we first analyze the following BP:

$$\Delta u = 0 \ \text{in } \Omega, \qquad u(x) \to 0 \ \text{as } x \to \infty, \qquad |\nabla u|^2 = 1 + 3t^2 \ \text{on } S. \tag{52}$$

A solution of this problem is the dipole potential $u^* = z/r^3$, where $z = r\cos\theta$. This function does not belong to K because $\partial u^*/\partial r\big|_{r=1} = -2t$ changes its sign on S vanishing on the equator $t = 0$. In addition, u^* is not positive and therefore it cannot be the maximal solution of the problem (52).

For the function $f = 1 + 3t^2$ we have

$$f_0 = \frac{1}{4\pi}\int_S f\, ds = \frac{1}{2}\int_{-1}^{1}\left(1+3t^2\right)\mathrm{d}t = 2 \quad \text{and} \quad \mu_0 = \frac{1}{2}. \tag{53}$$

Hence

$$v_1 = \frac{1}{6}\left(\frac{1}{r}\right)^3 P_2(t) \quad \text{and} \quad \mu_1 = \frac{13}{24}. \tag{54}$$

Rounding to six decimal places, it follows that

$$u_1 = \mu_1^{-1/2}\left(\frac{1}{r} + v_1\right) = 1.358732\left(\frac{1}{r}\right) + 0.226455\left(\frac{1}{r}\right)^3 P_2(t), \tag{55}$$

whose radial derivative on S is strictly less than zero, i.e. $u_1 \in K$. We now compute v_n, μ_n and u_n, from $n = 2$ to $n = 9$. We obtain:

a) Sequences μ_n and $\mu_n^{-1/2}$ (Table 1):

Here $\mu_n^{-1/2}$ is the sequence of successive approximations to the zero order spherical harmonic coefficient of the limiting function $u := \lim_{n\to\infty} u_n$ (assumed to exist), i.e.

$$\lim_{r\to\infty} ru = \lim_{n\to\infty} \mu_n^{-1/2} \approx 1.35826. \tag{56}$$

Table 1. Successive approximations to the constants μ and $\mu^{-1/2}$.

n	μ_n	$\mu_n^{-1/2}$
1	0.541667	1.358732
2	0.540808	1.359811
3	0.542183	1.358085
4	0.541950	1.358377
5	0.542064	1.358234
6	0.542030	1.358277
7	0.542043	1.358260
8	0.542038	1.358267
9	0.542040	1.358264

b) The functions v_n and u_n are of the form:

$$v_n = \sum_{m=1}^{N} a_{2m}^{(n)} \left(\frac{1}{r}\right)^{2m+1} P_{2m}(t) \quad \left(N(n) = 2^{n-1}\right), \tag{57}$$

and

$$u_n = \frac{\mu_n^{-1/2}}{r} + \sum_{m=1}^{N} b_{2m}^{(n)} \left(\frac{1}{r}\right)^{2m+1} P_{2m}(t), \tag{58}$$

where $b_{2m}^{(n)} = \mu_n^{-1/2} a_{2m}^{(n)}$. For example, the coefficients $b_{2m}^{(2)}$ and $b_{2m}^{(3)}$ of u_2 and u_3, respectively, are:

$$[0.223937, -0.009713],$$
$$[0.227508, -0.008016, 0.000702, -0.000013].$$

In all cases $u_n \in K$. For n = 6, 7, 8, in Fig. 4 we show the functions $\Delta_n := |u_{n+1} - u_n|$ evaluated on the unit sphere. Note that the maximum value of Δ_n tends to be zero as n increases. This is an indication of the uniform convergence of the sequence $\{u_n\}$.

Fig. 5 displays the functions $|G(u_n) - f|$ for n = 7, 8, 9, where $G(u_n) = |\nabla u_n|^2$. Since the maximum value of $|G(u_9) - f|$ is less than 10^{-4}, we can consider $u_9 \in K$ a good

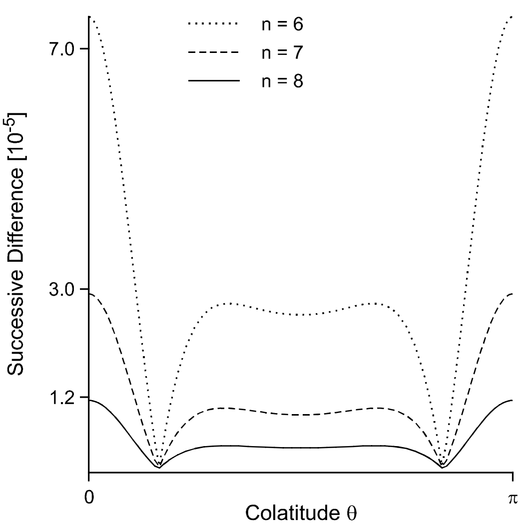

Fig. 4. Numerical convergence: absolute value of the successive differences $u_{n+1} - u_n$.

approximation of the maximal solution of the problem (52). From now on, for abbreviation, we write u instead of u_9. In Fig. 6 are plotted the functions $u^* = z/r^3$, the minimal ($-u$) and the maximal (u) solutions of the problem (52) on the unit sphere.

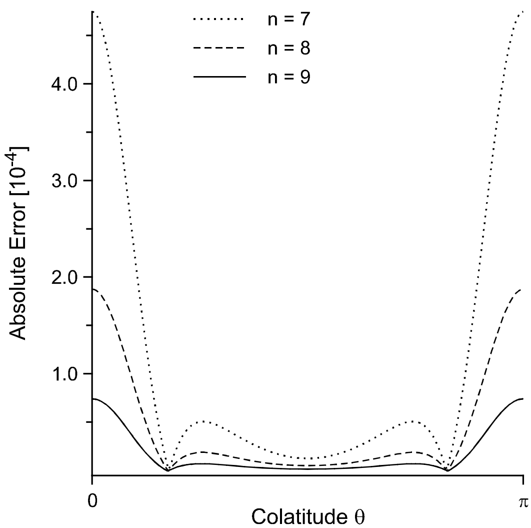

Fig. 5. Absolute error $\left|G(u_n) - f\right|$.

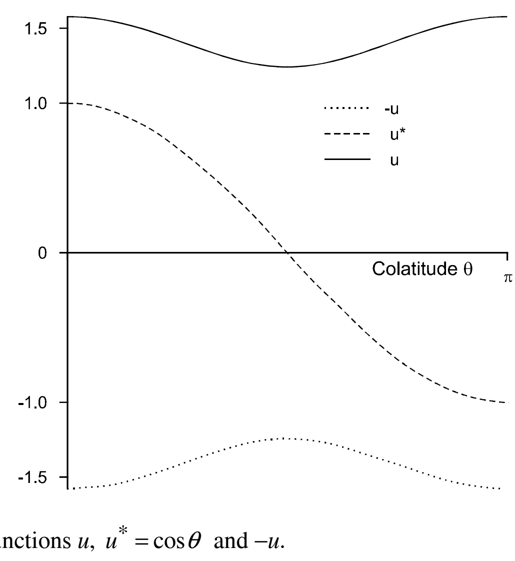

Fig. 6. Plot of the functions u, $u^* = \cos\theta$ and $-u$.

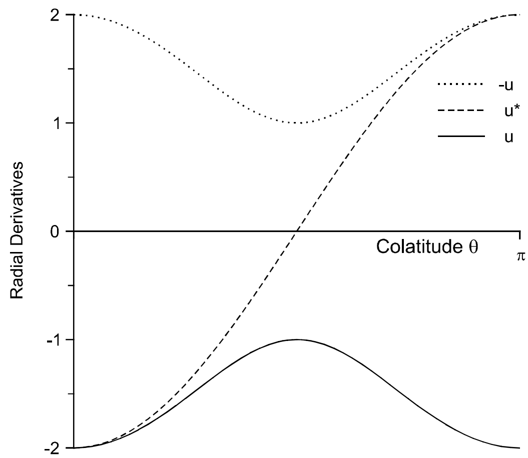

Fig. 7. Radial derivatives on S of the functions u, $u^* = \cos\theta$ and $-u$.

Finally, we draw in Fig. 7 the radial components of ∇u, ∇u^* and $-\nabla u$ on the unit sphere. It is interesting to observe that $\partial u^* / \partial r = \partial u / \partial r < 0$ at $\theta = 0$ and that $\partial u^* / \partial r = -\partial u / \partial r > 0$ at $\theta = \pi$. This is to illustrate a general property of the maximal solution of BP, namely (*Díaz et al., 2006, Theorem 2.4*): let $g \in C_+(S)$ and let $u \in K$ satisfy $G(u) = g$. If $u^* \neq -u$ is other solution of BP then the radial component of ∇u^* changes its sign on S. In particular, there are points $x \in S$ and $y \in S$ such that

$$\frac{\partial u^*}{\partial r}(x) = \frac{\partial u}{\partial r}(x) \leq 0 \quad \text{and} \quad \frac{\partial u^*}{\partial r}(y) = -\frac{\partial u}{\partial r}(y) \geq 0 \,. \tag{59}$$

Example 2.2. (Known maximal solution). We now consider the BP:

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega, \quad u(x) \to 0 \quad \text{as } x \to \infty, \\ &|\nabla u|^2 = f(t) := \left(1 - \frac{3}{2}c\right)^2 + 9c\left(1 - \frac{1}{2}c\right)t^2 + \frac{45}{4}c^2 t^4 \quad \text{on } S, \end{aligned} \tag{60}$$

where c is an arbitrary constant. A solution of this problem is

$$u = \frac{1}{r} + \frac{c}{r^3} P_2(t) \,. \tag{61}$$

A verification shows that $\partial u / \partial r < 0$ if, and only if, $c \in (-1/3, 2/3)$. In addition, if $c = -1/3$ then $f(\pm 1) = 0$, and if $c = 2/3$ we have $f(0) = 0$. Therefore for these values of c the function $u \in K$ is the maximal solution of the problem (60). For the singular case $c = 2/3$ see Remark 2.2.

Fix $c = 1/3$. In this case $\mu = 1$ and

$$v = \frac{1}{3}\left(\frac{1}{r}\right)^3 P_2(t) . \tag{62}$$

Since $f = \left(1 + 10t^2 + 5t^4\right)/4$, we get $f_0 = 4/3$ and $\mu_0 = 0.75$. Hence

$$v_1 = \frac{25}{84}\left(\frac{1}{r}\right)^3 P_2(t) + \frac{3}{140}\left(\frac{1}{r}\right)^5 P_4(t) \quad \text{and} \quad \mu_1 = \frac{233}{245} = 0.951020 . \tag{63}$$

We now compute v_n and μ_n from $n = 2$ to $n = 9$. In this example the convergence is slower than in the first one, and the computations from $n = 9$ are very time consuming because the functions v_n are of the form given in Eq.(57) with $N(n) = 2^n$ instead of $N(n) = 2^{n-1}$. This means that if $n = 9$ then $G(v_9)$ is now an even polynomial of degree 2048. However, since most of the coefficients of $G(v_n)$ (from $n \geq 4$) are very close to zero we can obtain further approximations by truncating these polynomials , i.e.

$$G(v_n) = \sum_{m=1}^{N} g_m^{(n)} t^{2m} \approx \sum_{m=1}^{t(n)} g_m^{(n)} t^{2m} , \tag{64}$$

where $t(n)$ is such that

$$\left| G(v_n) - \sum_{m=1}^{t(n)} g_m^{(n)} t^{2m} \right| \leq \sum_{m=t(n)+1}^{N} \left| g_m^{(n)} \right| < \varepsilon , \tag{65}$$

where ε is small (say $\varepsilon = 10^{-6}$). Proceeding in this way, for $n = 8$ we get $t(8) = 22$ with an error of approximation equals to 2.61×10^{-7}.

Without truncation the results are as follows. In Table 2 we give the values of $|\mu_n - 1|$. The functions $v_n - v$ are plotted in Fig. 8 where we see that $|v_n - v|$ tends to zero at least to this level of approximation.

Table 2. The absolute differences between μ_n and $\mu = 1$.

n	1	2	3	4	5	6	7	8	9
$\lvert\mu_n - 1\rvert \times 10^{-3}$	48.980	32.141	4.750	7.305	0.900	2.557	1.215	1.292	0.908

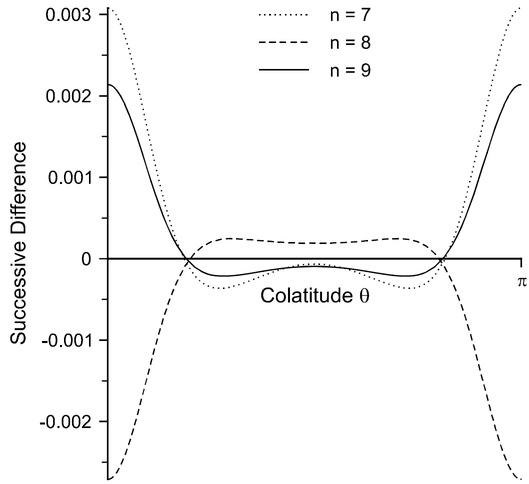

Fig. 8. Numerical convergence: successive differences $v_n - v$.

3. ON THE CONVERGENCE OF THE SEQUENCE OF SUCCESSIVE APPROXIMATIONS

Without loss of generality we can assume that

$$f_0 := \frac{1}{4\pi}\int_S f \, ds = 1 , \tag{66}$$

for if not, we replace the equation $G(u) = f$ by $G\left(f_0^{-1/2}u\right) = f / f_0$. Therefore one has $\mu_0 = 1$ too.

We recall the definition of the Hölder space $C^{0,\alpha}(S)$ with $0 < \alpha \leq 1$. Let φ be a function defined on *S*. When the quantity

$$[\varphi]_{\alpha;S} = \sup_{\substack{x,y\in S \\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{\alpha}} \tag{67}$$

is finite, the function φ is said to be Hölder continuous with exponent α on *S*. The Hölder space $C^{0,\alpha}(S)$ is defined as the subspace of $C^0(S)$ consisting of functions Hölder continuous with exponent α on *S* equipped with the norm

$$\|\varphi\|_{0,\alpha;S} = \|\varphi\|_{0;S} + [\varphi]_{\alpha;S} , \tag{68}$$

where

$$\|\varphi\|_{0;S} = \max_{x\in S}|\varphi(x)| \tag{69}$$

is the maximum norm of φ. We write $\|\varphi\|_0$, $[\varphi]_\alpha$ and $\|\varphi\|_\alpha$ instead of $\|\varphi\|_{0;S}$, $[\varphi]_{\alpha;S}$ and $\|\varphi\|_{\alpha;S}$, respectively, when no confusion can arise.

Let $M = \max_S f = \|f\|_0$ and $m = \min_S f$. By Eq.(66) we have $M \geq 1$ and $m \leq 1$. Since $\min(1-f) = 1-M \leq 0$ and $\max(1-f) = 1-m \geq 0$, from Eq.(21a) we get

$$1-M \leq 2\frac{\partial v_1}{\partial r} \leq 1-m \tag{70}$$

and

$$2\left\|\frac{\partial v_1}{\partial r}\right\|_\alpha = \|1-f\|_\alpha . \tag{71}$$

If $n \geq 1$, since $\mu_n \geq 1$, from Eq.(22) we get

$$m \leq \mu_n f \leq M\left(1+\|G(v_n)\|_0\right) . \tag{72}$$

Hence from Eq.(21b) we obtain

$$(1-M) - M\|G(v_n)\|_0 \leq 2\frac{\partial v_{n+1}}{\partial r} \leq (1-m) + \|G(v_n)\|_0 , \tag{73}$$

from which it follows that

$$2\left\|\frac{\partial v_{n+1}}{\partial r}\right\|_0 \leq \|1-f\|_0 + M\|G(v_n)\|_0 , \tag{74}$$

which is due to the facts that $\|1-f\|_0 = \max(M-1, 1-m)$ and $M \geq 1$.

Writing Eq.(21b) in the form

$$2\frac{\partial v_{n+1}}{\partial r} = (1-f) + G(v_n) - I_n f , \tag{75}$$

where $I_n = (4\pi)^{-1}\int_S G(v_n)\, ds \leq \|G(v_n)\|_0$, since $[\,\cdot\,]_\alpha$ is a seminorm, by the triangle inequality we have

$$2\left[\frac{\partial v_{n+1}}{\partial r}\right]_\alpha \leq [1-f]_\alpha + [G(v_n)]_\alpha + [f]_\alpha \|G(v_n)\|_0 . \tag{76}$$

Combining Eqs.(74) and (76) gives

$$2\left\|\frac{\partial v_{n+1}}{\partial r}\right\|_0 + 2\left[\frac{\partial v_{n+1}}{\partial r}\right]_\alpha \le \left(\|1-f\|_0 + [1-f]_\alpha\right) + \left[G(v_n)\right]_\alpha + \left(M + [f]_\alpha\right)\|G(v_n)\|_0 . \tag{77}$$

Hence

$$\begin{aligned} 2\left\|\frac{\partial v_{n+1}}{\partial r}\right\|_\alpha &\le \|1-f\|_\alpha + \left[G(v_n)\right]_\alpha + \|f\|_\alpha \|G(v_n)\|_0 \\ &\le \|1-f\|_\alpha + \max\left(1, \|f\|_\alpha\right)\left(\|G(v_n)\|_0 + \left[G(v_n)\right]_\alpha\right) \\ &= \|1-f\|_\alpha + \max\left(1, \|f\|_\alpha\right)\|G(v_n)\|_\alpha , \end{aligned} \tag{78}$$

and, since $\|f\|_\alpha \ge 1$, we obtain

$$2\left\|\frac{\partial v_{n+1}}{\partial r}\right\|_\alpha \le \|1-f\|_\alpha + \|f\|_\alpha \|G(v_n)\|_\alpha . \tag{79}$$

Let us now assume the following estimate for functions $u \in \mathcal{H}^1\left(\bar{\Omega}\right)$ with the property that $\partial u/\partial r \in C^\alpha(S)$ for some $\alpha \in (0,1]$:

$$\|\, |\nabla_t u| \,\|_\alpha \le c^{1/2}\left\|\frac{\partial u}{\partial r}\right\|_\alpha , \tag{80}$$

where c is a positive constant. Since

$$\|G(v_n)\|_\alpha \le \left\|\frac{\partial v_n}{\partial r}\right\|_\alpha^2 + \|\, |\nabla_t v_n| \,\|_\alpha^2 , \tag{81}$$

by Eq.(80) we have

$$\|G(v_n)\|_\alpha \le (1+c)\left\|\frac{\partial v_n}{\partial r}\right\|_\alpha^2 . \tag{82}$$

From Eqs.(79)–(82) we conclude

$$2\left\|\frac{\partial v_{n+1}}{\partial r}\right\|_\alpha \le \|1-f\|_\alpha + \|f\|_\alpha (1+c)\left\|\frac{\partial v_n}{\partial r}\right\|_\alpha^2 . \tag{83}$$

Theorem 1. Let $f \in C^{\alpha}(S)$ be such that $(4\pi)^{-1}\int_S f\, ds = 1$. Let $k = \|f\|_{\alpha}(1+c)\|1-f\|_{\alpha}$ where $\|f\|_{\alpha} \geq 1$ and c is the positive constant in Eq.(80). If $k \leq 1$, then

$$\left\|\frac{\partial v_n}{\partial r}\right\|_{\alpha} \leq \varepsilon , \tag{84}$$

for all $n \geq 1$, where

$$\varepsilon = \frac{1}{\|f\|_{\alpha}(1+c)}\left(1-\sqrt{1-k}\right) < 1 . \tag{85}$$

Moreover,

$$\|G(v_n)\|_{\alpha} \leq \varepsilon^2 (1+c) \quad \text{and} \quad \mu_n \leq 1 + 2\varepsilon^2 . \tag{86}$$

Proof. The proof follows *Sacerdote and Sansò (1989)*. Set $a_n = \|f\|_{\alpha}(1+c)\|\partial v_n/\partial r\|_{\alpha}$. Multiplying both sides in Eq.(83) by $\|f\|_{\alpha}(1+c)$ we have

$$2a_{n+1} \leq k + a_n^2 . \tag{87}$$

If $k \leq 1$ we can conclude that $a_n \leq 1-\sqrt{1-k}$ for all n, by induction on n. In fact, by Eq.(71) we have

$$a_1 = \frac{1}{2}\|f\|_{\alpha}(1+c)\|1-f\|_{\alpha} = \frac{k}{2} \leq 1-\sqrt{1-k} . \tag{88}$$

In addition, assuming that $a_n \leq 1-\sqrt{1-k}$ holds for degree $n > 1$, we see that

$$a_{n+1} \leq \frac{1}{2}\left[k + \left(1-\sqrt{1-k}\right)^2\right] = 1-\sqrt{1-k} , \tag{89}$$

as required. A trivial verification shows then that $\left\| |\nabla v_n|^2 \right\|_{\alpha} \leq \varepsilon^2 (1+c)$ by Eq.(82). Finally, the upper bound for μ_n in Eq.(86) follows from Eq.(22) observing that the inequality

$$\int_S |\nabla_t v|^2\, ds \leq \int_S \left|\frac{\partial v_n}{\partial r}\right|^2 ds , \tag{90}$$

holds for any $v \in \mathcal{H}^1(\bar{\Omega})$ (*Hörmander, 1976, Eq. 1.1.4*).

Remark 3.1. Let $b := (1+c)^{-1} < 1$ and let X be the subset of $C^{\alpha}(S)$ defined by

$$X = \left\{ f \in C^{\alpha}(S) : f > 0,\ f_0 = 1 \text{ and } \Phi(f) \le b \right\}, \tag{91}$$

where $f_0 = (4\pi)^{-1} \int_S f\, ds$ and $\Phi(f) = \|f\|_\alpha \|1-f\|_\alpha$. Note that $\Phi(f) \le b$ is the condition $k \le 1$ in Theorem 1. It is clear that X is not empty because $f \equiv 1 \in X$. In this remark, we want to show that this function is not the unique element of X. Since $\|1-f\|_0 = \max(M-1, 1-m)$ and $[1-f]_\alpha = [f]_\alpha$, if $M-1 \ge 1-m$, then we have

$$\begin{aligned} \Phi(f) &= \|f\|_\alpha \|1-f\|_\alpha = \left(M + [f]_\alpha\right)\|1-f\|_\alpha \\ &= \left(1 + \|1-f\|_0 + [f]_\alpha\right)\|1-f\|_\alpha = \left(1+\|1-f\|_\alpha\right)\|1-f\|_\alpha . \end{aligned} \tag{92}$$

Otherwise, i.e. if $M-1 \le 1-m$, then $M \le 1 + \|1-f\|_0$ and we get

$$\begin{aligned} \Phi(f) &= \|f\|_\alpha \|1-f\|_\alpha = \left(M + [f]_\alpha\right)\|1-f\|_\alpha \\ &\le \left(1 + \|1-f\|_0 + [f]_\alpha\right)\|1-f\|_\alpha = \left(1+\|1-f\|_\alpha\right)\|1-f\|_\alpha . \end{aligned} \tag{93}$$

Therefore

$$\Phi(f) \le \varphi\left(\|1-f\|_\alpha\right), \tag{94}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is the strictly increasing function defined by $\varphi(t) = (1+t)t$. In the Eq.(94) the equality holds if, and only if, $\|1-f\|_0 = M-1$. Since the solution of the equation $\varphi(t) = b$ is $t = \left(-1+\sqrt{1+4b}\right)/2$, we conclude that, if

$$\|1-f\|_\alpha \le -\frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \lessapprox 0.62, \tag{95}$$

then $\Phi(f) \le b$. In the case $\|1-f\|_0 = M-1$, the condition (95) is also necessary for $\Phi(f) \le b$.

For example, let $f = 1 + \varepsilon h$, where $\varepsilon \in \mathbb{R}^+$ and the function $h \ne 0$ is such that $\int_S h\, ds = 0$. Since $m_h := \min_S h < 0$ we have $f > 0$ if, and only if, $\varepsilon < -m_h^{-1}$. The condition (95) is satisfied if

$$\varepsilon \le \left(-1+\sqrt{1+4b}\right)\|2h\|_\alpha^{-1}. \tag{96}$$

Hence, if

$$\varepsilon < \min\left\{-m_h^{-1}, \left(-1+\sqrt{1+4b}\right)\|2h\|_\alpha^{-1}\right\}, \tag{97}$$

then $f = 1+\varepsilon h \in X$. For $h = \cos\theta$, where θ is the colatitude on the sphere, we have $[h]_\alpha = 1$, for any $\alpha \in (0,1]$, $\|h\|_0 = 1$ and $\|h\|_\alpha = 2$. Therefore, since $m_h = -1$, by Eq.(97) the function $f = 1+\varepsilon\cos\theta$ belongs to X if

$$\varepsilon < \min\left(1, \left(-1+\sqrt{1+4b}\right)/4\right) = \left(-1+\sqrt{1+4b}\right)/4 \lessapprox 0.31 . \tag{98}$$

Remark 3.2. In respect to the inequality (80), *Sacerdote and Sansò (1989)* estimate the seminorm $[\nabla_t u]_\alpha$ in terms of $\|\partial u/\partial r\|_\alpha$. For the maximum norm of the modulus of the tangential gradient we have (*Maergoiz, 1973*), using partly the notation of this author,

$$\|\,|\nabla_t u|\,\|_0 \le \frac{1}{1-\frac{3}{2}\arcsin\alpha}\left\{\left[\frac{5\left(4-\alpha^2\right)}{4\alpha^2}+\frac{1}{\alpha\sqrt{1-\alpha^2}}\right]\|u\|_0 + \|\,|\bar{\Phi}|\,\|_0\right\}, \tag{99}$$

for all $\alpha \in (0, \sin(2/3))$, where

$$\|u\|_0 \le \frac{5}{2\pi}\max_{x\in S}\left|\int_S \frac{\partial u}{\partial r}(y)\frac{ds_y}{|x-y|}\right|, \tag{100}$$

and

$$\bar{\Phi}(x) = \frac{1}{2\pi}\int_S \frac{\partial u}{\partial r}(y)\left[x, \nabla_x\left(\frac{1}{|x-y|}\right)\right] ds_y , \qquad x \in S . \tag{101}$$

Here $[\cdot\,,\cdot]$ denotes cross product. It can be proved that the singular integral (101) exists if $\partial u/\partial r \in C^{0,\lambda}(S)$ for some $\lambda \in (0,1]$.

To conclude a convergence theorem we first recall the definition and some properties of the spaces $C^{1,\alpha}(\bar{\Omega})$ with $0<\alpha\le 1$. (For a fuller treatment we refer the reader to *Kufner et al., 1977.*) By $C^{1,\alpha}(\bar{\Omega})$ we denote the subset of all functions $u \in C^1(\bar{\Omega})$ such that, where $D_i u = \partial u/\partial x_i$,

$$[D_i u]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha} < \infty \tag{102}$$

for all i = 1, 2, 3. The spaces $C^1(\bar{\Omega})$ and $C^{1,\alpha}(\bar{\Omega})$ are Banach spaces equipped with the norms

$$\|u\|_{1;\bar{\Omega}} = \sum_{i=1}^{3} \sup_{x \in \Omega} |D_i u(x)| \tag{103}$$

and

$$\|u\|_{1,\alpha;\bar{\Omega}} = \|u\|_{1;\bar{\Omega}} + \sum_{i=1}^{3} [D_i u]_{\alpha;\Omega} , \tag{104}$$

respectively. The imbedding from $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$ is compact, i.e. if M is a bounded set in $C^{1,\alpha}(\bar{\Omega})$, then every sequence in M contains a convergent subsequence in $C^1(\bar{\Omega})$ (*Kufner et al., 1977, Theorem 1.5.10*). This property is usually written as $C^{1,\alpha}(\bar{\Omega}) \hookrightarrow\hookrightarrow C^1(\bar{\Omega})$.

For the exterior Neumann problem we have the following estimate (cf. *Jorge, 1987*). Let $\varphi \in C^{\alpha}(S)$ and let u be the unique solution of the Neumann problem

$$\Delta u = 0 \ \text{ in } \Omega, \qquad u(x) \to 0 \ \text{ as } \ x \to \infty , \ \frac{\partial u}{\partial r} = \varphi \ \text{ on } S . \tag{105}$$

Then

$$\|u\|_{1,\alpha;\bar{\Omega}} \le c \|\varphi\|_{\alpha;S} \tag{106}$$

for some $c > 0$. For abbreviation, in the next theorem we write $\{v_n\}$ for both the sequence or some subsequence of the successive approximations defined in the Section 2.1.

Theorem 2. Under the assumptions of Theorem 1, the sequence $\{v_n\}$ contains a subsequence converging in $C^1(\bar{\Omega})$, i.e. there is a function $v \in C^1(\bar{\Omega})$ such that $\|v_n - v\|_{1;\bar{\Omega}} \to 0$ as $n \to \infty$. In addition, the function $u = \mu^{-1/2}(1/r + v)$, where $\mu = \lim_{n\to\infty} \mu(v_n)$, is the maximal solution of BP.

Proof. Combining the estimate (106) and Eq.(86) in Theorem 1 we see that $v_n \in C^{1,\alpha}(\bar{\Omega})$ and that the sequence is bounded, i.e. there exists a constant c such that $\|v_n\|_{1,\alpha;\bar{\Omega}} \le c$. Inasmuch as $C^{1,\alpha}(\bar{\Omega}) \hookrightarrow\hookrightarrow C^1(\bar{\Omega})$, it follows that $\{v_n\}$ contains a subsequence that converges in $C^1(\bar{\Omega})$ to some $v \in C^1(\bar{\Omega})$. Since the limit of a uniformly convergence sequence of harmonic functions is harmonic (*Gilbarg and Trudinger, 1983, Theorem 2.8*), the limit function v belongs to $\mathcal{H}^1(\bar{\Omega})$. By passing to the limit in Eqs.(21b)–(22) we have that v is a solution of the boundary-value problem (20).

Finally, since $\partial v_n/\partial r \le \varepsilon < 1$ on S by Eq.(84), we have $\partial v/\partial r < 1$ on S. Hence $u = \mu^{-1/2}\left(1/r + v\right)$, where $\mu = \lim_{n\to\infty} \mu_n$, belongs to K and it is the maximal solution of BP.

4. CONCLUSIONS

To get the solution of the BP such that its radial derivative is nonnegative, which turns out to be the maximal solution of this problem, we have used the decomposition $\mu^{1/2}u = 1/r + v$, with the harmonic function v satisfying $\int_S v\, ds = 0$. The method of successive approximations used to find v was motivated by *Sacerdote and Sansò (1989)*. Excluding the singular case where the data *f* vanishes at some point of the unit sphere, we have illustrated with two numerical examples that in fact we obtain the maximal solution of BP by the numerical approach here described. These examples, and others that we have worked out, allow us to conjecture that the sequence of successive approximations converge to the maximal solution for whatever given $f > 0$. To prove this conjecture we possibly need the a priori estimate (cf. *Kenig and Nadirashvili, 2001, Eq. 2.26*), where $u \in \mathcal{H}^1\left(\bar{\Omega}\right)$,

$$\left\|u\right\|_{0,\alpha;\bar{\Omega}} \le C\left\{\left\|\frac{\partial u}{\partial r}\right\|_{0;S} + \left\|u\right\|_{0;\Omega}\right\}. \tag{107}$$

In this paper we have proved the convergence to the maximal solution if *f* is closed to a constant function in a similar way as in *Sacerdote and Sansò (1989)* and *Jorge (1987)*. However, our result differs from those of these authors in that the convergence condition ensures that the limit function is the maximal solution of the Backus problem. We finally note that it is important to give a quantitative estimate of the constant *c* in Eq.(80). The research on this problem was initiated by *Sacerdote and Sansò (1989)*, and maybe someone will be able to continue looking into that in future.

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