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FINITE EXTINCTION TIME PROPERTY FOR A DELAYED LINEAR PROBLEM ON A MANIFOLD WITHOUT BOUNDARY

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Abstract. We prove that the mere presence of a delayed term is able to connect the initial state u_0 on a manifold without boundary (here assumed given as the set $\partial\Omega$ where Ω is an open bounded set in $\mathbb{R}^N$) with the zero state on it and in a finite time even if the dynamics is given by a linear problem. More precisely, we extend the states to the interior of Ω as harmonic functions and assume the dynamics given by a dynamic boundary condition of the type $\frac{\partial u}{\partial t}(t,x) + \frac{\partial u}{\partial n}(t,x) + b(t)u(t-\tau,x) = 0$ on $\partial\Omega$, where $b : [0,\infty) \to \mathbb{R}$ is continuous and $\tau > 0$. Using a suitable eigenfunction expansion, involving the Steklov BVP $\{\Delta\varphi_n = 0 \text{ in } \Omega,\ \partial_\nu\varphi_n = \lambda_n\varphi_n \text{ on } \partial\Omega\}$, we show that if $b(t)$ vanishes on $[0,\tau] \cup [2\tau,\infty)$ and satisfies some integral balance conditions, then the state $u(t,.)$ corresponding to an initial datum $u_0(t,\cdot) = \mu(t)\varphi_n(\cdot)$ vanish on $\partial\Omega$ (and therefore in Ω) for $t \geq 2\tau$. We also analyze more general types of delayed boundary actions for which the finite extinction phenomenon holds for a much larger class of initial conditions and the associated implicit discretized problem.

1. **Introduction.** We consider a linear problem with a time-delayed term in a manifold without boundary. More precisely, we shall assume that the manifold is given as the set $\partial\Omega$ where Ω is an open bounded set in $\mathbb{R}^N$. We shall also assume that the evolution of the states in such manifold is given through the trace of the solution of a boundary value problem in which the boundary condition contains the dynamic of the states, as, for instance, the following one

$$(P_N)\ \begin{cases} -\Delta u = 0 & \text{in } (0,+\infty)\times\Omega, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial n}(t,\mathbf{x}) + b(t)u(t-\tau,\mathbf{x}) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ u(s,\mathbf{x}) = u_0(s,\mathbf{x}) & \text{on } (-\tau,0)\times\partial\Omega. \end{cases} \quad (1)$$

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Here $u_0 \in C([-\tau,0], L^2(\partial\Omega))$, and $b:\mathbb{R} \to \mathbb{R}$ is assumed to be continuous (and with support in $[\tau, 2\tau]$). Our main goal is to prove that the mere presence of the delayed term is able to connect the initial state u_0 on the manifold $\partial\Omega$ with the zero state on it and in a finite time. Obviously the final state can be extended to zero in the whole domain $\overline{\Omega}$ and so this property can be regarded as another version of the so called "finite extinction time phenomenon" (*i.e.*, the existence of some instant $t_e > 0$ such that $u(t,x) \equiv 0$ for every $t \geq t_e$, and almost all $\mathbf{x} \in \Omega$).

It is well-known that such phenomenon is typical of some parabolic reaction-diffusion equations involving some non-Lipschitz nonlinear terms (see, *e.g.*, [1, Chapter 2], and the references therein, in particular [8]). The so-called "quenching phenomenon" is also related to this behavior. Interestingly, [10] also shows the somewhat surprising fact that the extinction process cannot be avoided by the inclusion of memory terms in the equation. On the other hand, in some previous work ([3], [4]), the authors have shown that a suitable delayed action may indeed induce quenching in semilinear parabolic equations with homogeneous boundary conditions. The main contribution of the present paper consists in proving that the phenomenon still holds when the delay term is only located on the manifold $\partial\Omega$ and nowhere else of $\overline{\Omega}$ and all that without the presence of any nonlinear term which in many cases is the cause of the phenomenon. We point out that problems with *dynamic boundary conditions* is an important area in which much recent study is being made (see [6] and the references therein; in particular [7] and [2]).

We end the paper with some comments showing that it is possible to prove the "finite extinction time phenomenon", again under the mere presence of a delay term, for the case of a discrete linear problem, related to problem (P_N), for instance, by means of the usual implicit Euler discretization.

2. **The simpler case of a suitable delaying coefficient.** We will use here a constructive approach, instead of using the semigroup generated on $L^2(\partial\Omega)$ by the dynamics of the problem (see to this respect [6], [2], [7]). As a matter of fact, we shall apply the classical separation of variables but now applied to the dynamic problem on the manifold $\partial\Omega$. We extend the initial state to the whole domain $\overline{\Omega}$ (as an harmonic function at the interior having u_0 as its trace on $\partial\Omega$). We assume now that the initial function $u_0(s,\mathbf{x})$ admits a separated expression of the type,

$$u_0(s,\mathbf{x}) = \mu(s)\psi(\mathbf{x}) \text{ for } -\tau \leq s \leq 0,\ x \in \Omega \tag{2}$$

where $\psi(\mathbf{x}) = \varphi_n(\mathbf{x})$ *is n-th eigenfunction* of the $-\Delta$ operator with **Steklov boundary condition**, *i.e.*

$$\begin{cases} -\Delta\varphi_n = 0 & \text{in } \Omega, \\ \dfrac{\partial\varphi_n}{\partial n} = \lambda_n\varphi_n & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Then any "separable solution" $u(t,\mathbf{x}) = \varphi_n(\mathbf{x})\, W_n(t)$ of [1] must satisfy (on $\partial\Omega$) :

$$\frac{\partial u}{\partial t}(t,x) - \frac{\partial}{\partial n}u(t,x) + b(t)u(t-\tau,\mathbf{x})) = \varphi_n(\mathbf{x})\left(W_n'(t) + \lambda_n W_n(t) + b(t)W_n(t-\tau)\right)$$

which means that $W_n(t)$ must be a solution of the delay equation

$$\begin{cases} W_n'(t) + \lambda_n W_n(t) + b(t)W_n(t-\tau) = 0, \\ W_n(s) = \mu(s), \quad \text{for} \quad s \in (-\tau, 0)\,. \end{cases} \tag{4}$$

We recall now some previous results in the literature for delayed ordinary differential equations. Perhaps, the simplest case corresponds to the *"pure delay"* equation

$$u'(t) = -b(t)u(t-\tau) \quad \text{for } t \geq 0 \tag{5}$$

where $b : [0,\infty) \to \mathbb{R}$ is a locally integrable function with support in $[\tau, T]$, where $T \leq 2\tau$. Since the general solution is given by

$$u(t) = u(0)\left(1 - \int_\tau^t b(s)ds\right), \tag{6}$$

the finite-time extinction property takes place if and only if $\int_\tau^T b(s)ds = 1$. Furthermore, the phenomenon is *global* in the sense that *all solutions become extinct* at the same time. Similarly, the solutions of

$$u'(t) + \lambda u(t) = -b(t)u(t-\tau) \tag{7}$$

satisfy

$$u(t) = e^{-\lambda t}u(0)\left\{1 - e^{\lambda\tau}\int_\tau^t b(s)ds\right\} \text{ for } t \geq 0$$

and therefore all of them vanish for $t \geq 2\tau$ if and only if

$$e^{\lambda\tau}\int_\tau^{2\tau} b(t)dt = 1 \tag{8}$$

Coming back to our problem we get that $W_n(t) = 0$ for $t \geq 2\tau$ if and only if one has the "balance equation"

$$e^{\lambda_n\tau}\int_\tau^{2\tau} b(s)ds = 1 \tag{9}$$

3. **Case of more general delaying coefficients: a "universal function".** Since the previous *balance equation* depends on n, only one "mode" $W_n(t)\varphi_n(x)$ can vanish for $t \geq 2\tau$ with the right choice of $b(t)$. We will see now that, by modifying the delay term in the original equation, keeping its essential structure, several eigenfunctions (in fact, all of them) can be dealt with *at the same time.*

Theorem 3.1. *Let $\tau > 0$ be given and assume that: $\{\tau_k\}$ is an increasing sequence such that $0 < \tau_1 < \tau_2 < \cdots < \tau = \lim \tau_k < \infty$ and $\tau_{k+1} - \tau_k < \tau_1$ for $k = 1, 2, ...$ Let $\{b_k\}$ be a sequence of continuous functions $b_k : \mathbb{R} \longrightarrow \mathbb{R}$ such that support$(b_k) \subset [\tau_k, \tau_{k+1}]$ and that $b_k(t) \geq 0$ or $b_k(t) \leq 0$ on $[\tau_k, \tau_{k+1}]$. Finally, we assume that*

$$\beta_k := \int_{\tau_k}^{\tau_{k+1}} b_k(s)ds$$

satisfy

$$\sum_{k=1}^\infty |\beta_k| < \infty \text{ and } \sum_{k=1}^\infty e^{\lambda_i\tau_k}\beta_k = 1 \quad i = 1, 2, \dots$$

Then the solution of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } (0,+\infty)\times\Omega, \\ \dfrac{\partial u}{\partial t}(t,\mathbf{x}) + \dfrac{\partial u}{\partial n}(t,\mathbf{x}) + \sum_{k=1}^\infty b_k(t)u(t-\tau_k,\mathbf{x}) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ u(s,x) = \mu(s)u_0(x) & \text{on } (-\tau, 0]\times\partial\Omega \end{cases}$$

vanish for $t \geq \tau$ independently of the initial data $\mu(s)$ and $u_0(x)$.

Proof. Let us first mention that the presence of infinitely many delays does not represent any problem as far as the existence and uniqueness of solutions of the ODE

$$u'(t) + \lambda_i u(t) = -\sum_{k=1}^{\infty} b_k(t) u(t - \tau_k), \; i = 1, 2, \dots \tag{10}$$

is concerned (see, *e.g.* [9, Theorem 6.1.1],). This is because the functional on the right-hand side of the equation satisfies for every $\phi(\theta)$, continuous on $[-\tau, 0]$, the following upper estimate

$$\left| -\sum_{k=1}^{\infty} b_k(t)\phi(-\tau_k) \right| \leq \left\{ \sum_{k=1}^{\infty} |b_k(t)| \right\} \max_{\theta\in[-\tau_\infty,0]} |\phi| = m(t) \|\phi\|$$

where the function $m(t) := \sum_{k=1}^{\infty} |b_k(t)|$ is *integrable* since

$$\int_0^{\tau_\infty} m(t)dt = \sum_{k=1}^{\infty} \int_{\tau_k}^{\tau_{k+1}} |b_k(t)| \, dt = \sum_{k=1}^{\infty} |\beta_k| < \infty.$$

by assumption. Observe that $b_k \geq 0$ or ≤ 0 on $[\tau_k, \tau_{k+1}]$ implies that

$$\int_{\tau_k}^{\tau_{k+1}} |b_k(t)| \, dt = \left| \int_{\tau_k}^{\tau_{k+1}} b_k(t) dt \right| = |\beta_k| \, .$$

The usual change of variables $v(t) = e^{\lambda t} u(t)$ gives the transformed equation

$$\begin{aligned} v'(t) = e^{\lambda t} \left(u' + \lambda u \right) &= e^{\lambda t} \left[-\sum_{k=1}^{\infty} b_k(t) e^{-\lambda t(t-\tau_k)} v(t - \tau_k) \right] \\ &= -\sum_{k=0}^{\infty} b_k(t) e^{\lambda \tau_k} v(t - \tau_k) \end{aligned}$$

Then we can proceed by the method of steps: For $t \in [0, \tau_1]$, $v'(t) = 0$ and then $v(t) = v(0)$. For $t \in [\tau_1, \tau_2]$, $v'(t) = -b_1(t) e^{\lambda\tau_1} v(t - \tau_1) = -b_1(t) v(0)$ since $t \in [\tau_1, \tau_2] \Longrightarrow t - \tau_1 \in [0, \; \tau_2 - \tau_1] \subset [0, \tau_1]$ (recall that $\tau_2 - \tau_1 \leq \tau_1$) and then the function $v(t - \tau_1)$ is constant ($= v(0)$). Therefore

$$v(t) = v(\tau_1) - \int_{\tau_1}^{t} b_1(s) e^{\lambda\tau_1} v(s - \tau_1) ds = v(0) \left\{ 1 - e^{\lambda\tau_1} \int_{\tau_1}^{t} b_1(s) ds \right\}$$

and

$$v(\tau_2) = v(0) \left\{ 1 - e^{\lambda\tau_1} \int_{\tau_1}^{\tau_2} b_1(s) ds \right\} = v(0) \left[1 - e^{\lambda\tau_1} \beta_1 \right]$$

In general,

$$v(t) = v(0) \left[1 - e^{\lambda\tau_1} \beta_1 - \; \dots \; - e^{\lambda\tau_k} \beta_k \right] \quad \text{for } t \in [\tau_k, \tau_{k+1}]$$

Hence

$$v(\tau) = \lim_{k \longrightarrow \infty} v(\tau_k) = v(0) \left[1 - \sum_{k=1}^{\infty} e^{\lambda\tau_k} \beta_k \right] = 0$$

by our assumptions on $\{\tau_k\}$ and $\{\beta_k\}$. And since $v'(t) = 0$ for $t \geq \tau$, the solution remains equal to zero, as we wanted to proved.

Finally, coming back to the solution of the elliptic equation, it can be written as

$$u(t, x) = \sum_{n=1}^{\infty} c_n W_n(t) \varphi_n(x)$$

where $\varphi_n(x)$ are the eigenfunctions of the Steklov problem (3) and

$$\sum_{n=1}^{\infty} c_n W_n(0)\varphi_n(x) = u(0,x) = \mu(0)u_0(x) \quad \text{for } x \in \partial\Omega$$

and $W_n(t)$ are solutions of the initial value problem

$$\begin{cases} W_n'(t) + \lambda_n W_n(t) = -\sum_{k=1}^{\infty} b_k(t) W_n(t-\tau_k) & \text{for } t \geq 0 \\ W_n(s) = \mu(s) \quad \text{for } -\tau \leq t \leq 0 \end{cases}$$

But we have just proven that $W_n(t) = 0$ for every $t \geq \tau$. Therefore,

$$u(t,x) = \sum_{n=1}^{\infty} c_n W_n(t)\varphi_n(x) = 0 \quad \text{for } t \geq \tau.$$

and the proof is finished. □

Remark 1. The existence of an ℓ^1 solution $\{\beta_k\}$ to the infinite system of linear equations is not an obvious matter. In the finite case it can be easily proven because the linear system has a *Casorati determinant* which does not vanish due to the linear independence of the functions $e^{\lambda_i z}$. In the 2×2 case there is a direct argument:

$$\begin{vmatrix} e^{\lambda_1\tau_1} & e^{\lambda_1\tau_2} \\ e^{\lambda_2\tau_1} & e^{\lambda_2\tau_2} \end{vmatrix} = e^{\lambda_1\tau_1+\lambda_2\tau_2} - e^{\lambda_1\tau_2+\lambda_2\tau_1} \neq 0$$

since $\lambda_1\tau_1 + \lambda_2\tau_2 - (\lambda_1\tau_2 + \lambda_2\tau_1) = (\lambda_1 - \lambda_2)(\tau_1 - \tau_2) \neq 0$. □

4. **Finite extinction phenomenon for some linear discrete problems with delay.** The above extinction time property is present also in some discrete problems. As in the preceding sections, we can reduce the implicit discretization of the problem to the case of a simpler discrete problem which depends only of a discrete time. To simplify our exposition we assume that the initial function is $u_0(s,\mathbf{x}) = \mu(s)\varphi_n(\mathbf{x})$ for $-\tau \leq s \leq 0,\ x \in \Omega$ (where φ_n is the n-th eigenfunction of the $-\Delta$ operator with a Steklov boundary condition). We assume to be given an arbitrary partition of of $[0,\infty)$ of size $h = \epsilon$, and let $b_\epsilon(t)$ be an approximation of $b(t)$. Consider the associated implicit discrete problem

$$\begin{cases} -\Delta u_\epsilon = 0 & \text{in } (0,+\infty)\times\Omega, \\ \dfrac{u_\epsilon(t,x) - u_\epsilon(t-\epsilon,x)}{\epsilon} + \dfrac{\partial u_\epsilon}{\partial n}(t,x) + b_\epsilon(t)u_\epsilon(t-\tau,x) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ u_\epsilon(s,x) = \mu(s)u_0(x) & \text{on } (-\tau,0]\times\partial\Omega \end{cases} \tag{11}$$

Theorem 4.1. *Let* $b(t) \in C([0,\infty)), b(t) \equiv 0$ *for* $t \in [0,\tau]\cup[2\tau,+\infty)$, *such that* $\int_0^\infty b(s)ds = 1$. *Then,*
i) there exists a function b_ϵ, *constant in each subinterval* $((n-1)h, h)$, *such that* $b_\epsilon \longrightarrow b$ *in* $L^2(0,T)$, *for any* $T > 0$, *as* $\epsilon \longrightarrow 0$, $b_\epsilon(s) \equiv 0$ *on* $[0,(n-1)h] \cup [kh,+\infty)$, $k > (n-1)$, *and* $\int_{(n-1)h}^{kh} b_\epsilon(s)ds = h\sum_{i=(n-1)h}^{kh} b_\epsilon(i) = 1$,
ii) $u_\epsilon \to u$ *on* $C([0,T] : L^2(\partial\Omega))$ *(and so at least in* $C([0,T] : L^2(\Omega)))$ *for any* $T > 0$, *as* $\epsilon \longrightarrow 0$,
iii) $u_\epsilon(t,x) \equiv 0$ *on* $\overline{\Omega}$ *after a finite time of iterations.*

Proof. i) For each step of the partition we can choose two different values for the b, the supremum and the infimum. The corresponding sums are in the following relation

$$1 = \int_0^\infty b(s)ds < h \sum_{i=(n-1)h}^{kh} \sup b(i), \text{ and } 1 = \int_0^\infty b(s)ds > h \sum_{i=(n-1)h}^{kh} \inf b(i)$$

So, for each point i of the partition we can choose a value $b_\epsilon(i)$ (different, in general, of $b(i)$) such that $\inf b(i) < b_\epsilon(i) < \sup b(i)$, which allows us to construct a step function $b_\epsilon(s)$, fulfilling the properties of i). The proof of ii) is an easy consequence of the fact that the abstract operator associated to the problem with $b \equiv 0$ is the subdifferential of a convex function (see [2]). Then, when $b \neq 0$, since on the interval $[0, \tau]$ the term $b_\epsilon(t)u_\epsilon(t-\tau, x)$ is known, we can apply the Crandall-Ligget's theorem leading to the convergence result. Finally to prove iii) we use the assumption on the initial datum and argue as in the preceding section. If we assume, for simplicity, that $\lambda_n \equiv 0$ (the general case, $\lambda_n \neq 0$ is absolutely similar), then we reduce the problem to the discrete dynamical system

$$\left\{ \begin{array}{l} \dfrac{u_\epsilon(t) - u_\epsilon(t-\epsilon)}{\epsilon} = -b_\epsilon(t)u_\epsilon(t-\tau) = 0, \ t \geq 0 \\ u_\epsilon(s) = \mu(s), \quad for \quad s \in (-\tau, 0)\,. \end{array} \right. . \tag{12}$$

Thus, $u_\epsilon(t) = u_\epsilon(t-\epsilon) - \epsilon b_\epsilon(t)u_\epsilon(t-\tau)$. Since $u_\epsilon(t) = u_{\epsilon 0}$ for $t \in [0, \tau]$, on $[\tau, 2\tau]$ we have

$$\begin{aligned} u_\epsilon(t-\epsilon) &= u_\epsilon(t-2\epsilon) - \epsilon\lambda b_\epsilon(t-\epsilon)u_{\epsilon 0}, \\ u_\epsilon(t-2\epsilon) &= u_\epsilon(t-3\epsilon) - \epsilon\lambda b_\epsilon(t-2\epsilon)u_{\epsilon 0} \\ u_\epsilon(t-3\epsilon) &= u_\epsilon(t-4\epsilon) - \epsilon\lambda b_{\epsilon\epsilon}(t-3\epsilon)u_{\epsilon 0}, ... \end{aligned}$$

and so

$$u_\epsilon(t) = u_{\epsilon 0} - \epsilon\left[b_\epsilon(0) + b_\epsilon(t-n\epsilon) + ... + b_\epsilon(t-\epsilon) + b_\epsilon(t)\right]u_{\epsilon 0} = \tag{13}$$

$$= u_{\epsilon 0}\left\{1 - \epsilon\left[b_\epsilon(0) + b_\epsilon(t-n\epsilon) + ... + b_\epsilon(t-\epsilon) + b_\epsilon(t)\right]\right\} \tag{14}$$

Observe that the term $\epsilon\left[b_\epsilon(0) + b_\epsilon(t-n\epsilon) + ... + b_\epsilon(t-\epsilon) + b_\epsilon(t)\right]$ is a (sort of) partial sum of the integral in (6). Thus, by property i) we get $u_\epsilon(t, x) \equiv 0$ on $\overline{\Omega}$ after a finite time of iterations.

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