

Homogenization of the p -Laplacian with Nonlinear Boundary Condition on Critical Size Particles: Identifying the Strange Term for the Some non Smooth and Multivalued Operators¹

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Abstract—We extend previous papers in the literature concerning the homogenization of Robin type boundary conditions for quasilinear equations, in the case of microscopic obstacles of critical size: here we consider nonlinear boundary conditions involving some maximal monotone graphs which may correspond to discontinuous or non-Lipschitz functions arising in some catalysis problems.

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Papers [2–10] were devoted to the study of asymptotic behavior of the solution to the boundary value problem for the p -Laplacian ($p \in [2, n)$) in ε -periodically perforated domain with nonlinear Robin-type boundary condition that contains function $\sigma(x, u)$. It was supposed there that $\sigma(x, u)$ is a smooth function of its arguments, monotone by variable u . In this paper we extend the method introduced in [3, 4, 7–10] to deal with the problems with more general conditions on the function $\sigma(x, u)$. As in all papers in which the holes are of critical size and the adsorption parameter has a critical power of ε (we will precise this later) we observe a change in the nature of the nonlinearity. Our aim is to present this change in the case $\sigma(u) = C|u|^{q-1}u$, $0 < q < 1$, which is not differentiable at 0, and in the case when σ is the maximal monotone operator for the Heaviside function, which is a multivalued operator, and $p \in [2, n)$. In a further paper [12] we extend the arguments to the case of general maximal monotone graphs σ and $p \in (1, n)$.

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Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$. Denote by G_0 the unit ball centered at the origin. For $\delta > 0$ and $0 < \varepsilon \ll 1$ we define sets $\delta B = \{x | \delta^{-1}x \in B\}$ and $\tilde{\Omega}_\varepsilon = \{x \in \Omega | \rho(x, \partial\Omega) > 2\varepsilon\}$. Let $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$ and C_0 is positive number. Define

$$G_\varepsilon = \bigcup_{j \in Y_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in Y_\varepsilon} G_\varepsilon^j,$$

where $Y_\varepsilon = \{j \in \mathbb{Z}^n | (a_\varepsilon G_0 + \varepsilon j) \cap \tilde{\Omega}_\varepsilon \neq \emptyset\}$, $|Y_\varepsilon| \cong d\varepsilon^{-n}$, $d = \text{const} > 0$, \mathbb{Z}^n is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in Y_\varepsilon$ and note that $\bar{G}_\varepsilon^j \subset Y_\varepsilon^j$ and center of the ball G_ε^j coincides with the center of the cube Y_ε^j . Define

$$\Omega_\varepsilon = \Omega \setminus \bar{G}_\varepsilon, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma} \sigma_q(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where $\Delta_p u \equiv \text{div}(|\nabla u|^{p-2} \nabla u)$, $p \in [2, n)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2} (\nabla u, \nu)$, ν is the outward unit normal vector to S_ε , $\gamma = \alpha(p-1)$. We suppose that $f \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$.

Function $\sigma_q(\lambda)$, $q \in [0, 1]$, is the maximal monotone continuous mapping [11], that depends on parameter q

$$\sigma_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda^q, & \lambda \in (0, 1), \\ 1, & \lambda > 1. \end{cases} \quad (2)$$

We note that $\sigma_0(\lambda)$ is the maximal monotone mapping for the Heaviside function, i.e. multivalued function

$$\sigma_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ [0, 1], & \lambda = 0, \\ 1, & \lambda > 0. \end{cases} \quad (3)$$

Boundary conditions of this type correspond to the presence of so called chemical reaction of order q on the boundary of cavities [5, 6]. The motivation to truncate the powers comes from the chemical modeling, in which concentrations impose range in $[0, 1]$, but it also corresponds to the case $f \in L^\infty(\Omega)$, for which the solution is bounded.

Applying monotonicity tools (see, e.g., [11]) it is easy to see that problem (1) is equivalent to ask for $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, satisfying the integral inequality

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla(\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi_q(\phi) - \Psi_q(u_\varepsilon)) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \quad (4)$$

for any arbitrary function $\phi \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, where $\Psi_q(\lambda)$ for $q \in (0, 1]$ is the primitive of σ_q . For $q \in (0, 1]$ we have that

$$\Psi_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \frac{\lambda^{q+1}}{q+1}, & \lambda \in (0, 1), \\ \lambda - \frac{q}{q+1}, & \lambda > 1 \end{cases} \quad (5)$$

and if $q = 0$ then

$$\Psi_0(\lambda) = \begin{cases} 0, & \lambda \leq 0, \\ \lambda, & \lambda > 0. \end{cases} \quad (6)$$

Space $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the closure in $W^{1,p}(\Omega_\varepsilon)$ of the set of infinitely differentiate functions in $\bar{\Omega}_\varepsilon$, that vanish near the boundary $\partial\Omega$.

It is well known that problem (1) has unique weak solution (see., e.g. [1, Theorem 8.5]). The following estimation is valid

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^p + \varepsilon^{-\gamma} \|u_\varepsilon\|_{L^1(S_\varepsilon \cap \{x|u_\varepsilon > 1\})} \leq K, \quad (7)$$

where constant K here and below is independent of ε .

Let $H_q(\lambda)$ be the function given through functional equation

$$B_0 |H_q|^{p-2} H_q \in \sigma_q(\lambda - H_q), \quad (8)$$

where $B_0 = \text{const} > 0$. Note that, for any prescribed λ equation (8) has unique solution. In the case if $q = 0$

$$H_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda, & 0 < \lambda < B_0^{\frac{1}{p-1}}, \\ B_0^{\frac{1}{p-1}}, & \lambda > B_0^{\frac{1}{p-1}} \end{cases} \quad (9)$$

and if $q \in (0, 1]$ then

$$H_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ (b_q)^{-1}(\lambda), & 0 < \lambda < 1 + B_0^{\frac{1}{p-1}}, \\ B_0^{\frac{1}{p-1}}, & \lambda > 1 + B_0^{\frac{1}{p-1}}, \end{cases} \quad (10)$$

where $b_q(s)$ is the strictly monotone function given for $s \geq 0$ by

$$b_q(s) \equiv s + B_0^{\frac{1}{p-1}} s^q = \lambda. \quad (11)$$

Note that, in both cases, $H_q(\lambda)$ is bounded and Lipschitz continuous.

Denote by \tilde{u}_ε a $W^{1,p}$ -extension of function u_ε (see [10]). Using estimate (7) we get following inequality

$$\|\tilde{u}_\varepsilon\|_{W^{1,p}(\Omega)} \leq K. \quad (12)$$

Therefore, there exists a subsequence (denoted as the original sequence), such that $\varepsilon \rightarrow 0$:

$$\tilde{u}_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega). \quad (13)$$

The following theorem gives a description of the limit function u .

Theorem 1. *Let $n \geq 3$, $p \in [2, n)$, $q \in [0, 1]$, $\alpha = \frac{n}{n-p}$, $\gamma = \alpha(p-1)$ and u_ε is the weak solution of the problem (1). Suppose that $H_q(\lambda)$ is the function given by equation (8), in which $B_0 = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$. Then the limit function u , introduced in (13), is the weak solution of the problem*

$$\begin{cases} -\Delta_p u + A(n, p) |H_q(u)|^{p-2} H_q(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (14)$$

which is understood as a function $u \in W_0^{1,p}(\Omega)$ satisfying the variational inequality

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A(n, p) \int_{\Omega} |H_q(v)|^{p-2} H_q(v) (v - u) dx \geq \int_{\Omega} f(v - u) dx,$$

for an arbitrary function $v \in W_0^{1,p}(\Omega)$. Here,

$$A(n, p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p} \omega_n,$$

with ω_n the surface area of the unit sphere in \mathbb{R}^n .

Proof. (a) Consider the case $q = 0$. Denote $B_1 = B_0^{-\frac{1}{p-1}}$. Note that the integral inequality in the case when $q = 0$ has the form

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\phi^+ - u_\varepsilon^+) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \tag{15}$$

where ϕ^+ is the positive part of function ϕ , $\phi = \phi^+ - \phi^-$. From (15) we conclude

$$\varepsilon^{-\gamma} \|u_\varepsilon^+\|_{L^1(S_\varepsilon)} \leq K. \tag{16}$$

Using the monotonicity of function $|\lambda|^{p-2}\lambda$ for $p > 1$ we derive inequality for u_ε

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla (\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\phi^+ - u_\varepsilon^+) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \tag{17}$$

that is valid for an arbitrary function $\phi \in W_0^{1,p}(\Omega)$.

Let us take a test function in inequality (17)

$$\phi(x) = v(x) - W_\varepsilon(x) H_0(v(x)),$$

where $v \in C_0^\infty(\Omega)$, $H_0(\lambda)$ is given by the formula (9), function $W_\varepsilon(x)$ is defined as follows

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \quad j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in \Upsilon_\varepsilon} T_\varepsilon^j. \end{cases} \tag{18}$$

Here, T_ε^j is the ball of radius $\varepsilon/4$ center of which coincides with the center P_ε^j of G_ε^j ,

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}{a_\varepsilon^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}.$$

Note that

$$W_\varepsilon \rightharpoonup 0 \quad \text{weakly in } W^{1,p}(\Omega) \tag{19}$$

as $\varepsilon \rightarrow 0$. Using that

$$v = H_0(v) \Leftrightarrow v \in [0, B_1],$$

$$v < H_0(v) \Leftrightarrow v < 0,$$

$$v > H_0(v) \Leftrightarrow v > B_1$$

and $\phi^+|_{S_\varepsilon} = (v - H_0(v))^+ = v - H_0(v)$ if $v > H_0(v)$ we get

$$\varepsilon^{-\gamma} \int_{S_\varepsilon} (\phi^+ - u_\varepsilon^+) ds \leq \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds. \tag{20}$$

Substituting introduced test function in inequality (17) and using (19) and (20), we get that the limit as $\varepsilon \rightarrow 0$ of the left-hand side of the inequality (17) doesn't exceed the limit of

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla (v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds - \sum_{j \in \Upsilon_\varepsilon \cap \partial G_\varepsilon^j} \int |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_0|^{p-2} \times H_0(v - H_0(v) - u_\varepsilon) ds - \sum_{j \in \Upsilon_\varepsilon \cap \partial T_\varepsilon^j} \int |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_0|^{p-2} H_0(v - u_\varepsilon) ds. \tag{21}$$

The limit of the right-hand side of inequality (17) is equal to

$$\int_{\Omega} f(v - u) dx. \tag{22}$$

Consider the integrals over S_ε , included in the expression (21):

$$\begin{aligned} & - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_0|^{p-2} H_0(v - H_0(v) - u_\varepsilon) ds \\ & + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds \\ & = \varepsilon^{-\gamma} \frac{B_0}{(1 - \alpha_\varepsilon)} \int_{S_\varepsilon} |H_0|^{p-2} H_0(v - H_0(v) - u_\varepsilon) ds + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds \end{aligned}$$

$$\begin{aligned}
 &= -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds \\
 &- \varepsilon^{-\gamma} B_0 \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) (v - H_0(v) - u_\varepsilon^+) ds \quad (23) \\
 &- \frac{B_0 \varepsilon^{-\gamma} \alpha_\varepsilon}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) (v - H_0(v) - u_\varepsilon^+) ds \\
 &- \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds \\
 &= -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds \\
 &- B_0 \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} B_0^{-1} (v - B_1 - u_\varepsilon^+) ds \\
 &+ \varepsilon^{-\gamma} B_0 \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} v^{p-1} u_\varepsilon^+ ds \\
 &- \frac{\alpha_\varepsilon B_0 \varepsilon^{-\gamma}}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) (v - H_0(v) - u_\varepsilon^+) ds \\
 &+ \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > B_1\}} (v - B_1 - u_\varepsilon^+) ds - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ ds \\
 &= -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds \\
 &- \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ (1 - B_0 v^{p-1}(x)) ds \\
 &- \frac{\alpha_\varepsilon B_0 \varepsilon^{-\gamma}}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) (v - H_0(v) - u_\varepsilon^+) ds \\
 &= J^\varepsilon - \frac{\alpha_\varepsilon B_0 \varepsilon^{-\gamma}}{1-\alpha_\varepsilon} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) (v - H_0(v) - u_\varepsilon^+) ds,
 \end{aligned}$$

where

$$\begin{aligned}
 J^\varepsilon &\equiv -\varepsilon^{-\gamma} \frac{B_0}{(1-\alpha_\varepsilon)} \int_{S_\varepsilon} |H_0(v)|^{p-2} H_0(v) u_\varepsilon^- ds \\
 &- \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, B_1)\}} u_\varepsilon^+ (1 - B_0 v^{p-1}) ds \leq 0, \quad (24)
 \end{aligned}$$

and $\alpha_\varepsilon \rightarrow 0$ if $\varepsilon \rightarrow 0$.

Using that $\varepsilon^{-\gamma} \|u_\varepsilon^+\|_{L^1(S_\varepsilon)} \leq K$, we get that the limit of the expression (21) doesn't exceed the limit of the following expression

$$\begin{aligned}
 &\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx \\
 &- \sum_{j \in Y_\varepsilon \partial T_\varepsilon^j} \int |\nabla w_\varepsilon^j|^{p-2} \partial_{\nu_p} w_\varepsilon^j |H_0(v)|^{p-2} H_0(v) (v - u_\varepsilon) ds. \quad (25)
 \end{aligned}$$

Using an equality proved in [3] we get

$$\begin{aligned}
 &\lim_{\varepsilon \rightarrow 0} \sum_{j \in Y_\varepsilon \partial T_\varepsilon^j} \int |\nabla w_\varepsilon^j|^{p-2} \partial_{\nu_p} w_\varepsilon^j |H_0(v)|^{p-2} H_0(v) (v - u_\varepsilon) ds \\
 &= -A(n, p) \int_{\Omega} |H_0(v)|^{p-2} H_0(v) (v - u) dx. \quad (26)
 \end{aligned}$$

It follows from (20)–(26) that u satisfies inequality, for any $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}
 &\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A(n, p) \\
 &\times \int_{\Omega} |H_0(v)|^{p-2} H_0(v) (v - u) dx \geq \int_{\Omega} f(v - u) dx. \quad (27)
 \end{aligned}$$

Taking $v = u + \lambda w$, with $w \in W_0^{1,p}(\Omega)$ arbitrary, and making $\lambda \rightarrow 0$, since H_0 is Lipschitz continuous and bounded, we get that u is a weak solution of problem (14) for $q = 0$ in the usual sense.

(b) Now we consider the case $q \in (0, 1]$. In this case we make similar reasoning. We set $\phi = v - W_\varepsilon H_q(v)$ in inequality (17) as a test function, where $H_q(\lambda)$ is defined by (10). Further, we only need to explain the method of the comparison of the integrals over S_ε , included in the obtained variational inequality. Note that in this case variational inequality has the form

$$\begin{aligned}
 &\int_{S_\varepsilon} |\nabla \phi|^{p-2} \nabla \phi (\nabla \phi - \nabla u_\varepsilon) dx + \varepsilon^{-\gamma} \\
 &\times \int_{S_\varepsilon} (\Psi_q(\phi) - \Psi_q(u_\varepsilon)) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \quad (28)
 \end{aligned}$$

where

$$\begin{aligned}
 &\varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi_q(\phi) - \Psi_q(u_\varepsilon)) ds \\
 &\leq \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \in (0, 1)\}} \left(\frac{(v - H_q(v))^{q+1}}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} \right) ds \\
 &+ \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon \leq 0\}} \frac{(v - H_q(v))^{q+1}}{q+1} ds \\
 &+ \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1+B_1)\} \cap \{u_\varepsilon > 1\}} \left(\frac{(v - H_q(v))^{q+1}}{q+1} - u_\varepsilon + \frac{q}{q+1} \right) ds \quad (29) \\
 &+ \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1+B_1\} \cap \{u_\varepsilon \in (0, 1)\}} \left(v - H_q(v) - \frac{q}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} \right) ds
 \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon \leq 0\}} \left(v - H_q - \frac{q}{q+1} \right) ds \\
 & + \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon > 1\}} \left(v - H_q - \frac{q}{q+1} - u_\varepsilon + \frac{q}{q+1} \right) ds.
 \end{aligned}$$

We substitute the test function in (17) and we consider the remaining integrals over S_ε in the left-hand side of variational inequality (17):

$$\begin{aligned}
 & - \sum_{j \in \Upsilon_\varepsilon} \int_{G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_q|^{p-2} H_q (v - H_q(v) - u_\varepsilon) ds \\
 & = - \varepsilon^{-\gamma} \int_{S_\varepsilon} B_0 H_q^{p-1}(v) (v - H_q(v) - u_\varepsilon^+) ds \\
 & - \frac{B_0 \varepsilon^{-\gamma} \alpha_\varepsilon}{1 - \alpha_\varepsilon} \int_{S_\varepsilon} H_q^{p-1}(v) (v - H_q(v) - u_\varepsilon^+) ds \\
 & - \varepsilon^{-\gamma} \frac{B_0}{1 - \alpha_\varepsilon} \int_{S_\varepsilon} H_q^{p-1} u_\varepsilon^- ds.
 \end{aligned} \tag{30}$$

Note that

$$\begin{aligned}
 & - \varepsilon^{-\gamma} \int_{S_\varepsilon} B_0 H_q^{p-1}(v) (v - H_q(v) - u_\varepsilon^+) ds \\
 & = - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon \in (0, 1)\}} (v - H_q(v))^q \\
 & \quad \times (v - H_q(v) - u_\varepsilon) ds \\
 & - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon > 1\}} (v - H_q(v))^q (v - H_q(v) - u_\varepsilon) ds \\
 & - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon \leq 0\}} (v - H_q(v))^q ds \\
 & - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon \leq 0\}} (v - B_1) ds \\
 & - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon \in (0, 1)\}} (v - B_1 - u_\varepsilon) ds \\
 & - \varepsilon^{-\gamma} \int_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon > 1\}} (v - B_1 - u_\varepsilon) ds.
 \end{aligned} \tag{31}$$

Next we compare integrals over the same subsets of S_ε in the left-hand side of inequality (17). We have

$$\begin{aligned}
 I_\varepsilon & \equiv \varepsilon^{-\gamma} \int_{M_\varepsilon} \left\{ \frac{(v - H_q(v))^{q+1}}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} \right. \\
 & \left. - (v - H_q(v))^{q+1} + (v - H_q(v))^q u_\varepsilon \right\} ds \\
 & = - \varepsilon^{-\gamma} \int_{M_\varepsilon} \left(\frac{q(v - H_q(v))^{q+1}}{q+1} + \frac{u_\varepsilon^{q+1}}{q+1} \right. \\
 & \quad \left. - u_\varepsilon (v - H_q(v))^q \right) ds,
 \end{aligned} \tag{32}$$

where $M_\varepsilon = S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon \in (0, 1)\}$. Using Young's inequality we get

$$u_\varepsilon (v - H_q(v))^q \leq \frac{q(v - H_q(v))^{q+1}}{q+1} + \frac{u_\varepsilon^{q+1}}{q+1}. \tag{33}$$

Therefore,

$$I_\varepsilon \leq 0. \tag{34}$$

The remaining integrals over subsets of S_ε are considered in the similar way and we establish that the sum of all integrals over the corresponding subsets of S_ε is non-positive. Therefore, the limit function u satisfy variational inequality

$$\begin{aligned}
 & \int_\Omega |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A(n, p) \\
 & \times \int_\Omega H_q^{p-1}(v) (v - u) dx \geq \int_\Omega f(v - u),
 \end{aligned}$$

for an arbitrary function $v \in W_0^{1,p}(\Omega)$. Again taking $v = u + \lambda w$, with $w \in W_0^{1,p}(\Omega)$ arbitrary, and making $\lambda \rightarrow 0$ we obtain that u is a weak solution the usual sense.

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