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SOLUTIONS WITH COMPACT SUPPORT FOR SOME DEGENERATE PARABOLIC PROBLEMS

J. ILDEFONSO DÍAZ DÍAZ
Departamento de Ecuaciones Funcionales, Facultad de Ciencias Matematicas,
Universidad Complutense de Madrid, Madrid, Spain

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1. INTRODUCTION

IT IS a well known fact that the mixed Cauchy problem for the heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) = 0 & \text{for } (t,x) \in (0,T) \times (0,+\infty) \\ u(t,0) = g(t) & \text{for } t \in (0,T) \\ u(0,x) = u_0(x) & \text{for } x \in (0,+\infty) \end{cases}$$

describes a process where disturbances move at infinite speed in the following sense: if $u(0, x)$ is a function of x with compact support, then this no longer holds for any $t > 0$.

However, the situation is different for some nonlinear parabolic problems of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2}{\partial x^2}\beta(u(t,x)) = 0 & \text{for } (t,x) \in (0,T) \times (0,+\infty) \\ u(t,0) = g(t) & \text{for } t \in (0,T) \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1.1}$$

where $\beta \in C^1(\mathbb{R})$ satisfies $\beta'(r) \geqslant 0$ for all $r \in \mathbb{R}$. This type of problem appears in the study of phenomena such as filtration of fluids through porous media and the flux of heat through a solid with a temperature depending on the thermal conductivity. There is a series of papers, published in the fifties, Zeldovich–Kompaneste [1], Barenblat [2], Barenblat–Vishik [3] and finally Oleinik–Kalashikov–Yui Lin [4], which prove that for every $t > 0$ the (generalized) solution $u(x, t)$ has compact support as a function of x, provided that u_0 is non-negative and has compact support and β satisfies:

$$\begin{cases} \beta \in C^1([0,+\infty)) \cap C^6((0,+\infty)) \\ \beta'(r) > 0,\ \beta''(r) > 0 \quad \text{if} \quad r > 0 \\ \beta'(0) = \beta(0) = 0 \end{cases} \tag{1.2}$$

$$\beta(r) \to +\infty \quad \text{if} \quad r \to +\infty \tag{1.3}$$

and

$$\int_0^r \frac{\beta'(s)}{s} \mathrm{d}s < +\infty \quad \text{for} \quad r > 0. \tag{1.4}$$

Later Kalashnikov [5] proved that under hypothesis (1.2) and (1.3), condition (1.4) is also necessary for the compactness of the above mentioned support.

More recently, Peletier [6] has obtained the preceding results under weaker hypothesis of regularity. He substitutes (1.2) by

$$\begin{cases} \beta \in C^1([0, +\infty)) \cap C^{2+\alpha}((0, +\infty)) & \text{for some } \alpha \in (0, 1] \\ \beta'(r) > 0 \quad \text{if} \quad r > 0 \\ \beta'(0) = \beta(0) = 0. \end{cases} \tag{1.5}$$

Under these same hypothesis, Knerr [7] has obtained several estimates on the evolution of the support of $u(t, \cdot)$

Problem (1.1) belongs to the class of degenerate parabolic problems. This means that the equation loses its parabolic character for some values of the unknown variable u.

The interest of this type of problems has increased considerably since very different problems such as the Stefan (Brezis [8], [9], Damlamian [10], [11]) and some "unilateral problems" (Brezis [12]) have been reformulated in a unified way as degenerate parabolic in a wider sense. This has been done in the context of the m-accretive operators theory (Brezis [13], [9], Benilan [14], Konishi [15]). More precisely, this unified formulation is

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) - \Delta\beta(u(t, x)) \ni f(t, x) & \text{on} \quad Q = (0, T) \times \Omega \\ \beta(u(t, x)) \ni g(t, x) & \text{on} \quad \Sigma = (0, T) \times \Gamma \\ u(0, x) = u_0(x) & \text{on} \quad \Omega \end{cases}$$

where Ω is a regular open set in $\mathbb{R}^N (N \geqslant 1)$, $\Gamma = \partial\Omega$, $T < +\infty$ and β a maximal monotone graph of $\mathbb{R}^2$ on which one can assume $0 \in \beta(0)$ without any loss of generality†. We shall denote this problem by $P(f, g, u_0, Q)$ or by P_D if no more precision is necessray.

Problems with non-linear Neumann boundary conditions such as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) - \Delta\beta(u(t, x)) \ni f(t, x) & \text{on} \quad Q \\ -\dfrac{\partial \beta}{\partial n}(u(t, x)) \in \gamma(u(t, x)) & \text{on} \quad \Sigma \\ u(0, x) = u_0(x) & \text{on} \quad \Omega \end{cases}$$

† A given mapping $\beta\colon \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is termed a "maximal monotone operator (or graph in $\mathbf{R}^2$)" if it satisfies (i) $(x - y).(a - b) \geqslant 0 \quad \forall x \in \beta(a), \forall y \in \beta(b)$: (ii) there is not other $\beta^*\colon \mathbf{R} \to \mathscr{P}(\mathbf{R})$ for which (i) holds and such that $\beta(a) \subset \beta^*(a)$ $\forall a \in \mathbf{R}$.

It will be useful to have in mind that every maximal monotone graph of $\mathbf{R}^2$ is given by: $\beta(r) = (-\infty, \theta(r+)]$ if $\theta(r-) = -\infty$, $\beta(r) = [\theta(r-), \theta(r+)]$ if $-\infty < \theta(r-) \leqslant \theta(r+) < +\infty$ and $\beta(r) = [\theta(r+), +\infty)$ if $\theta(r+) = +\infty$, for some non-decreasing function $\theta\colon \mathbf{R} \to \mathbf{R}$ (Brezis [13]).

(where $\partial/\partial n$ is the outward normal derivative and γ is another maximal monotone graph of $\mathbb{R}^2$ such that $0 \in \gamma(0)$), have also been already studied by several authors, (Friedman [16], Benilan [14], Damlamian [10]). We shall denote this problem by $P(f, \gamma, u_0, Q)$ or by P_N if no confusion arises.

The aim of this paper is to find out, for which conditions on β (β and γ, respectively) problem P_D (P_N) describes a finite speed propagation process, in the sense that compact support data provide compact support solutions $u(t, x)$, as functions of x for any $t \in [0, T]$. Ω is assumed to be unbounded regular and open in $\mathbb{R}^n$.

One of the chief applications of the compactness of the support of the solution (to be established *a priori*), is the extension to the case of unbounded Ω of existence results known only for bounded domains. Namely we will get under convenient hypothesis, that the stated problems may be considered formulated in bounded open sets, "inheriting" in this way a large number of already established properties for this case, such as regularity, maximum principle theorems and so on. (See, for instance, Aronson [17], Benilan [14], Brezis [9], Evans [18], Lions [19], etc.).

The rest of this paper can be summarized as follows. The main results are stated and commented with a few remarks in section 2. Some comparison results are proved in section 3. They are closely related to the uniqueness of P_D and P_N and will be systematically used in Section 4 where the main results are proved, considering previously the simple case $\Omega = (0, +\infty)$ and $f = 0$.

2. PREVIOUS DEFINITIONS AND STATEMENT OF THE MAIN RESULTS

In this paper we will stick throughout to the concept of solutions of problems P_D and P_N contained in the definitions to be given later and motivated by already known existence theorems (valid only for a bounded Ω) such as the two following†:

THEOREM A. (Damlamian [10].)

Let Ω be a bounded open regular set in $\mathbb{R}^n$. Let $f \in L^2(0, T: L^2(\Omega))$, $u_0 \in L^1(\Omega)$, $g \in V.B.\ (0, T: L^\infty(\Omega)) \cap L^2(0, T: H^1(\Omega))$ and suppose that

$$D(\beta) = R(\beta) = \mathbb{R}. \tag{2.1}$$

Then there exists a unique function $u \in C([0. T]: H^{-1}(\Omega)) \cap L^\infty(0, T: L^1(\Omega))$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, .) - \Delta w(t, .) = f(t, .) & \text{for a.e. } t \in (0, T) \text{ in } H^{-1}(\Omega) \\ w(t, x) = g(t, x) & \text{for a.e. } (t, x) \in \Sigma \\ u(0, x) = u_0(x) & \text{for a.e. } x \in \Omega \end{cases} \tag{2.2}$$

where w is a function satisfying

$$\begin{cases} w \in L^2(0, T: H^1(\Omega)), \quad \text{and} \\ w(t, x) \in \beta(u(t, x)) \quad \text{for a.e. } (t, x) \in Q. \end{cases} \tag{2.3}$$

† We assume the reader to be familiar with Sobolev Spaces $W^{k,2}(\Omega) = H^k(\Omega)$, $W_0^{k,2}(\Omega) = H_0^k(\Omega)$ for $k \geqslant 0$ a real number, $H^{-1}(\Omega)$ the dual topological spaces of $H_0^1(\Omega)$, and also with the spaces of vector valued functions $W^{m,p}(0, T; W^{s,q}(\Omega))$ for $m \in N$, $s \in \mathbb{R}$, $1 \leqslant p \leqslant +\infty$ and $1 \leqslant q \leqslant +\infty$, (see, e.g. Adams [20]).

The statement of some results of existence for the problem P_N requires some previous considerations. Such a problem can be properly treated in the framework of the m-accretive operators on $L^1(\Omega)$. In fact, in Benilan [21] it is shown that if Ω is a regular bounded open set and β and γ are maximal monotone graphs such that

$$D(\beta) + R(\gamma) = \mathbb{R} \tag{2.4}$$

then the operator $A = \{[u, v] \in (L^1(\Omega))^2 \,|\, \exists w \in W^{1,1}(\Omega)$ and $\exists z \in L^1(\Gamma)$ such that $w(x) \in \beta(u(x))$ a.e. $x \in \Omega$, $z(x) \in \gamma(u(x))$ a.e. $x \in \Gamma$ and such that $\forall \phi \in C^1(\overline{\Omega})$ on has

$$\int_\Omega v.\phi \, dx = \int_\Omega \operatorname{grad} \phi . \operatorname{grad} w \, dx + \int_\Gamma z.\phi \, d\Gamma \}$$

is m-accretive on $L^1(\Omega)$ with dense domain in $L^1(\Omega)$.

In this situation the Abstract Cauchy Problem

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni f(t) & t \in (0, T) \\ u(0) = u_0 \end{cases} \tag{A.C.P.}$$

can be handled, getting in this way the existence of a unique $u \in C([0, T] : L^1(\Omega))$ "integral solution" (see [21]), assuming $u_0 \in L^1(\Omega)$ and $f \in L^1(0, T : L^1(\Omega))$. Such a solution is characterized in [21] in the following way:

$$u(t) \to u_0 \text{ in } L^1(\Omega) \text{ when } t \to 0 \text{ essentially} \tag{2.5}$$

$$\begin{cases} \text{for all } g \in C(\overline{\Omega}) \text{ such that } \exists w_g \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \text{ with } w_g(x) \in \beta(g(x)) \text{ a.e. } x \in \Omega, \text{ such that} \\ \quad -\dfrac{\partial w_g}{\partial n} \in \gamma(g) \text{ a.e. in } \Gamma, \text{ and for all } \Theta \in \mathscr{D}^+((0, T)), \text{ there exists } \alpha \in L^\infty(Q) \text{ such that} \\ \alpha \in \operatorname{sign}(u - g) \text{ a.e. in } Q \text{ and } \displaystyle\int_Q \alpha.\{(u - g).\Theta' + (f - \Delta w_g).\Theta\} \, dx \, dt \geqslant 0. \end{cases} \tag{2.6}$$

It is clear that the condition (2.6) is not very handy in practice. For this reason it is interesting to emphasize some more precise results which are compiled in the following theorem.

THEOREM B. (Benilan [21].)

(a) Let us assume that β and β^{-1} are locally Lipschitzian, that β and γ satisfy (2.4) and that $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$. Then the integral solution u of the (A.C.P.) is a strong solution in $L^2(\Omega)$, that is:

$$u \in W^{1,2}(0, T : L^2(\Omega)) \quad \text{and} \quad w \in L^2(0, T : H^2(\Omega)) \quad \text{with} \quad w(t, x) \in \beta(u(t, x)) \quad \text{a.e.} \quad (t, x) \in Q$$

such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta w(t, \cdot) = f(t, \cdot) \text{ a.e. } t \in (0, T) \text{ in } L^2(\Omega) \\ -\dfrac{\partial w}{\partial n} \in \gamma(u) & \text{a.e. in } \Sigma \\ u(0, x) = u_0(x) & \text{a.e. in } \Omega. \end{cases}$$

(b) Let us assume $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$, $D(\gamma) = \mathbb{R}$, and both β and γ satisfy (2.4). Then the integral solution of the (A.C.P.) is such that $\exists w \in L^2(0, T: H^1(\Omega))$ such that $w(t,x) \in \beta(u(t,x))$ a.e. in Q and also u is a strong solution in $H^{-1}(\Omega)$ of $P(f, g, u_0, Q)$ being $g = w|_\Sigma$.

We define, regardless of the boundedness of Ω:

Definition 1. We say that a function $u \in C([0, T]: H^{-1}(\Omega)) \cap L^\infty((0, T: L^1(\Omega))$ is a "strong solution in $H^{-1}(\Omega)$ of $P(f, g, u_0, Q)$" if it satisfies (2.2) and (2.3).

Definition 2. We say that a function $u \in C([0, T]: L^1(\Omega))$ is an "integral solution in $L^1(\Omega)$ of $(P(f, \gamma, u_0, Q)$" if it satisfies (2.5) and (2.6).

The two main results of this paper can be formulated as follows.

THEOREM I. Let $f \in L^\infty(0, T: L^\infty(\Omega))$, $g \in V.B.(0, T: L^\infty(\Omega)) \cap L^2(0, T: H^1(\Omega))$ and $u_0 \in L^\infty(\Omega)$ such that $(\operatorname{supp} f(t, .) \cup \operatorname{supp} g(t, .) \cup \operatorname{supp} u_0) \subset \{x \in \overline{\Omega}: |x| \leqslant R\}$ for a.e. $t \in (0, T)$ and for some $R > 0$. We shall assume that β satisfies hypothesis (2.1) together with

$$\int_0^r \frac{ds}{\beta^{-1}(s)} < +\infty \quad \text{for all} \quad r \in \mathbb{R}. \tag{2.7}$$

Then problem $P(f, g, u_0, Q)$ has a unique strong solution in $H^{-1}(\Omega)$. Furthermore, there is a constant $K > 0$ such that

$$\operatorname{supp} u(t, .) \subset \{x \in \overline{\Omega}: |x| \leqslant R + 2K\sqrt{t}\} \quad \text{for all } t \in [0, T]. \tag{2.8}$$

THEOREM II. Let $f \in L^\infty(0, T: L^\infty(\Omega))$ and $u_0 \in L^\infty(\Omega)$ such that

$$(\operatorname{supp} f(t, .) \cup \operatorname{supp} u_0) \subset \{x \in \overline{\Omega}: |x| \leqslant R\} \quad \text{for a.e.} \quad t \in [0, T]$$

and for some $R > 0$. Let β be as in Theorem I and assume on γ the hypothesis $D(\gamma) = \mathbb{R}$ as well as

$$|\gamma^0(r)| \geqslant m.|r| \quad \text{if} \quad |r| < \delta \quad \text{for some } \delta > 0 \text{ and } m > 0\dagger. \tag{2.9}$$

Then the problem $P(f, \gamma, u_0, Q)$ as a unique integral solution in $L^1(\Omega)$. Furthermore, there are two positive constants K_1 and K_2 depending on $T. \|f\|_\infty$, $\|u_0\|_\infty$, β and γ, such that

$$\operatorname{supp} u(t, .) \subset \{x \in \overline{\Omega}: |x| \leqslant R + K_1 + K_2 t\} \quad \text{for all } t \in [0, T].$$

Theorem I generalizes the above mentioned result by Peletier [6] and Oleinik–Kalashnikov–Yui Lin [4] and besides, due to the equivalence of (2.7) and (1.4) ((1.5) assumed), it can be considered as the best possible. To the knowledge of the author there is no analogue of Theorem II in the existing literature.

The hypothesis (2.9) of Theorem II can be totally avoided if the domain Ω satisfies a certain geometric hypothesis (see Theorem III below which is satisfied, in particular, by every open set which complementary is a convex set of $\mathbb{R}^N$).

The general method to be used in Section 4 in order to prove both results consists of two steps. First the *a priori* boundedness of the support of the solution will be established using *ad hoc* comparison results (obtained in Section 3), applied to some super and sub-solutions (whose construction in Section 4 is made possible by (2.7)). Then by a simple argument we will be able to

† It is customary to associate different real functions to the maximal monotone graph γ of $\mathbb{R}^2$ such that $\gamma(0) = 0$. For instance, one defines $\gamma^+(r) = \operatorname{Max} \gamma(r)$; $\gamma^-(r) = \operatorname{Min} \gamma(r)$; and $\gamma^0(r) = \gamma^+(r)$ if $r < 0$, $\gamma^0(r) = \gamma^-(r)$ if $r > 0$ and $\gamma^0(0) = 0$.

extend Theorems A and B to the case of an unbounded Ω. Such a method has been used in many papers: Brezis [22], Benilan–Brezis–Crandall [23], Brezis–Friedman [24], Evans–Knerr [25], Knerr [7], Díaz [26], Díaz–Herrero [27] ....

However, there are other different methods in the existing literature on solutions with compact support, such as those used by Berkovitz–Pollard [28], Auchmuty–Beals [29], Bensoussan–Lions [30] and Redheffer [31].

Considering the kind of method sketched, one can see that Theorems I and II may be reformulated in many ways by just changing the functional framework or considering other definition of solutions different than I and II (see, e.g. Díaz [32]).

Let us point out that certain specific graphs β and γ provide a number of applications to very different concrete problems such as, for instance, those already mentioned in Section 1. (cf. Díaz [32]).

Related questions to those above, such that the compactness of support in t ($t \in [0, +\infty)$) of the solution of $P(f, g, u_0, Q)$ for some β maximal monotone graphs, the consideration of perturbations in the equations, etc., will be treated in a future paper.

3. THE COMPARISON OF SOLUTIONS

To obtain the results we show here, we use in an essential way the fact that the operator $-\Delta\beta$ is T-accretive in $L^1(\Omega)$ (see Benilan [21]). These results are closely related to the uniqueness of P_D and P_N.

LEMMA 1. Let β be as (2.1), let f_i, $f \in L^\infty(Q)$, g_i, $g \in B.V.(0, T: L^\infty(\Omega)) \cap L^2(0, T: H^1(\Omega))$ and u_{0i}, $u_0 \in L^\infty(\Omega)$ $i = 1, 2$, such that $f_1 \leqslant f \leqslant f_2$, $g_1 \leqslant g \leqslant g_2$ and $u_{01} \leqslant u_0 \leqslant u_{02}$ a.e. in their respective domains. Then, given u_i, strong solution in $H^{-1}(\Omega)$ of $P(f_i, g_i, u_{0i}, Q)$ $i = 1, 2$, with compact support, there exists a unique u, strong solution in $H^{-1}(\Omega)$ of $P(f, g, u_0, Q)$, such that $u_1(t, .) \leqslant u(t, .) \leqslant u_2(t, .)$ for all $t \in [0, T]$, a.e. in Ω.

Proof. Let $\tilde{\Omega}$ be open and bounded in Ω with regular boundary and such that $\tilde{\Omega}$ contains the set

$$S = \bigcup_{\substack{t \in [0, T] \\ i = 1, 2}} \operatorname{supp} u_i(t, .).$$

Let us denote by a $\sim$ on the respective symbol the restriction of the corresponding function to the set $\tilde{Q} = (0, T) \times \tilde{\Omega}$ or $\tilde{\Omega}$. Let $\tilde{u}$ be the unique solution of the problem $P(\tilde{f}, \tilde{g}, \tilde{u}_0, \tilde{Q})$ (such solution exists by Theorem A).

We start by stating Lemma 1 for the case of smooth data.

Let us approximate β by means of functions β_ε which are strictly increasing and such that $\beta_\varepsilon(r) \downarrow \beta(r)$ if $\varepsilon \downarrow 0$, with uniform convergence on compact sets. (Such an approximation can be obtained by considering $\Theta_\varepsilon(r) = 1/\varepsilon((r - (I + \varepsilon\beta^{-1})^{-1}(r))$ and taking then $\beta_\varepsilon(r) = \Theta_\varepsilon^{-1}(r) + \varepsilon r$. In this way, the functions $\eta_\varepsilon = \beta_\varepsilon^{-1}$ are Lipschitz continuous.

Let us denote by $\tilde{f}_{\varepsilon, i}$, $\tilde{f}_\varepsilon$, $\tilde{g}_{\varepsilon, i}$, $\tilde{g}_\varepsilon$, $\tilde{u}_{0, \varepsilon, i}$, $\tilde{u}_{0, \varepsilon}$ smooth approximations of the data.

By the general non-degenerate quasi-linear theory one can solve, the problem

$$\begin{cases} \eta'_\varepsilon(v_\varepsilon) . \dfrac{\partial v_\varepsilon}{\partial t} - \Delta v_\varepsilon = F_\varepsilon & \text{in} \quad \tilde{Q} = (0, T) \times \tilde{\Omega} \\ v_\varepsilon = G_\varepsilon & \text{on} \quad \tilde{\Sigma} = (0, T) \times \tilde{\Sigma} \\ v_\varepsilon(0, x) = \eta_\varepsilon(U_{0,\varepsilon}) & \text{on} \quad \tilde{\Omega} \end{cases}$$

obtaining then $u_\varepsilon = \eta_\varepsilon(v_\varepsilon) \in W^{1,2}(0, T: H^{-1}(\tilde{\Omega})) \cap L^2(0, T: L^2(\tilde{\Omega}))$. In this way we can approximate the functions $\tilde{u}_i$ and $\tilde{u}$ by taking suitable F_ε, G_ε and $U_{0,\varepsilon}$.

If α is a test function in $L^2(0, T: H_0^1(\tilde{\Omega}))$, we have

$$\int_0^T \langle \partial/\partial t(\tilde{u}_\varepsilon(t,.) - \hat{u}_{\varepsilon,2}(t,.)), \alpha(t,.)\rangle_{H^{-1}\times H_0^1}\, dt + \int_{\tilde{Q}} \operatorname{grad}(\tilde{v}_\varepsilon - \tilde{v}_{2,\varepsilon}).\operatorname{grad}\alpha\, dx\, dt = \int_{\tilde{Q}} (\tilde{f}_\varepsilon - \tilde{f}_{\varepsilon,2})\,\alpha\, dx\, dt.$$

Taking α given by $\alpha(\tau, x) = p(\tilde{v}_\varepsilon(\tau, x) - \tilde{v}_{\varepsilon,2}(\tau, x))\,\chi_{[0,t]}(\tau)$ with $p \in \mathscr{D}^+(\mathbb{R})$ such that $p(r) = 0$ if $r \leqslant 0$. $p' \geqslant 0$, then we have

$$\int_0^t \langle \partial/\partial t(\tilde{u}_\varepsilon(s,.) - \tilde{u}_{\varepsilon,2}(s,.)), \alpha(s,.)\rangle_{H^{-1}\times H_0^1}\, ds \leqslant \int_0^t \int_{\tilde{\Omega}} (\tilde{f}_\varepsilon(s,.) - \tilde{f}_{\varepsilon,2}(s,.)).\,p(\tilde{v}_\varepsilon(s,.) - \tilde{v}_{\varepsilon,2}(s,.))\, ds.$$

We now make p converge to the sign_0^+ function ($\operatorname{sign}_0^+(r) = 1$ if $r > 0$ and zero if $r \leqslant 0$). We get

$$\int_0^t \frac{d}{dt}\int_{\tilde{\Omega}} (\tilde{u}_\varepsilon(s,.) - \tilde{u}_{\varepsilon,2}(s,.))^+\, dx\, ds = \int_0^t \int_{\tilde{\Omega}} \frac{\partial}{\partial t}(\tilde{u}_\varepsilon(s,.) - \tilde{u}_{\varepsilon,2}(s,.)) \times \operatorname{sign}^+(\tilde{u}_\varepsilon(s,.) - \tilde{u}_{\varepsilon,2}(s,.))\, dx\, ds \leqslant \int_0^t \int_{\tilde{\Omega}} (\tilde{f}_\varepsilon(s,.) - \tilde{f}_{\varepsilon,2}(s,.))^+\, dx\, ds$$

and so

$$|(\tilde{u}_\varepsilon(t,.) - \tilde{u}_{\varepsilon,2}(t,.))^+|_{L^1(\tilde{\Omega})} \leqslant \int_0^t |(\tilde{f}_\varepsilon(s,.) - \tilde{f}_{\varepsilon,2}(s,.))^+|_{L^1(\tilde{\Omega})}\, ds + |(\tilde{u}_{0,\varepsilon} - \tilde{u}_{0,\varepsilon,2})^+|_{L^1(\tilde{\Omega})}.$$

Due to the fact that $\beta_\varepsilon \downarrow \beta$, $R(\beta) = \mathbb{R}$ and meas. $\tilde{\Omega} < +\infty$, the functions $\tilde{u}_\varepsilon$ and $\tilde{u}_{\varepsilon,2}$ are uniformly bounded in $W^{1,2}(0, T: H^{-1}(\tilde{\Omega})) \cap L^\infty(0, T: L^2(\tilde{\Omega}))$ (see Damlamian [10] (Remarque 6.16) and [11] Theorem 2.3). Then $\tilde{u}_\varepsilon$ and $\tilde{u}_{\varepsilon,2}$ converge weakly to $\tilde{u}$ and $\tilde{u}_2$ in $W^{1,2}(0, T: H^{-1}(\tilde{\Omega}))$ and also for all $t \in [0, T]$, $\tilde{u}_\varepsilon(t,.)$ and $\tilde{u}_{\varepsilon,2}(t,.)$ converge weakly in $L^2(\tilde{\Omega})$ to $\tilde{u}(t,.)$ and $\tilde{u}_2(t,.)$. Now using the weak l.s.c. of the function $|(.)^+|_{L^1}$ on $L^2(\tilde{\Omega})$ we get

$$|(\tilde{u}(t,.) - \tilde{u}_2(t,.))^+|_{L^1(\tilde{\Omega})} \leqslant |(\tilde{u}_0 - \tilde{u}_{0,2})^+|_{L^1(\tilde{\Omega})} + \int_0^t |(\tilde{f}(s,.) - \tilde{f}_2(s,.))^+|_{L^1(\tilde{\Omega})}\, ds.$$

Finally, if we define on Q the function

$$u(t,.) = \begin{cases} \tilde{u}(t,.) & \text{on } \tilde{\Omega} \\ 0 & \text{otherwise,} \end{cases}$$

we get the unique strong solution in $H^{-1}(\Omega)$ of $P(f, g, u_0, Q)$ such that $u(t,.) \leqslant u_2(t,.)$ for all $t \in [0, T]$. (In a similar way we can get $u_1(t,.) \leqslant u(t,.)$.)

Remark 1. Lemma 1 extends Theorem 4.7 of Damlamian [11] and generalizes Theorems 16, 17 and 18 of Oleinik–Kalashnikov–Yui Lin [4].

For the problem P_N, the theory of integral solutions for T-accretive operators contains abstract results of comparison that will be used later on.

LEMMA 2. (Benilan [14].)

Let Ω be a regular bounded open set and u be an integral solution of $P(f, \gamma, u_0, Q)$ with $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$. Then one has:

(i) If $\hat{u}$ is the integral solution of $P(\hat{f}, \gamma, \hat{u}_0, Q)$ with $\hat{f} \in L^1(Q)$ and $\hat{u}_0 \in L^1(\Omega)$ such that $f \leqslant \hat{f}$ and $u_0 \leqslant \hat{u}_0$ a.e. in Q and Ω respectively, then $u \leqslant \hat{u}$ a.e. in Q.

(ii) If $1 \leqslant p \leqslant \infty$, $f \in L^1(0, T: L^p(\Omega))$ and $u_0 \in L^p(\Omega)$, then u is such that $u(t) \in L^p(\Omega)$ for all $t \in [0, T]$ and also

$$\|u(t)\|_p \leqslant \|u_0\|_p + \int_0^t \|f(s)\|_p \, ds.$$

Proof. (See Benilan [14].)

To face later the problem P_N we shall need to construct supersolutions not defined on the whole Ω, for every t. This is the motivation for trying to obtain a comparison result relative to the following mixed-type problem:

$$\left\{\begin{array}{lll} \dfrac{\partial U}{\partial t}(t,.) - \Delta W(t,.) = F(t,.) & \text{for a.e. } t \in (0, T) \\ U(0,x) = U_0(x) & \text{in } \Omega \\ W(t,x) = G_-(t,x) & \text{in } \Sigma_- = (0,T) \times \Gamma_- \\ -\dfrac{\partial W}{\partial n}(t,x) = G_+(t,x) & \text{in } \Sigma_+ = (0,T) \times \Gamma_+ \end{array}\right. \tag{3.1}$$

$W \in L^2(0, T: H^1(\Omega))$, being such that $W(t,x) \in \beta(U(t,x))$ for a.e. $(t,x) \in Q$ and Γ_+, Γ_- a known partition of Γ.

This kind of problems have been considered in Damlamian [11]. There one can find results of existence and uniqueness in suitable functional spaces, as well as the following results of comparison.

LEMMA 3. (Damlamian [11].)

Let Ω be a regular bounded open set, and Γ_+, Γ_- a known partition of Γ. Let β, f_2, f, g_2, g, u_{02} and u_0 be as in Lemma 1. (Where G_- is the restriction of g (or g_2) on Σ_- and G_+ is $\partial g/\partial n$ (or $\partial g_2/\partial n$) on Σ_+)†. Then there exists u and u_2, solutions of (3.1) for their respective data such that

(i) u and u_2 are in $L^\infty(Q)$

(ii) $$|(u(t,.) - u_2(t,.))^+|_{L^1(\Omega)} \leqslant |(u_0 - u_{0,2})^+|_{L^1(\Omega)} + \int_0^t |(g_+(s,.) - g_{+,2}(s,.))^+|_{L^1(\Gamma_+)} \, ds + \int_0^t |(f(s,.) - f_2(s,.))^+|_{L^1(\Omega)} \, ds.$$

Proof. This is similar to the one of Lemma 1, taking now the test function α such that $\alpha \in L^\infty(0, T: H^1(\Omega))$ and vanishing on Γ_-. After having applied the Green's formula, one takes the limit as in Lemma 1.

† In (3.1) you should take as G_- the restriction of g and g to Σ_-, and as G_+ the restriction of $\partial g/\partial n$ and $\partial g_2/\partial n$ to Σ_+. Both restrictions should be understood in the sense of traces. (See Adams [20].)

Taking into account that the problem P_N is going to be solved in the sense of "integral solutions" in $L^1(\Omega)$, it is convenient to adapt Lemma 3 in the following way:

LEMMA 4. Let Ω be a bounded open set and Γ_+ and Γ_- a known partition of Γ. Let β be as in (2.1) and γ such that $D(\gamma) = \mathbf{R}$ and $0 \in \gamma(0)$. Let u be an "integral solution" in L^1 of $P(f, \gamma, u_0, Q)$ for $f \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\Omega)$. Let $u_i \in C([0, T]: H^{-1}(\Omega)) \cap L^\infty(Q)$ such that there exists

$$w_i \in L^2(0, T: H^1(\Omega)) \cap BV(0, T: L^\infty(\Omega))$$

with $w_1 \in \beta(u_i)$ a.e. in Q. $(i = 1, 2)$. Let us assume that the following is satisfied:

(a) $\dfrac{\partial u_1}{\partial t} - \Delta w_1 \leqslant f \leqslant \dfrac{\partial u_2}{\partial t} - \Delta w_2$ for a.e. in $(0, T)$, in $H^{-1}(\Omega)$

(b) $u_1(0, x) \leqslant u_0(x) \leqslant u_2(0, x)$ for a.e. $x \in \Omega$

(c) $w_1(t, x) \leqslant w(t, x) \leqslant w_2(t, x)$ for a.e. $(t, x) \in \Sigma_-$

(d) $-\dfrac{\partial w_i}{\partial n}(t, x) = \gamma_i(u_i(t, x))$ in Σ_+, where γ_i are real functions such that

$$\gamma_1(r_1) \geqslant \gamma^+(r) \geqslant \gamma^-(r) \geqslant \gamma_2(r_2)$$

for all $r_i \in D(\gamma_i)$, $r \in D(\gamma)$ such that $r_1 > r > r_2$, $i = 1, 2$. Then

$$u_1(t, .) \leqslant u(t, .) \leqslant u_2(t, .) \quad \text{for all} \quad t \in [0, T].$$

Proof. Due to the section (b) of Theorem B, u is a strong solution in $H^{-1}(\Omega)$ of $P(f, g, u_0, Q)$ with $g = w|_\Sigma$. Then the functions u and u_i satisfy a problem like (3.1), where now we take as G_+, the functions

$$g_+(t, x) = \frac{\partial w}{\partial n}(t, x) \in -\gamma(u(t, x)), \quad (t, x) \in \Sigma_+$$

and

$$g_{+,i}(t, x) = \frac{\partial w_i}{\partial n}(t, x) = -\gamma_i(u_i(t, x)) \quad (t, x) \in \Sigma_+, \quad i = 1, 2$$

respectively.

Then the result will be obtained with arguments similar to those used in Lemmas 1 and 3, with some adaptations that we outline in the following. We leave the details to the reader.

First of all, one states relation (ii) of Lemma 3 for the smooth functions u_ε, $u_{\varepsilon,i}$ and β_ε which approximate the data by smooth functions and β as in Lemma 1. Note that β_ε and β_ε^{-1} are locally Lipschitzian. There weak convergence in $L^2(\Omega)$ (and so in $L^1(\Omega)$) of $u_{\varepsilon,i}(t)$ to $u_i(t)$ is given as in Lemma 1. By Proposition 2.19, of [14] one has $u_\varepsilon(t) \to u(t)$ in $L^1(\Omega)$. Then, the relation (ii) of Lemma 3 is preserved in the limit due to the weak l.s.c. of the function $|(.)^+|_{L^1}$ in $L^1(\Omega)$. Finally, by the hypothesis (a)–(b), we have $(u(t, .) - u_2(t, .))^+ = (u_1(t, .) - u(t, .))^+ = 0$ for all $t \in [0, T]$, which completes the proof.

Remark 2. Similar results to the one in Lemma 4 have been given in Diaz [32] for stationary problems and in Brezis [12] for some unilateral problems.

4. COMPACTNESS OF THE SUPPORTS OF THE SOLUTIONS

4.1. Case $\Omega = (0, +\infty)$ and $f = 0$

The study of the problems $P(f, g, u_0, Q)$ and $P(f, \gamma, u_0, Q)$ reduces, by means of a simple argument, to the study of the same problems in the simpler case of $\Omega = (0, +\infty)$ and $f = 0$. So it seems reasonable to begin by treating this special situation.

The following example due to Lions [19] shows that we may expect that, for some β, the solutions of these problems have a compact support, assuming that the data have compact support.

Example 1. Let us consider $\Omega = (0, +\infty)$ and $x_0 > 0$ fixed. The non-linear problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2}{\partial x^2}(|u(t,x)|\,.\,u(t,x)) = 0 & \text{in } (0,T)\times(0,+\infty) \\ u(t,0) = \frac{1}{2}x_0 + t & \text{on } (0,T) \\ u(0,x) = \begin{cases} \dfrac{x_0 - x}{2} & \text{if } 0 < x < x_0 \\ 0 & \text{if } x \geqslant x_0 \end{cases} & \text{on } (0,+\infty) \end{cases}$$

admits as a "generalized solution" the function defined on $[0, T] \times [0, \infty)$ by

$$u(t,x) = \begin{cases} \left(\dfrac{x_0 - x}{2}\right) + t, & \text{if } 0 < x < x_0 + 2t \\ 0, & \text{if } x \geqslant x_0 + 2t \end{cases}$$

and, therefore, fixing $t \in [0, T]$, we have $\operatorname{supp} u(t,.) = [0, x_0 + 2t]$.

Other explicit examples illustrating the compactness of the support of the solution can be found in Knerr [7].

As a first positive result, we have:

THEOREM 0. Let $g \in C^1([0, T] \times (0, +\infty))$ and $u_0 \in L^\infty(0, +\infty)$ with

$$\Big(\big(\bigcup_{t\in(0,T)} \sup g(t,.)\big) \cup \operatorname{supp} u_0\Big) \subset [0, s_0]$$

Assume that β satisfies (2.1) and (2.7). Then the problem $P^* = P(0, g, u_0, (0, T) \times (0, +\infty))$ admits a unique strong solution in $H^{-1}(0, +\infty)$.

Moreover, there is a positive constant K depending on $\|u_0\|_\infty$, $\|g\|_\infty$ and β, such that

$$\operatorname{supp} u(t,.) \subset [0, s_0 + 2K\sqrt{t}] \quad \text{for all} \quad t \in [0, T]. \tag{4.1}$$

Proof. When dealing with the problem $P(f, g, u_0, Q)$ (even formulated with general conditions on Ω and f), for the purpose of obtaining an *a priori* bound of the support of the solution, we may always assume, $f \geqslant 0$, $g \geqslant 0$ and $u_0 \geqslant 0$ (similarly in case of negative signs) a.e. in Q, Σ and Ω, respectively. For, otherwise, it would be enough to take f^+, g^+ and u^+ the positive parts of those functions and also define $-f^- = f - f^+$, $-g^- = g - g^+$ and $-u_0^- = u_0 - u_0^+$. Since $-f^- \leqslant f \leqslant f^+$, $-g^- \leqslant g \leqslant g^+$ and $-u_0^- \leqslant u_0 \leqslant u_0^+$ a.e. in Q, Σ and Ω, respectively, we would have by, virtue of Lemma 1, that $u_1 \leqslant u \leqslant u_2$ a.e. if u_1, u and u_2 represent solutions of the respective problems.

Consider now the problem P^* in case $g \geqslant 0$ and $u_0 \geqslant 0$. One way of finding for all $t \in [0, T]$ a bound for the support of every solution is to build a "supersolution" v such that $v(t, .) \geqslant 0$ and v has compact support for all $t \in [0, T]$, since by Lemma 1 we would have supp $u(t, .) \subset$ supp $v(t, .)$ for all $t \in [0, T]$. For this purpose, consider the real non-decreasing function

$$\psi : r \to \int_0^r \frac{ds}{\beta^{-1}(s)}.$$

(Remember that the set of discontinuities of a real monotone function has measure zero and, therefore, the term $1/_{\beta^{-1}(s)}$ is well defined for almost every $s \in (0, \infty)$, even if the graph β is multi-valued.) The function ψ is one-to-one from $(0, +\infty)$ into $(0, +\infty)$ since by (2.1) $D(\beta^{-1}) = \mathbb{R}$ and besides, $\psi'(r) = 1/_{\beta^{-1}(r)} > 0$. Then the function $h = \psi^{-1}$ satisfies

$$h'(r) \in \beta^{-1}(h(r)) \quad \text{for a.e.} \quad r \in (0, +\infty). \tag{4.2}$$

Before giving explicitly the supersolution $v(t, x)$, let us define the function

$$w(t, x) = \begin{cases} h(-C_1(x - C_1 t - C_2)) & \text{if} \quad 0 \leqslant x \leqslant C_2 + C_1 t \\ 0 & \text{if} \quad x > C_2 + C_1 t \end{cases} \tag{4.3}$$

where C_1 and C_2 are positive constants that will be fixed later. Finally, let us take

$$v(t, x) = -\frac{1}{C_1} \cdot \frac{\partial w}{\partial x}(t, x) = \begin{cases} h'(-C_1(x - C_1 t - C_2)) & \text{if} \quad 0 \leqslant x \leqslant C_2 + C_1 t \\ 0 & \text{if} \quad x > C_2 + C_1 t. \end{cases} \tag{4.4}$$

(Note that by (4.2) it follows that $v(t, x) \in \beta^{-1}(w(t, x))$ for a.e. $(t, x) \in (0, T) \times (0, +\infty)$. The function v satisfies:

(a) $v \in L^2(0, T : L^2(0, +\infty))$; for

$$\int_{(0, +\infty)} v(t, x)^2 \, dx \leqslant (C_2 \leqslant C_1 t) . ((\beta^{-1})^+ h(C_1^2 t + C_1 C_2))^2 < +\infty$$

for a.e. $t \in (0, T)$ and

$$\int_0^T \| v(t, x) \|^2_{L^2(0, +\infty)} \, dt \leqslant T(C_2 + C_1 T) |(\beta^{-1})^+ h(C_1^2 T + C_1 C_2)|^2 < +\infty.$$

(b) $\beta(v(t, x)) \ni w(t, x)$ where $w \in L^2(0, T : H^1(0, +\infty))$ and we have

$$\frac{\partial v(t, .)}{\partial t} - \frac{\partial^2}{\partial x^2} w(t, .) = 0 \quad \text{a.e.} \quad t \in (0, T) \quad \text{in} \quad H^{-1}(0, +\infty); \tag{4.5}$$

for

$$\int_{(0, +\infty)} w(t, x)^2 \, dx \leqslant (C_2 + C_1 t) . |h(C_1^2 t + C_1 C_2)|^2 \quad \text{a.e.} \quad t \in (0, T),$$

$$\int_{(0, +\infty)} \left(\frac{\partial w(t, x)}{\partial x} \right)^2 dx = \int_{0 \leqslant x \leqslant C_2 + C_1 t} | h'(-C_1 x + C_1^2 t + C_1 . C_2)(-C_1)|^2 \, dx$$

$$\leqslant (C_2 + C_1 t) \, C_1^2 |(\beta^{-1})^+ h(C_1^2 t + C_1 . C_2)|^2$$

for a.e. $t \in (0, T)$ and

$$\int_0^T \|w(t,x)\|^2_{H^1(0,+\infty)}\,dt \leqslant T(C_2 + C_1 T)((h(C_1^2 T + C_1 . C_2)) + C_1^2((\beta^{-1})^+ h(C_1^2 T + C_1 C_2)^2)).$$

In order to check (4.5), we introduce $\xi = x - C_1 t$. By definition of v and w we have

$$v(t,x) = \tilde{v}(\xi) = \begin{cases} h'(-C_1(\xi - C_2)) & \text{if} \quad -C_1 T \leqslant \xi \leqslant C_2. \\ 0 & \text{if} \quad \xi > C_2. \end{cases}$$

and

$$w(t,x) = \tilde{w}(\xi) = \begin{cases} h(-C_1(\xi - C_2)) & \text{if} \quad -C_1 T \leqslant \xi \leqslant C_2 \\ 0 & \text{if} \quad \xi > C_2. \end{cases}$$

By using (4.2) it follows that

$$\frac{d\tilde{w}}{d\xi}(\xi) = -C_1 \tilde{v}(\xi) \quad \text{for a.e.} \quad \xi \in (-C_1 T, +\infty)$$

and, therefore, we have

$$\int_0^T \int_{(0,+\infty)} \left(\frac{\partial v}{\partial t}(t,x) - \frac{\partial^2 w}{\partial x^2}(t,x)\right) \phi(t,x)\,dx\,dt$$
$$= \int_0^T \int_{-C_1 T < \xi < +\infty} \left((-C_1) . \frac{d}{d\xi}\tilde{v}(\xi) - \frac{d^2}{d\xi^2}\tilde{w}(\xi)\right) . \phi(\xi, t)\,dt = 0$$

for every $\phi \in H^1((0,T) \times (0,+\infty))$ such that $\phi(0,.) = \phi(T,.) = \phi(.,0) = 0$ and ϕ has compact support. In particular, taking $\phi(t,x) = \theta(t) . y(x)$ with $\theta \in \mathscr{D}(0,T)$ and $y \in H_0^1(0,+\infty)$, we arrive at

$$\left\langle \frac{\partial v}{\partial t}(t,.) - \frac{\partial^2}{\partial x^2} w(t,.),\, y \right\rangle_{H^{-1}(0,+\infty) \times H_0^1(0,+\infty)} = 0 \quad \text{a.e.} \quad t \in (0,T)$$

for every $y \in H_0^1(0,+\infty)$, where the derivatives are to be interpreted in the sense of distributions. (Note that by (b) we have $v \in W^{1,2}(0,T: H^{-1}(0,+\infty))$.)

The constants C_1 and C_2 may be taken in such a way that

(c) $\beta^-(v(t,0)) \geqslant g(t)$ for a.e. $t \in (0,T)$,

and

(d) $v(0,x) \geqslant u_0(x)$ for a.e. $x \in (0,+\infty)$.

It is easy to see that the conditions (c) and (d) are obtained by imposing that $C_1(C_2 - s_0) \geqslant K$ where $K = \max\{\psi(\beta^+(\text{ess.}\sup u_0)), \psi((\beta^+)(\beta^{-1})^+(\text{ess}\sup g))$ and $s_0 \leqslant C_2$. It is also easy to verify that C_1 and C_2 are not uniquely determined by the preceding conditions, therefore, we obtain a whole family of supersolutions which give different estimates of the support, for each t, of every solution. Since, we have by definition that $\operatorname{supp} v(t,.) \subset \operatorname{supp} v(t',.)$ if $0 \leqslant t \leqslant t' \leqslant T$, C_1 and C_2 have to be determined in such a way that they provide the best estimate of the support of $v(T,.)$. This amounts to minimize $C_2 + C_1 T$, which can be obtained by taking $C_1 = \sqrt{K}/\sqrt{T}$ and $C_2 = \sqrt{K}.\sqrt{T} + s_0$. Such constants satisfy the desired properties. Then, Lemma 1 can be applied, getting so the existence and uniqueness of the solution u of P^*. One also has that

$$\operatorname{supp} u(t,.) \subset \left[0,\, s_0 + K\sqrt{T} + \frac{K}{\sqrt{T}}\,.\,t\right] \quad \text{for all} \quad t \in [0, T].$$

In particular, letting $t = T$ one has the estimate (4.1) for $t = T$, and thus for all $t \in [0, T]$.

4.2. *General case*

Assume that problems P_D and P_N are formulated in their greatest generality. We now proceed to prove Theorems I and II already stated in 2.

Proof of Theorem I. By Lemma 1 and arguing as in Theorem 0, it is sufficient to assume $f \geqslant 0$, $g \geqslant 0$ and $u_0 \geqslant 0$ a.e. in Q, Σ and Ω, respectively.

As before, we are going to get bounds for the support of every solution of $P(f, g, u_0, Q)$ by means of a suitable supersolution which is not defined on the whole of Ω, for every t.

Let R and $\tilde{R} = R + 2K\sqrt{T}$ be given as in the statement of Theorem I. Consider

$$\Omega^R_{R^*} = \{x \in \Omega : R < |x| < R^*\} \quad \text{for} \quad R^* > \tilde{R}.\dagger$$

Define now the function

$$w^*(t,x) = \begin{cases} h(-c_1(|x| - c_1 t - c_2)) & \text{if } \ R \leqslant |x| \leqslant c_2 + c_1 t; \ \ x \in \bar{\Omega} \\ 0 & \text{if } \ R^* \geqslant |x| > c_2 + c_1 t; \ \ x \in \bar{\Omega} \end{cases} \tag{4.6}$$

where c_1 and c_2 are positive constants, $c_2 > R$, to be determined later. The supersolution we seek will be of the type

$$v^*(t.\,x) = \begin{cases} h'(-c_1(|x| - c_1 t - c_2)) & \text{if } \ R \leqslant |x| \leqslant c_2 + c_1 t, \ \ x \in \bar{\Omega} \\ 0 & \text{if } \ R^* \geqslant |x| > c_2 + c_1 t, \ \ x \in \bar{\Omega}. \end{cases} \tag{4.7}$$

(Note that $w^*(t,x) = w(t,|x|)$ and $v^*(t,x) = v(t,|x|)$ where the functions w and v have been defined in Theorem 0.)

The function v^* satisfies:

(i) $v^* \in L^2(0, T : L^2(\Omega^R_{R^*})$;

(ii) $\beta(v^*(t,x)) \ni w^*(t,x)$ for a.e. $(t,x) \in Q^R_{R^*} = (0,T) \times \Omega^R_{R^*}$ with $w^* \in L^2(0, T : H^1(\Omega^R_{R^*}))$, moreover $f^* \in L^\infty(0, T : L^2(\Omega^R_{R^*}))$, $f^* \geqslant 0$ a.e. in $Q^R_{R^*}$ and

$$\frac{\partial v^*}{\partial t}(t,.) - \Delta w^*(t,.) = f^*(t,.) \quad \text{a.e.} \quad t \in (0.T), \text{in } H^{-1}(\Omega^R_{R^*}) \tag{4.8}$$

is satisfied.

(Let us take $f^*(t,x) = \partial v^*/\partial t(t,x) - \Delta w^*(t,x)$, the relation (4.5) gives us

$$f^*(t,x) = ((N-1)/|x|)\, c_1 h'(-c_1(|x| - c_1 t - c_2))$$

for a.e. $(t,x) \in Q^R_{R^*}$.)

Let $\Omega_{R^*} = \{x \in \Omega : |x| < R^*\}$, $Q_{R^*} = (0,T) \times \Omega_{R^*}$, and $\tilde{f}$, $\tilde{u}_0$ be the restriction of f and u_0 to Q_{R^*} and Ω_{R^*}, respectively. Let $\tilde{g}$ be given by the restriction of g on $(0,T) \times (\Gamma \cap (\partial\Omega_{R^*}))$ and zero on the remainder of $(0,T) \times \partial\Omega_{R^*}$. Then, if $\tilde{u}$ is the solution of $P(\tilde{f}, \tilde{g}, \tilde{u}_0, Q_{R^*})$, we have that $\tilde{u} \leqslant M$ a.e. in $Q^R_{R^*}(M = T\,.\,(\operatorname{ess\,sup}_Q f) + (\beta^{-1})^+ (\operatorname{ess\,sup}_\Sigma g) + \operatorname{ess\,sup}_\Omega u_0)$,

†Arguing as in Lemma 1, this kind of sets can be assumed with smooth boundary.

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because the function

$$q(t,x) = t\,.(\operatorname{ess\,.\,sup}_{Q} f) + (\beta^{-1})^{+}(\operatorname{ess\,.\,sup}_{\Sigma} g) + \operatorname{ess\,sup} u_0$$

is obviously a supersolution. This is why we have to fix the constants c_1 and c_2 in such a way as we have:

(iii) $v^*(t,x) \geqslant M$ for a.e. $t \in (0,T)$ if $|x| = R$.

Denoting $B = \psi(\beta^{+}(M))$, the preceding condition is obtained by imposing

$$-c_1 R + c_2 c_1 \geqslant B. \tag{4.9}$$

It is then clear that the choice of c_1 and c_2 satisfying (4.9) shows that Lemma 1 can be applied on $Q^R_{R^*}$, getting that $\operatorname{supp} \tilde{u}(t,.) \subset \operatorname{supp} v^*(t,.)$ for all $t \in [0,T]$ and a.e. in $\Omega^R_{R^*}$.† If we now denote by u the funtion defined on Q by extending $\tilde{u}$ with zero on $[0,T] \times (\Omega - \Omega_{R^*})$, we will have obviously a solution of $P(f,g,u_0,Q)$ which satisfies.

$$\operatorname{Supp} u(t,.) \subset \{x \in \bar{\Omega} : |x| \leqslant R\} \cup \operatorname{supp} v^*(t,.) \quad \text{for every} \quad t \in [0,T].$$

The choice of R^* and the estimate (2.8) are due to the determination of $c_1 > 0$ and $c_2 > R$ satisfying (4.9). Note that these constants are not determined in a unique way by this condition; therefore, it will be useful to make a choice so as to obtain the sharpest estimate of the measure of the support of $v^*(T,.)$. This amounts to minimizing the expression $c_2 + c_1\,.\,T$, which can be done by taking

$$c_1 = \frac{\sqrt{B}}{\sqrt{T}} \quad \text{and} \quad c_2 = R + \sqrt{B}\,.\sqrt{T},$$

and letting $K = \sqrt{B}$ one obtains

$$\operatorname{supp} u(t,.) \subset \{x \in \bar{\Omega} : |x| \leqslant R + K\sqrt{T} + (K/\sqrt{T}).t\} \qquad \text{for all} \quad t \in [0,T].$$

To get the estimate (2.8) it suffices to take first $t = T$ in the estimate above and make use of the increasing dependence of K on T.

The uniqueness is obtained in an obvious way from Lemma 1.

Proof of Theorem II. Adapting Lemma 2 to the case of Ω unbounded and data with compact support, one can assume, without loss of generality, that $f \geqslant 0$ and $u_0 \geqslant 0$ a.e. in Q and Ω, respectively.

In order to get a bound of the support of every possible solution of $P(f,\gamma,u_0,Q)$ we take the functions v^* and w^* given in the Proof of Theorem I by (4.6) and (4.7), respectively, where now $R^* > \tilde{R} = R + K_1 + K_2 T$. Those functions satisfy

(a) $v^* \in L^2(0,T : L^\infty(\Omega^R_{R^*}))$ and $v^*(t,x) \geqslant 0$ for a.e. $(t,x) \in Q^R_{R^*}$.

(b) $\beta(v^*(t,x)) \ni w^*(t,x)$ for a.e. $(t,x) \in Q^R_{R^*}$ and $w^* \in L^2(0,T : H^1(\Omega^R_{R^*})) \cap B.V.(0,T : L^\infty(\Omega^R_{R^*}))$.

(c) There exists $f^* \in L^\infty(0,T : L^\infty(\Omega^R_{R^*}))$ with $f^*(t,x) \geqslant 0$ for a.e. $(t,x) \in Q^R_{R^*}$ and such that $\partial v^*/\partial t\,(t,.) - \Delta w^*(t,.) = f^*(t,.)$ for a.e. $t \in (0,T)$ in $H^{-1}(\Omega^R_{R^*})$.

Furthermore the constants c_1, c_2 will be determined in such a way that:

(d) $v^*(t,x) \geqslant \bar{M}$ for a.e. $t \in (0,T)$ and $x \in \Omega^R_{R^*}$ with $|x| = R$, where

$$\bar{M} = \|u_0\|_{L^\infty(\Omega)} + \int_0^T \|f(s,.)\|_{L(\Omega)} ds.$$

†It is easy to see that $w \in W^{1,2}(0,T : L^2(\Omega_{R^*}))$ and thus $w \in B.V.(0,T : L^\infty(\Omega_{R^*}))$.

(e) There exists a function γ_2 such that

$$-\frac{\partial w^*}{\partial n}(t,x) = \gamma_2(v^*(t,x))$$

for a.e. $(t,x)\in(0,T)\times(\Gamma\cap\partial(\Omega^R_{R^*}))$ and with $\gamma_2(r_2)\leqslant\gamma^-(r)$ for every $r_2\in D(\gamma_2)$, $r\in D(\gamma)$ such that $r>r_2$.

The condition (d) leads us to take c_1 and c_2 satisfying

$$-c_1R + c_1c_2 \geqslant \bar{K}(\bar{K} = \psi(\beta^+(\bar{M}))). \tag{4.10}$$

On the other hand, it is easy to see that we have

$$\left|-\frac{\partial w^*}{\partial n}(t,x)\right| \leqslant c_1 . v(t,|x|) \quad \text{for a.e.} \quad (t,x)\in(0,T)\times(\Gamma\cap\partial(\Omega^R_{R^*})).$$

where v is defined by (4.4). Hence, by hypothesis (2.9), we have (e) if

$$c_1 . \bar{M} \leqslant m . \delta. \tag{4.11}$$

A choice of c_1 and c_2 which satisfies both (4.10) and (4.11) is

$$c_1 = \frac{m\delta}{\bar{M}}, \quad c_2 = R + \frac{\psi(\beta^+(\bar{M})) . \bar{M}}{m . \delta}.$$

If we denote by $\tilde{u}$ the unique solution of $P(\tilde{f},\gamma,\tilde{u}_0,Q_{R^*})$. Theorem B tells us that $\tilde{u}\leqslant\bar{M}$ a.e. in Q_{R^*}, and so properties (a) through (e) of v^* allow us to apply Lemma 4, getting that

$$0\leqslant\tilde{u}(t,.)\leqslant v^*(t,.) \quad \text{a.e. in} \quad \Omega^R_{R^*}.$$

Thus, the function u defined on Q by extending $\underset{\sim}{u}$ with zero on $[0,T]\times(\Omega-\Omega_{R^*})$, is obviously a solution of P_N which satisfies the wanted estimate. We obtain the uniqueness from Lemma 2 by arguing as in Lemma 1.

Similar results to the one of Theorem II, avoiding hypothesis (2.9) on γ, can be obtained assuming on the domain Ω a geometric condition.

THEOREM III. Let Ω be an open set of $\mathbb{R}^N$ of smooth boundary Γ and such that satisfy the hypothesis

$$(H_\Omega)\equiv\begin{cases}\exists x_0\in\mathbb{R}^N \quad \text{and} \quad \exists R_0>0 \quad \text{such that} \quad \sum_{i=1}^N (x_i-x_{0,i}).\cos(\vec{n}(x),\vec{x}_i)\leqslant 0\\ \text{for a.e.} \quad x\in\Gamma, \quad \text{and} \quad |x-x_0|\geqslant R_0,\end{cases}$$

being $\vec{n}(x)$ the normal unitary exterior vector to Γ in point x. Let $f\in L^\infty(0,T:L^\infty(\Omega))$ and $u_0\in L^\infty(\Omega)$ such that $(\operatorname{supp} f(t,.)\cup\operatorname{supp} u_0)\subset\{x\in\bar{\Omega}:|x-x_0|\leqslant R\}$ for a.e. $t\in(0,T)$ and for some $R\geqslant R_0$. Let β be as in Theorem I and assume on γ the hypothesis $D(\gamma)=\mathbb{R}$. Then problem $P(f,\gamma,u_0,Q)$ has a unique integral solution in $L^1(\Omega)$. Furthermore, there is a positive constant $\bar{K}$ depending on $\|f\|_\infty . T$, $\|u_0\|_\infty$, β (and independent of γ) such that

$$\operatorname{supp} u(t,.)\subset\{x\in\bar{\Omega}:|x-x_0|\leqslant R+2\bar{K}\sqrt{t}\} \quad \text{for all} \quad t\in[0,T]. \tag{4.12}$$

Proof. Once more, let us consider the functions v^* and w^* given by (4.6) and (4.7) but changing $|x|$ for $|x - x_0|$ in the case in which x_0 (of the hypothesis H_Ω) is not the origin. For simplicity we will assume in the following that $x_0 = 0$. As in Theorem II, there function satisfy (a), (b) and (c) and so it suffices to take the constants c_1 and c_2 in such a way as (d) and (e) hold.

Condition (d) holds if (4.10) is satisfied.

On the other hand, it is easy to check that

$$- \partial w^*/\partial n\,(t, x) \leqslant 0 \quad \text{for a.e.} \quad (t, x) \in (0, T) \times (\Gamma \cap \partial(\Omega^R_{R^*})),$$

v^* being non-negative in that set; in fact, it suffices to observe that $h'(r) \geqslant 0$ and, thus, by hypothesis H_Ω, $-\partial w^*/\partial n \leqslant 0$. Thus, condition (e) is satisfied independently of the constant c_1 and c_2, which can be finally determined as in Theorem I as $c_1 = \sqrt{K}/\sqrt{T}$ and $c_2 = R + \sqrt{K}\,.\sqrt{T}$.

Remark 3. As it was indicated in Section 2, it is easy to see that the hypothesis (H_Ω) is satisfied by every open set which complementary is a convex set of $\mathbb{R}^N$. Roughly speaking, the hypothesis (H_Ω) includes all open sets in $\mathbb{R}^N$ which boundary (except perhaps a bounded part of it) can be "illuminated" from some point of the space. It is also easy to check that if

$$\sum_{i=1}^{N} (x_i - x_{0,i})\,.\cos(\vec{n}(x), \vec{x}_i) > 0 \quad \text{for a.e.} \quad x \in \Gamma \quad \text{and} \quad |x - x_0| \geqslant R_0$$

for some $x_0 \in \mathbb{R}^N$ and some $R_0 \geqslant 0$ then no decreasing (increasing) function of $|x - x_0|$ can be supersolution (subsolution) for the problem $P(f, \gamma, u_0, Q)$ except if γ satisfy hypothesis (2.9).

Remark 4. If Γ is bounded, the hypothesis H_Ω is unnecessary as it is enough to choose an R such that the set $\Gamma \cap \partial(\Omega^R_{R^*}) = \phi$ and so the condition (e) is unnecessary.

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