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Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems

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Synopsis

We obtain existence and uniqueness of solutions with compact support for some nonlinear elliptic and parabolic problems including the equations of one-dimensional motion of a non-newtonian fluid. Precise estimates for the support of these solutions are obtained, and the optimality of our hypotheses is discussed.

1. Introduction and preliminaries

In this paper we will be concerned with the following nonlinear problems:

$$\text{(EP)}\begin{cases} -\Delta_p u + \beta(u) \ni f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

$$\text{(PP)}\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = F & \text{on } (0, T)\times\Omega \\ u = G & \text{on } (0, T)\times\partial\Omega \\ u(0, \cdot) = u_0(\cdot) & \text{on } \Omega \end{cases}$$

where Ω is a regular open subset of $\mathbb{R}^N$ $(N \geqq 1)$, $0 < T < +\infty$. β is a maximal monotone graph (see [4] for definition and properties) such that $0 \in \beta(0)$ (for example, $\beta(s) = |s|^{q-2}\cdot s$ with $1 < q < +\infty$), f, g, F, G are given (real) functions and

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right), \quad 1 < p < +\infty.$$

The operator $\Delta_p u$ has been widely considered in the literature of PDE. It represents one of the simpler examples of degenerate nonlinear elliptic operators for which a theory of classical solutions is not available. It also appears in several physical problems. For instance, it is known ([15], [16]) that the velocity of a conducting, incompressible non-newtonian fluid with a rheological power law in plane motion satisfies (if suitable units are selected) the equation:

$$\frac{d}{dx}\left(\left|\frac{du}{dx}\right|^{p-2}\frac{du}{dx}\right) - M^2\cdot u = 0 \tag{1.1}$$

for the stationary case, and

$$\rho \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(K \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right) + GB_0^2 u = 0 \tag{1.2}$$

for the evolutionary case. Here M^2, ρ, K, G, B_0^2 are nonnegative physical parameters. When $p>2$ the fluids are called dilatants, and for $1<p<2$ pseudoplastics. The case $p=2$ corresponds to newtonian fluids and equations (1.1), (1.2) then become linear.

Our main results will be established for not necessarily bounded Ω and can be stated, in an abridged way, as follows:

Elliptic problem (EP): there is a (unique) compactly supported solution, corresponding to compactly supported and bounded data f and g, if and only if the following condition holds:

$$\int_{-1}^{1} (j(s))^{-1/p}\, ds < +\infty, \tag{C}$$

where $\beta = \partial j$ (the subdifferential of j). We give also precise estimates for the size of the support of u, according to the size of the supports of f and g (see (2.4)).

Parabolic problem (PP): If the data F, G, u_0 are compactly supported and bounded, there is a unique solution with compact support when $p>2$. There are counterexamples in the case $1<p<2$, the case $p=2$ being well known. When the differential equation in (PP) is replaced by

$$\frac{\partial u}{\partial t} - \Delta_p u + \beta(u) = F \quad \text{on} \quad (0, T)\times\Omega, \quad 0<T\leqq+\infty \tag{1.3}$$

we show that the support of the solution is localized, i.e. the depth of penetration of signals in the medium is limited, if and only if (C) holds. In both cases $(\beta\equiv 0, \beta\neq 0)$ we obtain estimates for the size of the support of u (see (3.2), (3.10)).

As an example we show that, if $N=1$, $\Omega=(0, 1)$ $\beta(s)=M^2\cdot s$, estimate (2.3) gives the results obtained in [15] for (1.1) with $g(0)=0$, $g(1)=1$, whereas for $N=1$ and equation (1.3) our results extend to general domains and data, those given in [16] for $\Omega=(0, +\infty)$ and some particular initial and boundary conditions. In fact, in these papers the method employed is explicit integration of the problems under consideration, and hence it relies heavily on the one-dimensional character of the spatial variable and the selection of suitable initial and boundary data. Our technique, which does not depend on dimension, is essentially based on the maximum principle and construction of suitable supersolutions: in this way we are able to obtain *a priori* bounds on the support of the solution. In particular, by this procedure, problems on unbounded domains are reduced to others posed on bounded ones, where general theorems on existence and uniqueness are known. For completeness we quote here the basic results on that point which will be used later on.

THEOREM 0. (a) *Let Ω be bounded assume $1<p<+\infty$ and β maximal monotone graph in $\mathbb{R}^2$ such that $0\in\beta(0)$ and $R(\beta)=D(\beta)=\mathbb{R}$. Then for each $f\in L^2(\Omega)$ and $g\in H^1(\Omega)$ there is a unique $u\in W^{1,p}(\Omega)$, $u=g$ on $\partial\Omega$, such that there is $w\in L^2(\Omega)$*

with $w(x)\in\beta(u(x))$ almost everywhere on Ω and $-\Delta_p u+w=f$ in $D'(\Omega)$ (the usual space of distributions on Ω).

(b) *Let G be defined on $(0,T)\times\partial\Omega$, Ω bounded, and such that there is $\tilde{G}\in L^p(0,T;W^{1,p}(\Omega))$ with*

$$\frac{\partial\tilde{G}}{\partial t}\in L^{p'}(0,T\colon W^{-1,p'}(\Omega)),\quad \frac{1}{p}+\frac{1}{p'}=1,$$

and satisfying $\tilde{G}(t,x)=G(t,x)$ almost everywhere on $(0,T)$. Assume also that $f\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $u_0\in L^2(\Omega)$. Then there is a unique $u\in L^p(0,T);W^{1,p}(\Omega))$ with $\partial u/\partial t\in L^{p'}(0,T;W^{-1,p'}(\Omega))$ (and hence $u\in C((0,T)\colon L^2(\Omega))$ such that $\partial u/\partial t-\Delta_p u=f$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, $u=G$ on $(0,T)\times\partial\Omega$ and $u(0,x)=u_0(x)$ on Ω.

Part (a) of Theorem 0 is a simple adaptation of the results of [2]. Part (b) can be found in [14].

Finally, our plan is as follows: Section 2 is devoted to the elliptic problem, and Section 3 deals with the parabolic problem. Some of the results of this paper were announced in [10] and made a part of [12].

2. The elliptic problem

We shall use throughout the notation $\Omega_R=\{x\in\bar{\Omega}\colon |x|\leqq R\}\Omega^R=\{x\in\Omega\colon |x|>R\}$ for $R>0$. Let $j\colon\mathbb{R}\to[0,\infty]$ be a convex, lower semicontinuous (l.s.c.) function with $j(0)=0$ and such that $\partial j=\beta$. We define $\psi\colon\mathbb{R}\to[0,\infty]$ by:

$$\psi(\tau)=\int_0^\tau\frac{c\,ds}{(j(s))^{1/p}},\quad\text{where}\quad c=\left(\frac{p-1}{p}\right)^{1/p}\cdot N^{1/p}\ \text{for all}\ \tau\in\mathbb{R}. \tag{2.1}$$

Now we have:

THEOREM 1. *Let Ω be an open regular subset of $\mathbb{R}^N$, $1<p<\infty$, and let β be a maximal monotone graph such that $0\in\beta(0)$, $D(\beta)=R(\beta)=\mathbb{R}$. Assume that $f\in L^\infty(\Omega)$, $g\in H^1(\Omega)$ and $g\,|_{\partial\Omega}\in L^\infty(\partial\Omega)$, both with compact support. Then, the following is a necessary and sufficient condition for the existence of a unique solution $u\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ of (EP) with compact support:*

$$\int_{-1}^{1}(j(s))^{-1/p}\,ds<+\infty,\quad\textit{where}\quad \partial j=\beta, \tag{2.2}$$

Moreover, if (2.2) is satisfied, $\operatorname{supp} f\cup\operatorname{supp} g\subset\Omega_{R_0}$ for some $R_0\geqq 0$ (here $\operatorname{supp} v$ means support of v) and $R(\psi)=\mathbb{R}$, the following estimate holds:

$$\operatorname{supp} u\subset\Omega_{R_1},\quad\textit{where:} \tag{2.3}$$

$$\begin{aligned}R_1=R_0+\max\{&\psi(\beta^{-1})(\max(\|f^+\|_\infty,\beta^-(\|g^+\|_\infty))),\\ &\psi(\beta^{-1})^-(\max(\|f^-\|_\infty,\beta^-(\|g^-\|_\infty)))\}.\end{aligned} \tag{2.4}$$

Remark 1. By assumption, $j(s)^{-1/p}=0$ if $j(s)=+\infty$ and $j(s)^{-1/p}=+\infty$ if $j(s)=0$. Condition (2.2) is fulfilled if, for example, $\beta(s)=|s|^{q-2}\cdot s$ where $q<p$, and when $0\in\operatorname{Int}\beta(0)$ for each p, $1<p<+\infty$.

Proof of Theorem 1 (*Sufficiency*). We can suppose β single-valued, the changes corresponding to the general case being notational. By the maximum principle (see [12]) we can suppose $f \geqq 0$, $g \geqq 0$ almost everywhere. Let Ω^* be an open, regular and bounded subset of Ω such that $\Omega_{2R_1} \subset \Omega^* \subset \Omega$ (R_1 given in (2.4)). Let u^* be the solution of (EP) on Ω^* corresponding to data $f^* = f|_{\Omega^*}$ and $g^* = g$ on $\partial\Omega^* \cap \partial\Omega$ and $g^* = 0$ elsewhere. The function u defined on Ω by $u = u^*$ on Ω, $u = 0$ on $\Omega - \Omega^*$ will be the (unique) solution of (EP) satisfying (3.4) provided that $u^* = 0$ out of Ω_{R_1}. To see this last point, we construct a suitable supersolution $v(x)$ in the following way. Let $h(x)$ be the inverse function of $\psi(x)$ given in (2.1); h is nondecreasing, and $h(0) = 0$. In addition,

$$\begin{gathered} h'(x) = \frac{j(h(x))^{1/p}}{c} \text{ almost everywhere on } R(\psi) \\ N^p(|h'(x)|^{p-2} \cdot h'(x))' = \beta(h(x)) \text{ almost everywhere on } R(\psi). \end{gathered} \tag{2.5}$$

Assume for the moment that $R(\psi) = \mathbb{R}$. We define

$$v(x) = \begin{cases} h(R_1 - |x|) & \text{for } R_0 \leqq |x| \leqq R_1 \\ 0 & \text{for } |x| \geqq R_1. \end{cases}$$

Then $v \in C^1(\bar{\Omega})$ and after some straightforward calculations we have:

$$\begin{aligned} -\Delta_p v = &-(h'(R_1 - |x|)^{p-1})' \sum_{i=1}^{N} \left(\frac{|x_i|}{|x|}\right)^p \\ &+ h'(R_1 - |x|)^{p-1} \cdot \sum_{i=1}^{N} \frac{1}{|x|} \left(\left(\frac{|x_i|}{|x|}\right)^{p-2} - \left(\frac{|x_i|}{|x|}\right)^p\right). \end{aligned}$$

So we get:

$$-\Delta_p v + \beta(v) \geqq 0 \text{ almost everywhere on } \bar{\Omega}. \tag{2.6}$$

On the other hand, $v \geqq 0$ on $\bar{\Omega}$ and $v \geqq K$ on $\{x \in \bar{\Omega}: |x| = R_0\}$, where $K = (\beta^{-1}) \max(\|f^+\|_\infty, \beta^-(\|g^+\|))$. In addition $v(x) = 0$ if $|x| \geqq R_1$. Hence $v \geqq u^*$ for $|x| = R_0$, R_1 so that, by the maximum principle (see [12]), it follows that supp $u^* \subset \Omega_{R_1}$ as needed. In the case $R(\psi) \neq \mathbb{R}$, we can truncate β (by the maximum principle, u is bounded provided that the data are) and so obtain a reduction to the previous case. Finally, uniqueness for compactly supported solutions is a consequence of uniqueness of the solutions in bounded domains, as obtained in [2].

Proof of Theorem 1 (*Necessity*). Assume $\int_0^1 j(s)^{-1/p}\, ds = +\infty$, $\beta^{-1}(0) = 0$, β single-valued. Let $f = 1$ on $[0, 1]$ and $f = 0$ on $(1, +\infty)$, $g \in \mathbb{R}$ such that $0 < g \leqq \beta^{-1}(1)$ and let us suppose that there is a compactly supported u solution of the corresponding one-dimensional problem (EP). By the maximum principle, $0 \leqq u \leqq (\beta^{-1}) \max(\|f\|_\infty, \beta(\|g\|_\infty))$. As $\phi(x) = (|u'(x)|^{p-2}u'(x)) \in L^{p'}(0, +\infty)$ and $\phi'(x) \in L^2(0, +\infty)$, the solution u will be in $C^1(0, +\infty)$. On the other hand

$$\beta(u(x)) = f(x) + (|u'(x)|^{p-2}u'(x)|^{p-2}u'(x))' \equiv \Theta(x) \leqq 1 \text{ almost everywhere } x > 0. \tag{2.7}$$

Let $R = \max\{x: u(x) > 0\}$. R is positive and finite (by assumption). In addition, $R \geq 1$ (otherwise $1 = \beta(0)$). Actually, $R > 1$ since from (2.7)

$$(|u'(x)|^{p-2}u'(x)' \leqq 1 - f(x) \text{ almost everywhere } x > 0. \tag{2.8}$$

Hence $|u'(x)|^{p-2}u'(x))' \leqq 0$ on $(0, 1)$. If $R=1$, then $u(1)=u'(1)=0$ and by (2.8), $u'(x) \geqq 0$ if $x \leqq 1$ which would imply $u=0$ on $(0, 1)$, a contradiction. We have also $u(x)>0$ on $(1, R)$ and $(|u'(x)|^{p-2}u'(x))'>0$ almost everywhere on $(1, R)$. As $u'(R)=0$, it follows $u'<0$ on $(1, R)$. On the other hand, by assumption:

$$\int_0^{u(1)} \frac{ds}{j(s)^{1/p}} = -\int_1^R \frac{u'(r)\,dr}{j(u(r))^{1/p}} = +\infty. \tag{2.9}$$

We will get a contradiction by estimating $-u'(r)/j(u(r))^{1/p}$ on $(1, R)$. Let $w(x)=(-u'(x))^p$. We have $(j(u))' = (p-1/p) \cdot w'$ on $(1, R)$. Bearing in mind that $w(R)=0$, $j(u(R))=0$, we have, integrating the preceding equality between r and R

$$j(u(x)) = \frac{p-1}{p} \cdot w(x)$$

and finally

$$\int_1^R \frac{-u'(r)\,dr}{j(u(r))^{1/p}} = \left(\frac{p}{p-1}\right)^{1/p} \int_1^R ds < +\infty. \tag{2.10}$$

If $\beta^{-1}(0) \neq 0$, we can reduce to the previous case by arguing as in [3, Lemma 6.5].

Remark 2. When $p=2$ and $\Omega=\mathbb{R}^N$ the preceding result is valid even for $f \in L^1(\mathbb{R}^N)$ with compact support ([3]). Other results for $p=2$, Ω unbounded can be found in [8].

Remark 3. If $\beta(r)=r$, then (2, 2) gives $p>2$, and we obtain Theorem 1 in [10].

Example 1. Let us consider the flow of a conducting field with a power rheological law when the nonconducting walls of a plane channel ($z=\pm 1$) move along the x-axis with velocity ± 1. There is no pressure gradient, and no electric field, and the external magnetic field of induction is perpendicular to the walls (Magnetohydrodynamic Couette flow). After some changes, the problem can be formulated (see [15]) as:

$$\begin{cases} (\mathrm{E})\ \dfrac{d}{dz}\left(\left|\dfrac{du}{dz}\right|^{p-2}\dfrac{du}{dz}\right) - M^2u = 0 \quad \text{on } (0, 1) \\ \qquad\qquad\qquad\qquad u(1)=1, \quad u(0)=0. \end{cases} \tag{2.11}$$

By means of effective integration, it is shown in [15] that for $p<2$, $u>0$ on $(0, 1)$, whereas for $p>2$ there exists a critical value of M (generalized Hartmann number),

$$M_c^2 = 2\frac{p-1}{p-2}\left(\frac{p}{p-2}\right)^{p-1}$$

such that for $M^2 \leqq M_c^2$, u vanishes only at $x=0$, but for $M^2 > M_c^2$ u vanishes on $[0, z_0]$, where

$$z_0 = 1 - \frac{R_c}{R}, \quad R_c = \frac{p}{p-2} \quad \text{and} \quad R = \left(\frac{M^2}{2} \cdot \frac{p}{p-1}\right)^{1/p}.$$

Let us particularize Theorem 1 to the problem (2.11). If $p>2$, hypothesis (2.2) is fulfilled with

$$\psi(s)=\left(\frac{2}{M^2}\cdot\frac{p-1}{p}\right)^{1/p}\cdot\frac{p}{p-2}\cdot s^{1-2/p}$$

If we make the change $\xi=1-x$ in (2.11), estimate (2.4) says that the support of $u(\xi)$ is contained in $[0, \psi(1)]$. From the very definition of $\psi(s)$ it follows that when $M^2>M_c^2$, $\psi(1)<1$ and $u(\xi)=0$ on $[\psi(1), 1]=[R_c/R, 1]$. When $M^2\leqq M_c^2$, $u(\xi)>0$ on $(0, 1)$, as can be shown by noting that, h being the inverse of ψ given in (2.1), $v(\xi)=h(1-\xi)$ is a subsolution of (E) with $v(0)=0$, $0<v(1)\leqq 1$, $v>0$ on $(0, 1)$. We obtain again the estimates of [15] for $p>2$.

Remark 4. If β is such that $0\in \operatorname{Int}\beta(0)$ (for instance that is the case for some problems known as variational inequalities, see [14]), Theorem 1 may be extended to cover some cases where f is not compactly supported. For example, it can be shown that, if $0\in\beta(0)=[\beta^-(0), \beta^+(0)]$ where $\beta^-(0)<0<\beta^+(0)$, $D(\beta)=R(\beta)=\mathbb{R}$ and if $g=0$, $f\in L^\infty(\Omega)$ is such that there is $\varepsilon>0$ with $\Omega-D(f, \varepsilon)$ bounded, where:

$$D(f, \varepsilon)=\{x\in\Omega: \beta^-(0)+\varepsilon\leq f(x)\leq\beta^+(0)-\varepsilon\}$$

then there is a unique compactly supported $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution of (EP). As to the estimates, it can be shown that, if $D(f, \varepsilon)$ is connected:

$$\operatorname{supp} u\subset\{x\in\bar{\Omega}\cap D(f, \varepsilon): d(x, \partial D(f, \varepsilon)-\partial\Omega)\geqq K\}$$

where k is a nonnegative constant depending on $\|f\|_\infty$, N, p and ε. The proofs of these facts will appear elsewhere. Suitable modifications make the results valid when $g\neq 0$, $g\in L^\infty(\partial\Omega)$ with compact support. Such results extend those previously known in the literature (see [8] and the references therein).

3. The parabolic problem

To introduce our first result, let us consider (PP) with $p=2$, $\Omega=\mathbb{R}^N$. We then have the Cauchy problem for the heat equation. It is well known that such a problem propagates signals with infinite speed in the sense that, if $u_0(x)\geqslant 0$ $u_0\neq 0$ and $\operatorname{supp} u_0\subset B_R$ for some $R>0$, then for each $t>0$, $u(t, x)$ is positive for all $x\in\mathbb{R}^N$. The situation is very different if $p>2$, as Theorem 2 asserts.

THEOREM 2. *Let* $p>2$, $F(t, x)\in L^\infty((0, T)\times\Omega)$, $u_0\in L^\infty(\Omega)$ *and* $G\in L^\infty((0, T)\times\partial\Omega)$ *satisfying the hypothesis in Theorem* 0, (b). *Assume that there is* R, $0<R<+\infty$ *such that:*

$$(\operatorname{supp} u)\cup\Big(\bigcup_{t\in(0,T)}\operatorname{supp} G(t,\cdot)\Big)\cup\Big(\bigcup_{t\in(0,T)}\operatorname{supp} F(t,\cdot)\Big)\subset\Omega_R. \tag{3.1}$$

Then there is a unique compactly supported $u\in L^\infty((0, T)\times\Omega)$ *solution of* (PP), *and there is a nonnegative constant* $M(T)$ (*depending on* T) *such that the following estimate holds*:

$$\operatorname{supp} u(t,\cdot)\subset\{x\in\bar{\Omega}: |x|\leqq R+M(T)\cdot t^{1/p}\},\quad \text{for}\quad t\in(0, T). \tag{3.2}$$

Proof. We proceed as in Theorem 1 by comparing out of the origin with a suitable supersolution. Let $U(t, x)$ be defined by:

$$U(t,x)=\begin{cases} K(c^p t-c|x|+cA)^{\alpha} & \text{for} \quad R\leqq|x|\leqq c^{p-1}t+A \\ 0 & \text{for} \quad c^{p-1}t+A\leqq|x| \end{cases}$$

where:

$$\alpha=\frac{p-1}{p-2}, \quad K=\left(\frac{p-2}{p-1}\right)^{p-1/p-2} \cdot \left(\frac{1}{N}\right)^{1/p-2};$$

c, A will be chosen later (see (3.6), (3.7)).

By straightforward calculations we get:

$$\frac{\partial U}{\partial t}-\Delta_p U\geqq c^p\alpha K\cdot(c^p t-c|x|+cA)^{\alpha-1}\cdot[1-K^{p-2}(\alpha-1)(p-1)N]\geqq 0$$

$$\text{on} \quad R\leqq|x|\leqq c^{p-1}t+A. \qquad (3.4)$$

We want to show that $U(t, x)$ is a supersolution for (PP) in $\{R\leqq|x|\leqq c^{p-1}T+A\}\times(0, T)$. To compare on the lateral boundary, we need:

$$(-cR+cA)^{p-1/p-2}\geqq\tilde{M}(T)=(T\|F\|_{\infty}+\max(\|G\|_{\infty}, \|u_0\|_{\infty})).$$

Hence:

$$A\geq\frac{M(T)^{p-2/p-1}}{\tilde{M}(T)}+R. \qquad (3.5)$$

On the other hand we are interested, in order to give as accurate an estimate as possible, in minimizing $c^{p-1}T+A$. Noting that c will be chosen nonnegative, we can write by means of (3.5)

$$c^{p-1}\cdot T+A\geqq c^{p-1}\cdot T+\frac{\tilde{M}(T)^{p-2/p-1}}{c}+R\equiv\phi(c).$$

The minimum of $\phi(c)$ is achieved at

$$c=\left(\frac{1}{p-1}\right)^{1/p}\cdot\frac{\tilde{M}(T)^{p-2/p(p-1)}}{T^{1/p}}. \qquad (3.6)$$

And now (3.5) reads

$$A\geq R+(p-1)^{1/p}\cdot\tilde{M}(T)^{p-2/p}\cdot T^{1/p}. \qquad (3.7)$$

So we have

$$c^{p-1}\cdot T+A=R+\left(\frac{1}{p-1}\right)^{p-1/p}\cdot T^{1/p}\cdot\tilde{M}(T)+(p-1)^{1/p}\cdot\tilde{M}(T)^{p-2/p}\cdot T^{1/p}$$

$$\leqq R+2(p-1)^{1/p}\cdot\tilde{M}(T)^{p-2/p}\cdot T^{1/p},$$

and

$$c^{p-1}\cdot T+A\leqq R+M(T)\cdot T^{1/p}, \quad \text{where} \quad M(T)=2(p-1)^{1/p}\cdot\tilde{M}(T)^{p+2/p}$$

Noting that $M(t)\leqq M(T)$ when $t\leqq T$ we obtain the desired estimate (3.2).

The preceding result is optimal in the sense that it is no longer true in general when $1<p\leq 2$ (see [12]).

Let us consider now the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}-\Delta_p u+\beta(u)\ni F & \text{on } (0,T)\times\Omega \\ u=0 & \text{on } (0,T)\times\partial\Omega \\ u(0,x)=u_0(x) & \text{on } \Omega \end{cases} \tag{3.8}$$

where β is a m.m.g. on $\mathbb{R}^2$ with $0\in\beta(0)$. In this case, for bounded Ω and $D(\beta)=\mathbb{R}$, we can use the Crandall-Liggett theorem to assert the existence of a unique solution in some (weak) sense in $L^1(\Omega)$ (see [11]). On the other hand, if v (respectively u) is a solution of (3.8) corresponding to $\beta\equiv 0$ (respectively $\beta\not\equiv 0$), and we suppose F, u_0 nonnegative, it is clear (by the maximum principle) that $0\leq u\leq v$ almost everywhere. Hence we can extend to (3.8) the results obtained in Theorem 2. Moreover, we can state in this case a more precise result, which includes localization of the support of the solution in time.

THEOREM 3. *Let Ω be an open, regular subset of $\mathbb{R}^N$ $(N\geq 1)$, $\beta=\partial j$ m.m.g. in $\mathbb{R}^2$ with $0\in\beta(0)$ and $D(\beta)=\mathbb{R}$. Let $u_0\in L^\infty(\Omega)$, $F\in L^\infty((0,T)\times\Omega)$ be such that there is R_0, $0<R_0<+\infty$ satisfying:*

$$\Big(\bigcup_t \operatorname{supp} F(t,\cdot)\Big)\cup\operatorname{supp} u_0\subset\Omega_{R_0} \quad \text{for each } t>0. \tag{3.9}$$

Then (2.2) is a necessary and sufficient condition for the existence of a unique $u(t,x)\in C(0,\infty))$, $L^1(\Omega)\cap L^1(\Omega)\cap L^\infty((0,T)\times\Omega)$ solution of (3.8) with uniformly localized support for every t (i.e. $\operatorname{supp} u(t,\cdot)\subset\Omega_R$ for some R and all $t>0$). In addition, if (2.2) is fulfilled and $R(\psi)=\mathbb{R}$ (ψ given in (2.1)), the following estimate holds:

$$\operatorname{supp} u(t,\cdot)\subset\Omega_R \text{ for each } t>0, \tag{3.10}$$

where

$$R=R_0+\max\left\{\psi\Big(\|u_0^+\|_\infty+\int_0^{+\infty}\|F^+(s)\|_\infty\,ds\Big),\ \Big(\|u_0^-\|_\infty+\int_0^{\infty}\|F^-(s)\|_\infty\,ds\Big)\right\}.$$

Proof (*Sufficiency*). Without loss of generality we can suppose the data to be nonnegative, and β single-valued. Let $v(t,x)=\Theta(x)$ solution of the stationary problem:

$$\begin{cases} -\Delta_p\Theta+\beta(\Theta)=0 \quad \text{on} \quad \Omega^{R_0}-\Omega^{2R} \\ \Theta(x)=\begin{cases} \|u_0\|_\infty+\displaystyle\int_0^{+\infty}\|F(s,\cdot)\|_\infty\,ds & \text{for } |x|=R_0,\ x\in\partial\Omega^{R_0} \\ 0 & \text{elsewhere on } \partial\Omega^{R_0}. \end{cases} \end{cases} \tag{3.11}$$

It is clear that Θ is a supersolution on Ω^{R_0} since the solution u^* of (3.7) corresponding to $\Omega=\Omega_{2R}$ is such that

$$\|u^*(t,\cdot)\|_\infty\leqq\|u_0\|_\infty+\int_0^\infty\|F(s,\cdot)\|_\infty\,ds. \tag{3.12}$$

(see [1], Prop. 2.14]). So we have $0 \leqq u^*(t, x) \leqq \Theta(x)$ for each $t>0$ and almost everywhere $x \in \Omega$ with $R_0<|x|<2R$. But by Theorem 1, we have $\operatorname{supp} \Theta(x) \subset \Omega_R$ and the required result follows extending u^* by zero out of Ω_{2R}.

Proof (*Necessity*). Let $N=1$, $F(t, x)=F_\infty(x)=1$ on $[0, 1]$, $F_\infty(x)=0$ elsewhere. Assume that (2.2) does not hold, but the support of the solution of (3.8) is localized, $\operatorname{supp} u(t, \cdot) \subset \Omega_{R^*}$ for some $R^*>0$. Then u^*, restriction of u to Ω_{2R^*} satisfies (3.7) on Ω_{2R^*}. Hence we can write our problem:

$$\begin{cases} \dfrac{\partial u^*}{\partial t}+\partial\phi(x, u^*)=0 & \text{on } (0, T)\times\Omega_{2R^*} \\ u^*(0)=u(x) & \text{on } \Omega_{2R^*} \end{cases}$$

where

$$\phi(x, u^*)=\frac{1}{p}\int_{\Omega_{2R^*}}\left|\frac{\partial u^*}{\partial x}\right|^p+\int_{\Omega_{2R^*}}(j(u^*)-F_\infty(u^*))$$

if the integrals make sense, and zero elsewhere. If $u_0(x) \geqq 0$, we can suppose $\phi(x, u^*)$ to be even in u^* without loss of generality. Then, the results of [6] apply, and $u^*(t, x) \to u^*(x)$ in $L^2(\Omega_{2R^*})$ when $t \to \infty$, where $u^*(x)$ solves:

$$\begin{cases} -(|u_\infty^{*\prime}|^{p-2}(u_\infty^{*\prime}))+\beta(u_\infty^*)=F_\infty & \text{on } \Omega_{2R^*} \\ u_\infty^*=0 & \text{on } \partial\Omega_{2R^*} \end{cases}$$

In fact, $u_\infty^*(x)=0$ almost everywhere on $R^*<|x|<2R^*$. The contradiction follows now from the necessity of (2.2) in Theorem 1.

Example 2. It is known that the equation of the motion of a conducting, incompressible non-newtonian fluid with a rheological power law can be written (cf. [16]):

$$\rho\frac{\partial u}{\partial t}-\frac{\partial}{\partial z}\left(K\left|\frac{\partial u}{\partial z}\right|^{p-2}\frac{\partial u}{\partial z}\right)+GB_0^2u=0,\ t>0,\ z>0, \tag{3.13}$$

where K, ρ are the rheological constants of the medium, G is the conductivity, B_0 is the induction of the external transverse magnetic field and $1<p<+\infty$. In [16] equation (3.13) is integrated for some particular initial and boundary conditions. Then the following effects are obtained:

(i) In dilatant fluids (p>2), the front of the shear stress wave propagates with finite speed. This is not the case for pseudoplastic fluids ($p<2$).

(ii) When $p>2$ and $GB_0^2>0$, there occurs spatial localization of shear disturbances; in addition the depth of penetration into the medium tends to infinity as σB_0^2 goes to zero.

If we take $N=1$ in Theorem 2, we obtain (i) even for $GB_0^2=0$. In fact we have seen in the proof that finite propagation property does not depend on dimension nor on the existence of the perturbation term $\beta(u)$ on (3.8), but on the very properties of $\Delta_p u$ for $p>2$. Concerning (ii) we note that Theorem 3 characterizes the relationship among p and β which gives localization (condition (2.2)). In fact one can have localization for $p<2$ and suitable β, and there is not that effect, in general, in the case $GB_0^2=0$. The dependence on the depth of penetration upon the term $\beta(u)$ is given by estimate (3.10); it agrees with (ii) in the hypothesis there assumed. We emphasize again that no restrictions on dimension or data are needed, except those listed in Theorems 2 and 3.

Remark 5. There are analogous results to those quoted in Remark 4 for the case $0 \in \beta(0) = [\beta^-(0), \beta^+(0)]$ with $\beta^-(0) < 0 < \beta^+(0)$. In that case, it can be shown that the support of the corresponding solution is localized in t, provided that there is $\varepsilon > 0$ such that $\beta^-(0) + \varepsilon \leq F(t, x) \leq \beta^+(0) - \varepsilon$ for $x \in \Omega^{R_0}$, $t > 0$, where R_0 is such that supp $u_0 \subset \Omega_{R_0}$. For such problems (evolution variational inequalities) and several qualitative properties of their solutions, see [5], [9].

Remark 6. The fact that signals are propagated with finite speed was shown in [17] for another kind of nonlinear degenerate parabolic equations, namely $u_t = (\phi(u))_{xx}$, provided that a condition similar to (2.3) was satisfied. The corresponding result for $N > 1$ and weaker regularity hypotheses on ϕ can be found in [7]. Note also a recent result of Kalashnikov [13] concerning a large class of equations in one spatial dimension.

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