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On the Behaviour and Cases of Nonexistence of the Free Boundary in a Semibounded Porous Medium

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1. INTRODUCTION

This article deals with the Cauchy–Dirichlet problem associated to the nonlinear Fokker–Planck equation:

$$u_t = (u^m)_{xx} + b_0 \cdot (u^\lambda)_x \qquad \text{in } \mathbb{R}^2_+ \tag{1}$$

$$u(x, 0) = u_0(x) \qquad x > 0, \tag{2}$$

$$u(0, t) = u_1(t) \qquad t > 0, \tag{3}$$

where $m \geqslant 1$, $\lambda > 0$, $b_0 \in \mathbb{R}$ and $\mathbb{R}^2_+ = \{(x, t): x > 0,\ t > 0\}$. We shall also assume that the functions $u_0(x)$ and $u_1(t)$ satisfy the hypothesis

$$\begin{aligned} &u_0 \in C(\mathbb{R}^+), \qquad u_0 = 0 \text{ on } (l, +\infty), \\ &\qquad \text{and} \qquad u_0 > 0 \text{ on } [0, l], \qquad \text{for some } l > 0, \\ &u_1 \in C(\mathbb{R}^+), \qquad u_1(t) > 0\ \forall t \geqslant 0, \\ &\qquad \text{and} \qquad u_0(0) = u_1(0). \end{aligned} \tag{4}$$

Equation (1) arises in the study of the flow of a fluid through a homogeneous isotropic rigid porous medium where $u(x, t)$ represents the

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volumetric moisture content, $b_0 u^\lambda$ is the hydraulic conductivity, and the value of m is related to the capillary section of the medium (see [10, 11]).

It is well known that if $m > 1$, in general we cannot expect to have classical solutions of (1) because the equation degenerates where $u = 0$. Existence, regularity, comparison, and uniqueness results on a weaker notion of solution, called a generalized solution, can be found in [3], assuming u_0^β is Lipschitz, $\beta = \max\{(m-1), (m-\lambda)^+\}$.

The aim of this article is to study several qualitative properties of the solution of (1)–(3) and, in particular, the occurrence or not of the free boundaries, or interfaces, separating the sets where $u > 0$ and $u = 0$. We shall concentrate our attention on the case of a singular transport term $0 < \lambda < 1$. The case in which $\lambda \geqslant 1$ is of a very different nature.

The study of the free boundary for the Cauchy problem associated to Eq. (1) is well known for $b_0 = 0$ [1, 9, 12]. The case $b_0 \neq 0$ is fairly well studied (see, for example, [4, 5, 6] for $\lambda \geqslant 1$ and [3] for $0 < \lambda < 1$).

The main goal of this work is to study the dependence of the behaviour of the free boundary with respect to the behaviour of the boundary datum $u_1(t)$. Before describing our results, let us remark that the presence of the convection term ($b_0 \neq 0$) leads to obvious modifications of the support of the solution with respect the case $b_0 = 0$: the support shrinks when $b_0 > 0$ and spreads out when $b_0 < 0$.

We start by considering the case of $b_0 > 0$. When $u_1(t)$ goes to zero as $t \to \infty$ (in some regular sense) then we show, in Section 2, that the boundary

$$\zeta(t) = \sup\{x > 0 : u(t, x) > 0\} \tag{5}$$

goes to zero as $t \to +\infty$, and so, every point x of the medium becomes dry ($u(x, t) = 0$) after some finite time $t_0(x)$.

In Section 3 we show how the above behaviour of $u(x, t)$ is peculiar to boundary datum $u_1(t)$ goint to zero as $t \to +\infty$. In particular we show that if $u_1(t) \to +\infty$ as $t \to +\infty$ then the free boundary $\zeta(t) \nearrow +\infty$ as $t \nearrow +\infty$, so, even in the case of $\lambda < 1$, there is an infinite penetration and every point of the medium becomes wet ($u(x, t) > 0$) after some finite time.

Finally, in Section 4 we consider the case of $b_0 < 0$, $0 < \lambda < 1$. The singularity and sign of the transport coefficient leads to another interesting phenomenon in contrast with the case $\lambda \geqslant 1$; now there is no free boundary (i.e., the solution is positive everywhere and, therefore is classical) although Eq. (1) seems to be, formally, degenerate.

2. Reversing Free Boundary

We shall assume in this section that

$$b_0 > 0, \qquad 0 < \lambda < 1, \qquad \text{and} \qquad m \geqslant 1, \tag{6}$$

and that $u_1(t) \to 0$ as $t \to +\infty$. In order to control the convergence of $u_1(t)$ to 0 we introduce the following set of growing functions:

Definition 1. We say that a function Ψ belongs to the set Φ if $\Psi \in C^0([0, \infty)) \cap C^1((0, \infty))$, $\Psi(t) > 0\ \forall t \geqslant 0$, $\Psi'(t) > 0\ \forall t > 0$, $\Psi(t) \to +\infty$ as $t \to +\infty$, and Ψ satisfies that for every $\delta > 0$ there exists $C > 0$ such that

$$\Psi'(t)\, \Psi^{-1-\delta}(t) \leqslant C \qquad \forall t > 0. \tag{7}$$

Remark 1. It is easy to see that set Φ includes a large class of functions as for instance, $(t + C)^p$, $(Ln(t + C))^p$, e^{pt} for $p > 0$, and many others. We remark that there exist functions satisfying all the conditions of Definition 1 but (7).

Theorem 1. *Assume that there exists $\Psi \in \Phi$ such that $u_1(t) \leqslant (\Psi(t))^{-1}$ $\forall t > 0$. Then for every $x_0 > 0$ there exists $t_0 = t_0(x_0) > 0$ such that $u(x_0, t) = 0$ for $t \geqslant t_0$. In other words, $\zeta(t) \to 0$ as $t \to +\infty$.*

Proof. The main idea of the proof is to construct a supersolution $v(x, t)$ such that its free boundary $\tilde{\zeta}(t) = \sup\{x > 0 : v(x, t) > 0\}$ satisfies that $\tilde{\zeta}(t) \to 0$ as $t \to +\infty$. In this case, by the comparison principle (see Theorem 4.3 of [2]), we obtain that $0 \leqslant u(x, t) \leqslant v(x, t)$ on $\mathbb{R}^2_+$ and, in particular, $0 \leqslant \zeta(t) \leqslant \tilde{\zeta}(t)$ for every $t > 0$ which gives the result. We shall construct $v(x, t)$ in the following way

$$v(x, t) = \begin{cases} (1/s\Psi(\varepsilon t))\, w(x, t)^q & \text{when } w(x, t) := r - x\Psi(\varepsilon t)^p > 0 \\ 0 & w(x, t) \leqslant 0, \end{cases}$$

where p, q, r, s, and ε will be suitably chosen.

In order to show that $v(x, t)$ is a supersolution, we need to check the following conditions:

(i) $v(x, 0) \geqslant u_0(x)$, $\forall x > 0$

(ii) $v(0, t) \geqslant u_1(t)$, $\forall t > 0$

(iii) $Lv := -v_t + (v^m)_{xx} + b_0(v^\lambda)_x \leqslant 0$, in $\mathscr{D}'(\mathbb{R}^2_+)$.

To verify the above inequalities we shall assume, for the sake of notation, that $\Psi(0) = 1$. It is clear this is no restriction, and obvious modifications lead to the result if $\Psi(0) \neq 1$. We also denote by $M_0 := \max u_0(x)$. Then condition (i) reads

$$s^{-1}(r-x)^q \geqslant u_0(x) \qquad \text{for} \qquad x>0.$$

This holds if we have

$$s^{-1}(r-l)^q \geqslant M_0, \qquad \text{i.e.,} \qquad s^{-1}r^q \geqslant \left(M_0^{1/q} + \frac{l}{s^{1/q}}\right)^q.$$

Now we fix q such that

$$q > \frac{1}{m-\lambda}. \tag{8}$$

Let $s > s_0 = l^q$. So $ls^{-1/q} < 1$. In what follows we shall relate s and r by the expression

$$s^{-1}r^q = (M_0^{1/q}+1)^q := M \tag{9}$$

and then condition (i) is fulfilled.

Condition (ii) is equivalent to

$$s^{-1}\Psi^{-1}(\varepsilon t)\, r^q \geqslant u_1(t) \qquad \text{for} \qquad t>0.$$

From the assumption $\Psi \in \Phi$ we have $\Psi^{-1}(\varepsilon t) \geqslant \Psi^{-1}(t)$ provided $\varepsilon \leqslant 1$. On the other hand, from (9) we see that $s^{-1}r^q > 1$. Thus, (ii) follows from the assumption $u_1(t) \leqslant \Psi^{-1}(t)$.

To check condition (iii) we note that it is enough to verify that

(a) $Lv \leqslant 0$ on the set where $w > 0$, and

(b) $(v^m)_x$ is continuous in $\mathbb{R}^2_+$.

On the set $r > x\Psi(\varepsilon t)^q$ we have

$$\begin{aligned} Lv &\leqslant s^{-1}\varepsilon \Psi^{-2}\Psi' w^q + s^{-1} qp\varepsilon \Psi^{-2}\Psi' w^{q-1} \\ &\quad + s^{-m}(qm-1)\, \Psi^{2p-m} w^{qm-2} - s^{-\lambda} b_0 \lambda q \Psi^{q-\lambda} w^{q\lambda-1} \\ &= (s^{-\lambda} w^{q\lambda-1} \Psi^{q-\lambda})\, I_1, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= s^{\lambda-1}\varepsilon \Psi^{-q+\lambda-2}\Psi' w^{1+q(1-\lambda)} + s^{\lambda-1} qp\varepsilon \Psi^{-q+\lambda-2}\Psi' w^{q(1-\lambda)} \\ &\quad + s^{\lambda-m} qm(qm-1)\, \Psi^{q+\lambda-m} w^{q(m-\lambda)-1} - b_0\lambda q. \end{aligned}$$

In that set we have $w < r$, then $I_1 \leqslant I_2$, where

$$\begin{aligned} I_2 &:= (s^{-1}r^q)^{1-\lambda}\, r\varepsilon \Psi' \Psi^{\lambda-q-2} + (s^{-1}r^q)^{1-\lambda}\, qp\varepsilon \Psi' \Psi^{\lambda-q-2} \\ &\quad + r^{-1} qm(qm-1)(s^{-1}r^q)^{m-\lambda}\, \Psi^{q+\lambda-m} - b_0\lambda q. \end{aligned}$$

Now we choose p such that $0<p<m-\lambda$ and we take $\delta:=1+p-\lambda$. Then by (9) and the fact that $\Psi\in\Phi$ we have

$$I_2\leqslant M^{1-\lambda}Cr\varepsilon+M^{1-\lambda}Cqp\varepsilon+M^{m-\lambda}qm(qm-1)\,r^{-1}-b_0\lambda q.$$

Finally, we shall choose our parameter in the following way: first we fix r in such a way that

$$M^{m-\lambda}qm(qm-1)\,r^{-1}<b_0\lambda q/2.$$

After this, we choose s from (9). Finally, we take $\varepsilon<1$ such that

$$\varepsilon(M^{1-\lambda}Cr+M^{1-\lambda}Cqp)<b_0\lambda q/2.$$

For such a choice of parameters we have $Lv<0$ on the set $w>0$. Moreover, condition (b) holds because $qm>1$ and the proof ends.

Remark. From the proof we note that, if $u_1(t)^{-1}\in\Phi$, we have the estimate

$$\zeta(t)\leqslant ru_1(\varepsilon t)^p$$

for $0<p\leqslant m-\lambda$, ε small and r large enough, respectively.

The following result shows that the free boundary $\zeta(t)$ attains the point $x=0$ in a finite time when $u_1(t)$ vanishes after some time.

PROPOSITION 1. *Assume $b_0>0$, $0<\lambda<1$, and $u_1(t)$ such that $u_1(t)=0$ for $t\geqslant t_0$. Then for any $\varepsilon>0$ there exists $k>0$ such that*

$$\operatorname{supp} u(x,t)\subset P:=\left\{(x,t)\in\mathbb{R}^2_+:\frac{x}{k^\alpha}+\frac{t}{k^\beta}<1\right\},$$

where $\alpha=m-\lambda+\varepsilon$ and $\beta=m+1-2\lambda+2\varepsilon$.

Proof. Define the function w by

$$w:=\begin{cases} k\left(1-\dfrac{x}{k^\alpha}-\dfrac{t}{k^\beta}\right)^{1/(m-\lambda)} & \text{on } P\\ 0 & \text{on } \mathbb{R}^2_+\setminus P.\end{cases}$$

It is clear that we can choose $k\geqslant k_0$ such that $w(x,0)\geqslant u_0(x)$ and $w(0,t)\geqslant u_1(t)$ for $x>0$ and $t>0$. On the other hand, $Lw\leqslant 0$ in $\mathscr{D}'(\mathbb{R}^2_+)$ because

$$Lw\leqslant\left(1-\frac{x}{k^\alpha}-\frac{t}{k^\beta}\right)^{(2\lambda-m)/(m-\lambda)}k^{\lambda-\alpha}$$
$$\times\left[-\frac{b_0\lambda}{m-\lambda}+\frac{1}{k^\varepsilon}+\frac{m\lambda}{(m-\lambda)^2k^\varepsilon}\right]<0$$

on P, if $k \geqslant k_1$ for some suitable k_1. We also note that $(w^m)_x$ is continuous in $\mathbb{R}^2_+$. Then choosing $k = \max\{k_0, k_1\}$, by the comparison principle we conclude that $0 \leqslant u(x, t) \leqslant w(x, t)$ on $\mathbb{R}^2_+$, and the result follows.

3. Localization and Positivity

Gien $m \geqslant 1$ and λ such that $m > \lambda > 0$ and $b_0 > 0$, it is not difficult to show that the function

$$v(x) = \left[r - \frac{b_0(m-\lambda)}{m} x \right]_+^{1/(m-\lambda)}$$

is a generalized solution of Eq. (1) for any $r > 0$ (here $[h(x)]_+ = \max\{0, h(x)\}$). From this fact the next result follows.

Proposition 2. (i) *Assuming $u_1(t)$ is bounded there exists $r_0 > 0$ such that $u(x, t) = 0$ for any $x \geqslant r_0$ and any $t > 0$, i.e., $0 \leqslant \zeta(t) \leqslant x_1$ for any $t \geqslant 0$ and for some $x_1 > 0$ (localization property).*

(ii) *If there exists $\varepsilon > 0$ such that $\varepsilon < u_1(t)$ for any $t \geqslant 0$ then there exists $x_2 > 0$ such that $x_2 \leqslant \zeta(t)$ for $t \geqslant 0$.*

Now we consider the case of $u_1(t) \to +\infty$ as $t \to \infty$. When $\lambda \geqslant 1$, although it is not in the literature, it is known that there is no localization of the free boundary, i.e., $\zeta(t) \to +\infty$ as $t \to +\infty$ (see [7, 8] for some related results). Here we shall show that this lack of localization still holds, even for a singular convection $(0 < \lambda < 1)$.

Theorem 2. *Let $b_0 > 0$ and $0 < \lambda < 1$. Assume that there exists $\Psi \in \Phi$ such that $u_1(t) \geqslant \Psi(t)$ for any $t > 0$. Then for any $x_0 > 0$ there exists $t_0 = t_0(x_0)$ such that $u(x_0, t) > 0$ $\forall t \geqslant t_0$, i.e., $\zeta(t) \to +\infty$ as $t \to \infty$.*

Proof. Consider the function

$$w(x, t) = r - (x/\Psi(t)^p),$$

where $p \in (0, (m-1)/2)$ and r will be chosen later. Define the set

$$P := \{(x, t) : x < r\Psi(t)^p\}$$

and let $v(x, t)$ be given by

$$v(x, t) = \Psi(t)\, w(x, t)^q$$

for $q \in (1/m, 1/(m-\lambda))$. We shall show that we can choose r such that

$v(x, t)$ is a subsolution and then, by the comparison result, we conclude that $r\Psi(t)^p \leqslant \zeta(t)$, which gives the result.

As in the proof of Theorem 1, for the sake of notation, we shall assume $\Psi(0) = 1$. Condition $v(x, 0) \leqslant u_0(x)$ for any $x > 0$ holds if $r \leqslant r_0$, for some r_0 small enough; indeed, we know that $u_0(0) > 0$ so, by continuity, it is clear that $[r - x]_+^q \leqslant u_0(x)$ for any $x > 0$ if r is small enough.

The boundary condition $v(0, t) \leqslant u_1(t)$ for $t > 0$ comes from the assumption of $u_1(t)$ when $r \leqslant 1$.

Since $(v^m)_x$ is continuous on $\mathbb{R}_+^2$, in order to show that v is a subsolution we only need to prove that $Lv \geqslant 0$ in P. But for $r \leqslant 1$ we have

$$\begin{aligned} Lv &\geqslant -\Psi' w^q - qp\Psi' w^{q-1} + qm(qm-1)\,\Psi^{m-2p} w^{qm-2} - b_0 q\lambda \Psi^{\lambda - p} w^{q\lambda - 1} \\ &= \Psi^{m-2p} w^{qm-2}(qm(qm-1) - \Psi^{2p-m}\Psi' w^{2-q(m-1)} \\ &\quad - qp\Psi^{2p-m}\Psi' w^{1-q(m-1)} - b_0 q\lambda \Psi^{\lambda+q-m} w^{1-q(m-\lambda)}). \end{aligned}$$

By the choice of q and p we know that $1 - q(m-1) > 0$, $1 - q(m-\lambda) > 0$, $\lambda + p - m < 0$, and we can define $\delta := m - 1 - 2p$ with $\delta > 0$ (note that $2p - m = -1 - \delta$). Then by using the fact that $\Psi \in \Phi$, in order to have $Lv \geqslant 0$ in P it is enough to have

$$qm(qm-1) - Cr^{2-q(m-1)} - qpCr^{1-q(m-1)} - b_0 q\lambda r^{1-q(m-\lambda)} > 0.$$

This is satisfied if $r \leqslant r_1$ for some $r_1 > 0$. Taking $r \leqslant \min\{1, r_0, r_1\}$ all the conditions are fullfilled and $v(x, t)$ is a subsolution. ∎

Remark. From the proof we note that, if $u_1(t) \in \Phi$, then we have the estimate

$$ru_1(t)^p \leqslant \zeta(t)$$

for $0 < p < (m-1)/2$ and $r > 0$ small enough.

4. Nonexistence of the Free Boundary: Everywhere Positivity of the Solution

In this section we consider the case of

$$b_0 < 0 \qquad \text{and} \qquad 0 < \lambda < 1. \tag{10}$$

It is well known (see, e.g., [6, 4]) that for $b_0 < 0$, $m > 1$, and $\lambda \geqslant 1$, the free boundary $\zeta(t)$ does exist starting for $t = 0$ from the point $x = l$ (recall assumption (4)). In the case of $0 < \lambda < 1$ the convection coefficient becomes infinity when $u = 0$ and, due to the sign of b_0, we shall be able to show that

$\zeta(t)$ does not exist. In [3] a similar behaviour for solutions of the Cauchy problem was shown (nonexistence of the right-side interface). There, the conclusion comes from the fact that the solution satisfies a differential inequality. This method does not work directly for the case of the Cauchy–Dirichlet problem. So, here we follow another idea.

THEOREM 3. *Assume* (10) *and consider problem* (1), (2), (3) *in the strip* $S_T = \mathbb{R}_+ \times (0, T)$ *for any fixed* $T > 0$. *Then, if* $u_1(t) > 0$ *for* $t \in (0, T]$ *we have* $u(x, t) > 0$ *in* S_T.

Proof. Let $f(t)$ be a monotone increasing C^1 function on $\mathbb{R}_+$ such that $f(0) > 1$. Let $a > 0$. We are going to show that the function

$$v(x, t) := f(t)^{-1/(m-\lambda)} \left(\frac{t}{(x/|b_0|\,\lambda) + a} \right)^{1/(1-\lambda)} \tag{11}$$

is a subsolution of our problem in S_T, if a is large enough. Obviously, $0 = v(x, 0) \leqslant u_0(x)$. On the other hand, condition $v(0, t) \leqslant u_1(t)$ for $0 \leqslant t \leqslant T$ holds, if $a = a(T)$ is large enough.

To conclude the proof we have to show that $Lv \geqslant 0$ in S_T.
We have

$$\begin{aligned} Lv = & -\frac{t^{\lambda/(1-\lambda)}}{(1-\lambda)} f^{-(1/(m-\lambda))} \left(\frac{x}{|b_0|\,\lambda} + a \right)^{-(1/(1-\lambda))} \\ & + \frac{1}{m-\lambda} t^{1/(1-\lambda)} f' f^{-((m-\lambda+1)/(m-\lambda))} \left(\frac{x}{|b_0|\,\lambda} + a \right)^{-(1/(1-\lambda))} \\ & + \frac{m}{(1-\lambda)} \left(\frac{m}{1-\lambda} + 1 \right) |b_0|^{-2} \lambda^{-2} t^{m/(1-\lambda)} f^{-(m/(m-\lambda))} \\ & \times \left(\frac{x}{|b_0|\,\lambda} + a \right)^{-(m/(m-\lambda))-2} \\ & + \frac{1}{1-\lambda} t^{\lambda/(1-\lambda)} f^{-(\lambda/(m-\lambda))} \left(\frac{x}{|b_0|\,\lambda} + a \right)^{-(1/(1-\lambda))}. \end{aligned}$$

The second and the third terms are non-negative. Thus, $Lv > 0$ if

$$\frac{1}{1-\lambda} t^{\lambda/(1-\lambda)} f^{-(\lambda/(m-\lambda))} \left(\frac{x}{|b_0|\,\lambda} + a \right)^{-(1/(1-\lambda))} (f^{(1-\lambda)/(m-\lambda)} - 1) > 0.$$

which is true by the choice of $f(t)$, and this ends the proof.

Remark. Once that we know that $u(x, t) > 0$ in S_T, Eq. (1) is not degenerate and, by standard results, u is a classical solution (and, in fact, $u \in C^\infty(S_T)$).

Remark. As it was kindly pointed out to us by the referee, Theorem 3 can also be proved by using our previous results in [3] for the Cauchy problem. Indeed, taking the $v(x, t)$ solution of the Cauchy problem such that $v(x, 0) \leqslant u_0(x)$ for any $x \geqslant 0$ and $v(0, 0) < \frac{1}{2}u_1(0)$ by the comparison principle we have $v(x, t) \leqslant u(x, t)$ for any $x \geqslant 0$ and $t \leqslant \tau$ provided $v(0, t) \leqslant u_1(t)$ for $t \leqslant \tau$. Choosing $v(x, 0) > 0$ on $(-L, 0)$ for L large enough, from Theorem 1 of [3] we conclude that $v(x, t) > 0$ for any $x \geqslant 0$ and $t \leqslant \tau$. Finally by continuity $u(x, t) > 0$ for any $x \geqslant 0$ and $t \leqslant T$. We remark that the first proof of Theorem 3 in this paper does not use any gradient estimate (as it does the proof of Theorem 1 of [3]) and that the subsolution given in (11) can be of interest for some other purposes.

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