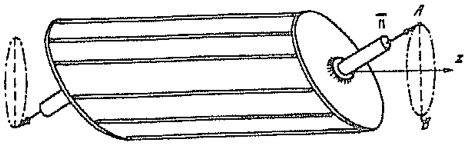

FIG. 3

filaments to be small but of finite size, then the shear of the sample will cause their relative slippage. For different generatrices of a filament this slippage differs considerably: greatest slippage for generatrices tangent to the planes orthogonal to AB, zero slippage for generatrices tangent to the planes parallel to AB, and intermediate slippage occurs in other cases. Solution (3) gives a new type of shear. Here the vector $\bar{n}$ moves along a conic surface and all its positions are equivalent, so that a transition from A to B and back to A occurs as t increases monotonically. Such shear leads to the same slippage or uniform work of all the generatrices of any inner filament.

Our experiment was based on the scheme in Fig. 1. The lateral surface is made from a set of rigid rods on which rubber is stretched. The tests were made with dry quartz sand. A flow similar to (3) was produced. The approximation was linked with the influence of weight and with the fact that the condition of dry friction, rather than adhesion, was provided at the contact with the surface. We observed a differential rotation: Each cross section z = const rotates approximately as a rigid whole and the angular velocity depends only slightly on z. The diametral plane of the sample (colored black) therefore acquires a helicoidal shape with time (Fig. 3; the material is impregnated with gelatin, and after solidification the right half of the sample is removed).

In summary, all steady, uniform flows are divided into elliptical, hyperbolic, and degenerate flows. Confined flows are contained in the class of elliptical flows, while other flows are unconfined. In addition, each class has flows in which one of the principal values ε_i equals zero. The motion of particles in them obeys Kepler's second law. This also includes plane flows. A particular case of degenerate plane flows is Couette flow between parallel plates (the fact that the sectorial velocity is constant is not accidental). The position of (3) in the adopted classification is as follows: The motion of particles obeys Kepler's law, the flow is elliptical (circular) and confined, but not plane. It can therefore be called circular plane-parallel flow. Here a complex loading[4] occurs, with continuous rotation of the axes of the stress tensor, as does a special type of "isotropic" shear deformation.

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Applications of an energy method in the localization of the solutions of the equations of continuous-medium mechanics

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In this paper we are reporting some new results on the spatial and temporal localization of the solutions of complex systems of equations of a composite type which arise in the mechanics of a continuous medium. In such systems the components of the unknown solution (the velocity, the density, the degree of saturation), etc. may satisfy equations of different types.

Existing methods for studying the localization of the solutions of nonlinear degenerate equations (see the review by Kalashnikov,[8] for example) cannot be applied to such systems.

In the present study we use a method of integral energy estimates which was proposed and validated in Refs. 1-4, 7, 12, and 13 for equations of the elliptical, parabolic, and composite types. The idea of the method is to derive and study ordinary differential inequalities for energy functions.

The basic purpose of this study, and also its basic new aspect, was the demonstration by the method of energy estimates that the time scale for the localization (vanishing) of the solution is finite and that its spatial localization is metastable for certain models of the mechanics of a continuous medium which contain "sources," i.e., given right-hand sides. This study is an application of the results reported in Refs. 3, 4, 7, 12, and 13.

The finite localization time for the solution and the finite velocity at which perturbations propagate away from initial perturbations were demonstrated for corresponding models without such sources in Ref. 3.

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In this paper we will not discuss the existence of the corresponding solutions; we are interested only in their properties.

1. Incompressible inhomogeneous viscoplastic media. The system of equations is of the composite type and can be written in the form[3,9]

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0, \quad \nabla \rho = \left(\frac{\partial \rho}{\partial x_1}, \ldots, \frac{\partial \rho}{\partial x_n}\right); \tag{1}$$

$$\operatorname{div} v = 0, \quad v = (v_1, \ldots, v_n); \tag{2}$$

$$\rho \frac{dv}{dt} \equiv \rho\left(\frac{\partial v}{\partial t} + (v\nabla) v\right) = \operatorname{div} P + \rho f; \tag{3}$$

$$P = -pE + 2(\mu + \tau |D|^{\sigma-1})D, \quad 0 \leqslant \sigma \leqslant 1,$$

$$D_{lk}(v) = \frac{1}{2}\left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right),$$

$$|D^2| = D_{lk}:D_{lk}, \quad x \in \Omega \subset R^n, \quad t \in (0, T). \tag{4}$$

Here v(t, x), ρ(t, x), and p(t, x) are the unknown velocity, density, and pressure, respectively, in the liquid; P is the stress tensor; D is the rate-of-strain tensor; f(t, x) is a given body force (the "source"); μ = const > 0 is the viscosity; and τ = const > 0 is the limiting shear stress.

For w = (v, ρ, p) we consider the following initial-boundary-value problem:

$$v(t, x) = 0, \quad (x, t) \in Q = \Omega \times (0, T); \tag{5}$$

$$v(0, x) = v_0(x), \quad \rho(0, x) = \rho_0(x), \quad x \in \Omega. \tag{6}$$

We assume

$$\frac{1}{M} \leqslant \rho_0(x) \leqslant M, \quad \|v_0(x)\|_{2,\Omega} \leqslant c_v = \text{const}; \tag{7}$$

$$\|f(t, \cdot)\|_{2,\Omega}^{q/(q-1)} \leqslant c_f (T_f - t)_+^{q/(2-q)}, \quad c_f = \text{const}, \tag{8}$$

$$u_+ = \max(0, u), \quad q \in (\max(1+\sigma, 2n/(n+2)), 2).$$

A theorem concerning the existence of a generalized solution $w = (v, \rho, p) \in V$, where

$$V = \Big\{ w \mid v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_0^{1,2}(\Omega)),$$

$$\rho \in L^\infty(Q), \frac{1}{M} \leqslant \rho \leqslant M \Big\}.$$

was proved for problem (1)-(6) in Ref. 11. Let us examine the qualitative properties of the solution.

Theorem 1 (finite localization time). We assume that $w = (v, \rho, p) \in V$ is a generalized solution of problem (1)-(6), and we assume that conditions (7) and (8) hold. Then for any $T_f \in (0, T)$ there exist constants c_v and c_f [which are small in comparison with $\mu^\lambda \tau^{1-\lambda}$, where $\lambda = (p - 1 - \sigma)/(1 - \sigma)$] and also a c such that

$$\|v(t, \cdot)\|_{2,\Omega}^2 \leqslant c(T_f - t)_+^{q/(2-q)}. \tag{9}$$

In particular, we assume

$$v(t, x) \equiv 0, \quad x \in \Omega, \quad t \geqslant T_f. \tag{10}$$

The proof of Theorem 1 is a complete repetition of the basic ideas of the method of energy estimates, which are presented in Refs. 1-4, 7, 12, and 13.

Multiplying (3) by v(t, x), and integrating by parts, we find the following energy relation, using (1), (2), and (4):

$$\frac{1}{2}\frac{dE}{dt} \equiv \frac{1}{2}\frac{d}{dt}(\rho v, v)_\Omega = (\operatorname{div} P, v)_\Omega + (\rho f, v)_\Omega \tag{11}$$

$$= -(\mu |D|^2 + \tau |D|^{\sigma+1}, 1)_\Omega + (\rho f, v)_\Omega, \quad (u, v)_\Omega = \int_\Omega uv\, dx.$$

Now using Young's and Korn's inequalities,[10]

$$K\|v\|_{m,\Omega}^p \leqslant \|D(v)\|_{p,\Omega}^p, \quad m \leqslant np/(n-p) \tag{12}$$

we find an ordinary differential inequality for the energy function E(t):

$$\frac{dE}{dt} + aE^{q/2} \leqslant b(T_f - t)_+^{q/(2-q)}. \tag{13}$$

Here the constants a and b are calculated in terms of μ, τ, σ, K, q, and n. An analysis of the solutions of inequality (13) completes the proof of the theorem, as was shown in Ref. 7.

Comment 1. Theorem 1 can be interpreted in the following way. The flow of a viscoplastic liquid which is initiated by initial given and body forces (a source) reaches a state of rest (v ≡ 0) at the time T_f.

Comment 2. The result of Theorem 1 can also be formulated in the following way. For any constant $c_v \in (0, \infty)$ in (7), and for a sufficiently small $c_f > 0$ in (8), there exists a $T_f \in (0, \infty)$ such that (9) and (10) hold. A corresponding assertion for the case f ≡ 0 was proved in Ref. 3.

Comment 3. The constant K in Korn's inequality (12) does not depend on Ω if m = 2, q = 2n/(n + 2), and $q \in (1+\sigma, 2)$. Consequently, the assertions made above are also valid in the case of the Cauchy problem

$$v(0, x) = v_0(x), \quad \rho(0, x) = \rho_0(x), \quad x \in R^n,$$

$$\|v_0\|_{2,R^n} < \infty, \quad \frac{1}{M} \leqslant \rho_0 \leqslant M$$

for system (1)-(4) if 0 ≤ σ < (n − 2)/(n + 2). Functional dependences P = P(D), more general than (4), could be dealt with in a corresponding way.

2. Joint flows of surface and underground water. Mathematical models of joint flows of underground and surface water, not under pressure, based on the equations of planned filtration and the hydraulics of open channels were discussed in Refs. 3 and 5. In the simplest case (the channel has a rectangular cross section and a constant width; the confining layer and the bottom of the channel are horizontal; etc.), the corresponding system of equations and the internal coupling conditions are[5]

$$\frac{\partial H}{\partial t} = \operatorname{div}(H\nabla H) + f_\Omega(t, x), \quad x \in \Omega^\Pi = \Omega \backslash \Pi; \tag{14}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}(\psi(s, u)|u_s|^{-1/2}u_s) - \left[H\frac{\partial H}{\partial n}\right]_\Pi + f_\Pi(t, x), \quad x \in \Pi; \tag{15}$$

$$H\frac{\partial H}{\partial n}\bigg|_{\Pi_\pm} = \sigma_\pm(u - H_\pm), \quad x \in \Pi, \quad 0 \leqslant \sigma_\pm < \infty, \tag{16}$$

$$\psi = \psi_0(s, u)u^{5/3}, \quad |\ln\psi_0| \leqslant M.$$

Here H(t, x) is the level of the underground water in the region $\Omega \subset R^2$; u(t, s) is the water level in the channel, which corresponds to curve Π in Ω; s is the arc length along Π; n is the normal to Π; $H_\pm$ are the values of H as Π is approached from the different sides; correspondingly, $[H]_\Pi = H_+ - H_-$; and $f_\Omega(t, x)$, and $f_\Pi(t, s)$ are given external inflows of water: the "sources."

We will be studying the local properties of the solution w = (H(t, x), u(t, x)) of system (14)-(16) in the circle $B_\rho(x_0) = \{x \in \Omega: |x - x_0| < \rho\}$, $x_0 \in \Pi$, assuming without any loss of generality that we have $x_0 = 0$:

$$\Pi_\rho = \{x \in \Omega: x_2 = 0, |x_1| < \rho\}, \quad s = x_1,$$
$$B_\rho(x_0) = B_\rho, \quad B_\rho^\pm = \{x \in B_\rho: 0 \lessgtr x_2\}. \tag{17}$$

The existence of a generalized solution $w = (H, u) \in V$, where

$$V = \{(H,u) | \ 0 \leqslant (H,u) \leqslant M, \ \sqrt{H} \nabla H \in L^2(0,T; L^2(B_R)),$$
$$\psi^{2/3} |u_s| \in L^{3/2}(0,T; L^{3/2}(\Pi_R))\}, \ 0 < R < \infty,$$

was proved for system (14)-(16) in Ref. 5 for the basic initial-boundary-value problems.

Theorem 2. We assume that $w = (H, u) \in V$ is a generalized solution of system (14)-(16) and

$$(\|H(0,\cdot)\|^2_{2,B_\rho^\pm} + \|u(0,\cdot)\|^2_{2,\Pi_\rho} + \int_0^T (\|f_\Omega\|^2_{2,B_\rho^\pm} + \|f_\Pi\|^2_{2,\Pi_\rho}) d\tau)$$
$$\leqslant c(\rho - \rho_0)_+^{1/(1-\beta)},$$
$$u_+ = \max(0, u), \quad \rho \in (0,R), \quad \beta = 5/6, \quad c = \text{const}, \quad \rho_0 < R.$$

Then there exists a $t_0 = t_0(R, C, M) > 0$ such that

$$w = (H, u) \equiv 0, \quad x \in B_{\rho_0}, \quad t \leqslant t_0. \tag{18}$$

To prove the theorem, we introduce the notation

$$A(t,\rho) = (\|H(t,\cdot)\|^2_{2,B_\rho^+ \cup B_\rho^-} + \|u(t,\cdot)\|^2_{2,\Pi_\rho}),$$
$$E(t,\rho) = \int_0^t \{(H\nabla H, \nabla H)_{B_\rho^+ \cup B_\rho^-} + (\psi |u_s|^{-1/2} \cdot u_s; u)_{\Pi_\rho}\} d\tau,$$
$$D^2 = \int_0^t \sum_\pm (\sigma_\pm, (u - H_\pm)^2)_{\Pi_\rho} d\tau,$$
$$F = \int_0^t \{(f_\Omega, H)_{B_\rho^+ \cup B_\rho^-} + (f_\Pi, u)_{\Pi_\rho}\} d\tau$$

and we use an energy relation which follows from (14)-(16):

$$\frac{1}{2}(A(t,\rho) - A(0,\rho)) + E(t,\rho) + D^2$$
$$= F + \int_0^t \left\{ \left(H \frac{\partial H}{\partial n}, H \right)_{\partial B_\rho} + \psi |u_s|^{-1/2} u_s u |_{x_1 = -\rho}^{x_1 = \rho} \right\} d\tau. \tag{19}$$

[..]ng (17), and working by analogy with Refs. [.], 4, and 7, we thus find an ordinary differential inequality for an energy function

$$\bar{E}(\rho) \leqslant a t_0^\nu \left(\frac{\partial \bar{E}}{\partial \rho} \right)^{1/\beta} + b(\rho - \rho_0)_+^{1/(1-\beta)}, \tag{20}$$

where $\bar{E} = \sup_{0 \leqslant t \leqslant t_0} E(t,\rho)$, $\nu = 2/5$, $\beta = 5/6$ and a, b, and t_0 are calculated in terms of c, M, and R. According to the results of Refs. 4 and 6, it follows from (20) that we have

$$\bar{E}(\rho) \equiv 0, \quad 0 \leqslant \rho \leqslant \rho_0.$$

This completes the proof of Theorem 2.

Comment 4. The result of Theorem 2 has the following physical interpretation. The region B_{ρ_0} which is not occupied by water (the dry region) at the initial time, t = 0, remains dry at $t \leq t_0$, regardless of the boundary conditions and regardless of sources outside B_R.

3. Two-phase filtration of nonmixing incompressible liquids. The time-dependent filtration of two nonmixing and incompressible liquids in a homogeneous and isotropic porous medium is described by a system of equations of a composite type[6]:

$$\frac{\partial s}{\partial t} = \operatorname{div}(a(s)\nabla s + k_1(s)\nabla p), \quad (x,t) \in Q = \Omega \times (0,T); \tag{21}$$

$$\operatorname{div}(k(s)\nabla p + f(s)) = 0 \tag{22}$$

for the unknown saturation s(t, x) and the unknown "reduced" pressure p(t, x). The coefficients of system (21), (22) are determined by[6]

$$a(s) = |p_k'(s)| \cdot k_1 \cdot k_2/k, \quad k = k_1(s) + k_2(s), \quad f = k_2(\rho_2 - \rho_1)g,$$
$$k_i(s) > 0, \quad 0 < s < 1; \quad k_1(0) = k_2(1) = 0, \tag{23}$$

where $p_k(s)$ is the capillary pressure ($p_k' \leq 0$); $\bar{k}_i = k_i \cdot \mu_i$ are the relative phase permeabilities; μ_i and ρ_i are the viscosities and densities, respectively, of the liquids; and g is the acceleration due to gravity. The functional parameters of the model ($\bar{k}_1$, $\bar{k}_2$, p_k) are given, as usual, by the functions

$$k_1 = f_1(s)s^{\lambda_1}, \quad k_2 = f_2(s)(1-s)^{\lambda_2}, \quad p_k = f_3 s^{\lambda_3}(1-s)^{\lambda_4}, \tag{24}$$
$$f_i \in c^1, \quad |\ln f_i| \leqslant M, \quad 0 \leqslant (\lambda_1, \lambda_2).$$

A theorem concerning the existence of a generalized solution $w = (s,p) \in V$, $V = \{(s,p) | \ 0 \leqslant s \leqslant 1, \sqrt{a}\nabla s \in L^2(Q), \nabla p \in L^\infty(0,T; L^q(\Omega))\}$, n < q, was proved for system (21), (22) in Ref. 6. The finite localization time for the solution in the case of a boundary-value problem and the finite propagation velocity of perturbations from given initial perturbations [s(0, x) = 0 or s(0, x) = 1] were established for w(t, x) in Ref. 3. We are interested in the local properties of the solution of system (21), (22) under the assumption that the initial data satisfy the condition

$$\|s(0,\cdot)\|^2_{2,B_\rho} \leqslant c(\rho - \rho_0)_+^{1/(1-\lambda)}, \quad \rho \in (0,R). \tag{25}$$

We also restrict the analysis to cases of planned flow [in the $x_3 = 0$ plane; g = (0, 0, g)] or the case of liquids with identical densities. In these cases, according to (23), we have

$$f = 0. \tag{26}$$

Theorem 3. We assume that $w = (s,p) \in V$ is a generalized solution of system (21), (22), that (26) is satisfied, and that, in addition,

$$0 < \lambda = (\lambda_1 + \lambda_3 - 1),$$
$$|(k_1/k)'|^2/a \cdot s^\lambda \leqslant M(\delta), \quad 0 \leqslant s \leqslant 1 - \delta < 1.$$

There then exists a $t_0 = t_0(M, c, k)$ such that

$$s(t,x) = 0, \quad x \in B_{\rho_0}, \quad t \leqslant t_0$$

and, correspondingly, $\Delta p(t, x) = 0$.

The proof follows from the energy relation

$$\frac{1}{2}(A(t,\rho) - A(0,\rho)) + E = \int_0^t \left\{ \left(\left(\frac{k_1}{k} \right)' k \nabla p \nabla s, s \right)_{B_\rho} + \left(a \frac{\partial s}{\partial n}, s \right)_{\partial B_\rho} \right\} d\tau,$$

$$A(t,\rho) = \|s(t,\cdot)\|^2_{2,B_\rho}, \quad E(t,\rho) = \int_0^t (a\nabla s, \nabla s)_{B_\rho} d\tau, \tag{27}$$

which leads to an inequality of the type in (20) for the function $\bar{E}(\rho) = \sup_{t \leqslant t_0} E(t,\rho)$. An analysis of inequality (20) completes the proof.

Comment 5. A corresponding assertion is valid for the functions s = 1 − s(t, x).

Comment 6. Theorem 3 can be given the following physical interpretation. We assume that at the initial time, t = 0, the region B_{ρ_0} is occupied by only one of the liquids [s(0, x) = 0 or s(0, x) = 1]. Then for any agents outside B_R the displacement of the given liquid from B_{ρ_0} begins no earlier than at $t_0 > 0$.

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