3. S.N. Antontsev, J.I. Díaz (Madrid), and S.I. Shmarev, "New applications of energy methods to parabolic problems with free boundaries".

1°. Introduction.

We present some recently obtained results on the application of various energy methods to the study of problems with free boundaries. Such methods provide an alternative approach in the absence of the maximum principle. They are therefore of special interest as a means of investigating higher-order equations and systems. These methods are also useful in the study of boundary-value problems for second order equations with a complex structure that makes the construction of the super- and subsolutions difficult. For example, when the boundary data are unbounded, when the independent variables occur explicitly in the non-linear functions, in the presence of terms with first derivatives in the equation, and so on. Many of the results in this direction are contained in a monograph to be published by the first two authors of this lecture. Applications to three different problems are given below.

2°. The flow of an immiscible liquid through a porous medium.

We consider the system

$$(1)\qquad \begin{cases} \varphi(x)\dfrac{\partial s}{\partial t} - \operatorname{div}(k_0(x)\,a(s)\,\nabla s) = \operatorname{div}(k_0(x)\,b(s)\,\nabla p) + f(x,t), \\ \operatorname{div}(k_0(x)\,d(s)\,\nabla p) = 0 \end{cases}$$

under the following hypotheses: $0 < c_1 \leqslant \varphi(x) \leqslant c_2$; $c_3|\xi|^2 \leqslant k_0(x)\xi\cdot\xi \leqslant c_4|\xi|^2$ for all $\xi \in \mathbb{R}^N\setminus\{0\}$; $d(s) \geqslant c_5 > 0$; $c_6 s^\alpha(1-s)^\beta \leqslant a(s) \leqslant c_7 s^\alpha(1-s)^\beta$. This system arises in the study of the motion of an immiscible fluid in a porous medium. References to the derivation of the system and the existence of weak solutions in it can be found in [1]. We note that the maximum principle does not hold for the system (1). Using energy methods for (1) for the case $\alpha > 0$, that is, in the presence of degeneracy, we have proved the following results.

Sommerfeld problem consists in finding the reflected and refracted waves under given Dirichlet or Neumann conditions on the semibounded screen $x_1 \geqslant 0$, $x_2 = 0$.

We consider the more general case when there are $N \geqslant 2$ parallel semibounded screens with end-points $(a_\nu, h_\nu) \in \mathbb{R}^2$, $\nu = 1, \ldots, N$, under boundary conditions and first-order junction conditions of general form, which may be different on different screens. In our investigation we apply a modern version of the Wiener–Hopf method in conjunction with the Fourier transformation and the use of Sobolev spaces. Let Σ_ν be a subset of the line $x_2 = h_\nu$, which is a screen, and let Σ_ν' be its complement, $\Omega = \mathbb{R}^2 \setminus (\bigcup_{\nu=1}^{n} \Sigma_\nu)$. The solution $u(x_1, x_2)$ is expressed as a Fourier integral with respect to x_1. If a solution with finite energy $u \in H^1(\Omega)$ is sought, then by the trace theorem $u(\cdot, h_\nu)$ and $\frac{\partial u}{\partial x_2}(\cdot, h_\nu)$ $(\nu = 1, \ldots, N)$ must belong to $H^{1/2}(\mathbb{R})$ and $H^{-1/2}(\mathbb{R})$ respectively. The requirement that the function $u(x_1, x_2)$ and its derivatives be continuous on Σ_ν' leads to compatibility conditions for the data on Σ_ν. For example, in the case of the standard cascade with $a_\nu = 0$ the relations

$$u(\cdot, h_\nu + 0) - u(\cdot, h_\nu - 0) \in H^{1/2}(\mathbb{R}_+),$$

$$\frac{\partial u}{\partial x_2}(\cdot, h_\nu + 0) - \frac{\partial u}{\partial x_2}(\cdot, h_\nu - 0) \in H^{-1/2}(\mathbb{R}_+), \quad \nu = 1, \ldots, N$$

must hold.

It is shown that the above problem for the Helmholtz equation in Ω is equivalent to a system of Wiener–Hopf integral equations in a product of spaces $L^2(\mathbb{R}_+)$ with piecewise continuous symbols in the form of $2N \times 2N$-matrices, which are of block form. The blocks of dimension 2×2 along the principal diagonal correspond to the boundary conditions and the junction conditions on the individual screens. The off-diagonal blocks correspond to the interactions between adjacent screens and contain coefficients proportional to $\exp[-(h_\nu - h_{\nu-1})(\xi_1^2 - k^2)^{1/2}]$, where $\mathrm{Re}(\xi_1^2 - k^2)^{1/2} > 0$ for $\xi_1 \in \mathbb{R}$. Thus they become small when the distances $h_\nu - h_{\nu-1}$ between screens are large. We arrive at a perturbation of the problem which splits into N problems with a single screen, previously solved by F.-O. Speck and the first-named author using matrix factorization.

6. T. Dudnikova, "The connection between matrix scattering and amplitude scattering for symmetric hyperbolic systems".

7. A.R. Shirikyan, "The continuous dependence of the scattering operator on time-periodic potentials for the wave equation".