C. R. Acad. Sci. Paris, t. 315, Série I, p. 787-792, 1992

Équations aux dérivées partielles/*Partial Differential Equations*

Nonlinear elliptic boundary-value problems in unbounded domains and the asymptotic behaviour of its solutions

Jesus Ildefonso Diaz and Olga Arsenievna Oleinik

Abstract – We use energy methods for the study of nonlinear boundary value problems with a superlinear term (e.g. $g(x, u)=a(x)|u|^{p-1}u$ with $p>1$) in unbounded domains. Suitable *a priori* estimates on $\Omega\cap B_R$ allow us to obtain a general existence and uniqueness result without growth restrictions at infinity on the solutions. We also obtain estimates on the asymptotic behaviour of solutions when $|x|\to+\infty$ and the data f vanishes outside of some ball $B_R=\{x:|x|<R\}$.

Problèmes aux limites elliptiques non linéaires dans des domaines non bornés et le comportement asymptotique de ses solutions

Résumé – *On utilise des méthodes d'énergie pour étudier les problèmes aux limites elliptiques non linéaires avec un terme surlinéaire* (e.g. $g(x, u)=a(x)|u|^{p-1}u$ *avec* $p>1$) *dans des domaines non bornés. Quelques estimations* a priori *nous permettent d'obtenir un résultat d'existence et d'unicité très général sans conditions de croissance à l'infini sur les solutions. On donne aussi des estimations sur le comportement asymptotique des solutions lorsque* $|x|\to\infty$ *et la donnée* f *s'annule dans* $\mathbb{R}^n-B_R$.

Version française abrégée – On considère le problème aux limites (P) sur un ouvert régulier non borné de $\mathbb{R}^n$. Dans tout le travail on suppose vérifiée l'hypothèse (1). On introduit les espaces $V(\Omega_R)=\{w\in H^1(\Omega_R): w=0 \text{ sur } \Gamma_D\cap B_R\}$, $V_{loc}(\Omega)=\{w: w\in V(\Omega_R) \text{ pour tout } R>0\}$ et ses espaces duaux $V^*(\Omega_R)$ et $V^*_{loc}(\Omega)$ respectivement. Pour $f\in V^*_{loc}(\Omega)$ on dit que $u\in V_{loc}(\Omega)$ est une solution faible de (P) si $a(x)|u|^{p-1}u\in L^1_{loc}(\Omega)$ et l'identité (3) est vérifiée pour tout $\zeta\in V_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$ à support compact. On définit la fonction de localisation $\Theta_R(x)=\theta^2(|x|/R)$ où $\theta\in C^\infty(\mathbb{R})$ est telle que $\theta(s)=1$ si $|s|\leqq 1/2$ et $\theta(s)=0$ si $|s|\geqq 1$.

Notre premier résultat (théorème 1) assure l'existence et l'unicité d'une solution faible u telle que $a(x)|u|^{p+1}\in L^1_{loc}(\Omega)$ et qui satisfait (3) pour $\zeta=u\Theta_R$. La démonstration utilise les estimations *a priori* (6) et (7) pour les solutions des problèmes (P_N) posés sur $\Omega_N=\{x\in\Omega:|x|<N\}$. L'hypothèse (1) permet de choisir θ tel que $\theta'(s)=O((1-s)^{m-1})$ et l'important lemme 1 est montré en prenant $u_N\Theta_R$ comme fonction test. La démonstration de l'existence d'une solution faible de (P) est alors basée sur un procédé d'extraction diagonale appliqué à la suite $\{u_N\}$. Pour montrer l'unicité on fait $v=u_1-u_2$ et en prenant $\zeta=v\Theta_R$ on voit que les estimations du lemme 1 (avec $f\equiv 0$) restent valables pour v ce qui permet de conclure si $1<p<(n+2)/(n-2)$ ou $n=2$. Si $p\geqq(n+2)/(n-2)$ on suppose, par l'absurde, que $v(x)>\alpha>0$ sur une boule $B_r(x_0)$. On prend $\zeta=T_m([v-\alpha]_+)\Theta_R(|x|)$ comme fonction test et la contradiction est obtenue avec des estimations similaires à celles du lemme 1 mais cette fois en remplaçant p par q avec $q\in(1, (n+2)/(n-2))$. Des informations sur quelques résultats précédents dans la littérature sont donnés dans la remarque 1. En particulier nos résultats généralisent ceux du travail pionnier de Brezis [3].

La deuxième partie de l'article est consacrée à l'étude du comportement des solutions quand $|x|\to\infty$. On fait l'hypothèse (10) et par simplicité on considère seulement le cas $f\equiv 0$ [équation (11)]. Dans le cas de conditions aux limites de type Dirichlet à l'infini, (12), on donne deux estimations. La première est obtenue par comparaison avec certaines supersolutions (*voir* théorème 2) et elle est exacte sous des hypothèses convenables ($\Omega=\mathbb{R}^n$ entre autres). La deuxième, démontrée par une méthode d'énergie, est du type exponentielle (15)

Note présentée par Haïm Brezis.

0764-4442/92/03150787 $ 2.00

grâce à une hypothèse géométrique sur le domaine Ω [*voir* (14)]. Une estimation asymptotique pour le cas du problème de Neumann à l'infini est donnée dans le théorème 4. On montre aussi que les solutions convergent vers zéro quand $|x| \to +\infty$.

---

1. EXISTENCE, UNIQUENESS AND SOME ESTIMATES. — Let Ω be a regular open *unbounded* set of $\mathbb{R}^n$. We consider the boundary value problem

$$\text{(P)} \qquad -\mathrm{L}u + a(x)|u|^{p-1}u = f(x) \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \Gamma_\mathrm{D}, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_\mathrm{N},$$

where $\partial\Omega = \Gamma_\mathrm{D} \cup \Gamma_\mathrm{N}$ and Γ_D or Γ_N can be the empty set. The operator L is given by

$$\mathrm{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

with $a_{ij} \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \lambda|\xi|^2$, a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$, and for some constant $\lambda > 0$. As usual $\partial u/\partial \nu = \sum_{i,j=1}^{n} a_{ij}(x)(\partial u/\partial x_j)\gamma_i$ if $\nu = (\gamma_1, \ldots, \gamma_n)$ is the outward unit normal vector to $\partial\Omega$. In the following we shall always assume

(1) $\quad p > 1$ and $a \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, $\quad a(x) \geqq a_0 > 0$ a.e. $x \in \Omega$ and some constant a_0.

We introduce the following notation:

$$\mathrm{B}_\mathrm{R} = \{x \in \mathbb{R}^n : |x| < \mathrm{R}\}, \qquad \Omega_\mathrm{R} = \Omega \cap \mathrm{B}_\mathrm{R},$$
$$\mathrm{V}(\Omega_\mathrm{R}) = \{w \in \mathrm{H}^1(\Omega_\mathrm{R}) : w = 0 \text{ on } \Gamma_\mathrm{D} \cap \mathrm{B}_\mathrm{R}\}$$

and

$$\mathrm{V}_{\mathrm{loc}}(\Omega) = \{w : w \in \mathrm{V}(\Omega_\mathrm{R}) \text{ for any } \mathrm{R} > 0\}.$$

We also consider the dual spaces (for the usual topologies), $\mathrm{V}^*(\Omega_\mathrm{R})$ and $\mathrm{V}^*_{\mathrm{loc}}(\Omega)$ respectively. Let

$$\text{(2)} \qquad f \in \mathrm{V}^*_{\mathrm{loc}}(\Omega).$$

DEFINITION. — A function $u \in \mathrm{V}_{\mathrm{loc}}(\Omega)$ is called a *weak solution* of problem (P) if $a(x)|u|^{p-1}u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and

$$\text{(3)} \qquad \sum_{i,j=1}^{n} \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \zeta}{\partial x_i}dx + \int_\Omega a(x)|u|^{p-1}u\zeta\,dx = \langle f, \zeta\rangle$$

for any $\zeta \in \mathrm{V}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, ζ with compact support.

Let us introduce the cut-off function

$$\text{(4)} \qquad \Theta_\mathrm{R}(x) = \theta^2(|x|/\mathrm{R}) \qquad \text{for} \quad \mathrm{R} > 0,$$

where $\theta \in \mathrm{C}^\infty(\mathbb{R})$ is such that $\theta(s) = 1$ if $|s| \leqq 1/2$ and $\theta(s) = 0$ of $|s| \geqq 1$.

THEOREM 1. — *Under the above conditions there exists a unique weak solution* $u(x)$ *of problem* (P) *such that* $a(x)|u|^{p+1} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ *and* (3) *holds for* $\zeta = u\Theta_\mathrm{R}$, *for any* $\mathrm{R} > 1$.

To carry out the proof, given $\mathrm{N} > 0$, we consider the problem

$$(\mathrm{P_N}) \qquad \begin{cases} -\mathrm{L}u_\mathrm{N} + a(x)|u_\mathrm{N}|^{p-1}u_\mathrm{N} = f(x) \quad \text{in } \Omega_\mathrm{N} \\ \dfrac{\partial u_\mathrm{N}}{\partial \nu} = 0 \quad \text{on } \Gamma_\mathrm{N} \cap \overline{\mathrm{B}}_\mathrm{N}, \qquad u_\mathrm{N} = 0 \quad \text{on } (\Gamma_\mathrm{D} \cap \overline{\mathrm{B}}_\mathrm{N}) \cup \{x \in \Omega : |x| = \mathrm{N}\} \end{cases}$$

The existence and uniqueness of a solution $u_N \in H^1(\Omega_N)$ satisfying (P_N), in the sense that $a(x)|u_N|^p \in L^1(\Omega_N)$ and the integral identity (3) holds when replacing Ω by Ω_N for any $\zeta \in H^1(\Omega_N) \cap L^\infty(\Omega_N)$ with $\zeta = 0$ on $(\Gamma_D \cap \bar{B}_N) \cup \{x \in \Omega : |x| = N\}$, is a consequence of the results of Brezis and Browder [4]. Their results also imply that $a(x)|u_N|^{p+1} \in L^1(\Omega_N)$ and that the integral identity also holds for $\zeta = u_N$. In fact, if $0 < R < N$ we have

$$\sum_{i,j=1}^{n} \int_{\Omega_R} a_{ij} \frac{\partial u_N}{\partial x_j} \frac{\partial (u_N \Theta_R)}{\partial x_i} dx + \int_{\Omega_R} a(x)|u_N|^{p+1} \Theta_R \, dx = \langle f, u_N \Theta_R \rangle. \tag{5}$$

Indeed, it suffices to take $\zeta = T_m(u_N)\Theta_R$ in the integral identity associated to (3) but on Ω_N, where $T_m(r) = \min(m, |r|) \operatorname{sign} r$, if $r \in \mathbb{R}$. Making $m \to \infty$ we obtain (5).

We extend the function u_N by zero over $\Omega - \Omega_N$ and, for simplicity, we denote again by u_N this extension. The convergence of u_N as $N \to \infty$ will be obtained from

LEMMA 1. — *Assume that* $u \in V(\Omega_N)$, $a(x)|u|^{p+1} \in L^1(\Omega_N)$ *and the inequality*

$$\sum_{i,j=1}^{n} \int_{\Omega_N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial (u \Theta_R)}{\partial x_i} dx + \int_{\Omega_N} a(x)|u|^{p+1} \Theta_R \, dx \leqq \langle f, u \Theta_R \rangle \tag{5_N}$$

holds for some $f \in V^*(\Omega_R)$ *for any* $R \leqq N$. *Then there exists* $C > 0$ *independent of* Ω, R *and u such that if* $R \in [1, N]$ *we have*

$$\int_{\Omega_{R/2}} |\nabla u|^2 dx \leqq R^{n-2-(4/(p-1))} (C + R^{(4p-n(p-1))/(p-1)} \|f\|_{V^*(\Omega_R)}), \tag{6}$$

$$\int_{\Omega_{R/2}} |u|^{p+1} dx \leqq R^{n-(2(p+1)/(p-1))} (C + R^{(4p-n(p-1))/(p-1)} \|f\|_{V^*(\Omega_R)}). \tag{7}$$

Proof. — Let $R = 1$. From (5_N) and assumption (1) we have

$$C_1 \int_{\Omega_1} |\nabla u|^2 \theta^2(|x|)\, dx + a_0 \int_{\Omega_1} |u|^{p+1} \theta^2(|x|)\, dx \leqq 2\varepsilon \int_{\Omega_1} |\nabla u|^2 \theta^2(|x|)\, dx$$
$$+ \delta \int_{\Omega_1} |u|^{p+1} \theta^2(|x|)\, dx + C_2(\varepsilon, \delta) \int_{\Omega_1} |\nabla \theta(|x|)|^{2(p+1)/(p-1)} \theta(|x|)^{-4/(p-1)} dx$$
$$+ C_3(\delta) \int_{\Omega_1} |\theta|^{4p/(p-1)} dx + C_4(\delta) \|f\|^2_{V^*(\Omega_1)},$$

for any ε and δ positive numbers, where we have used the Hölder and the Young inequalities. Choosing $\theta(s)$ such that $\theta'(s) = O((1-s)^{m-1})$, with $m > 0$, we have

$$I \equiv \int_{\Omega_1} |\nabla \theta(|x|)|^{2(p+1)/(p-1)} \theta(|x|)^{-4/(p-1)} dx \leqq C_4 \int_{B_1} (1-|x|)^{(2m(p-1)-2(p+1))/(p-1)} dx,$$

and thus I is finite if $m > (p+1)/(p-1)$. The conclusion, for $R = 1$, is now obvious. For the general case $R \geqq 1$ we introduce the change of variables $x = Rx'$ and $u(x) = hv(Rx')$ with $h = R^{-2/(p-1)}$. Estimates (6), (7) are obtained from the result for the case $R = 1$. ∎

Proof of the Theorem 1. — *Existence.* — By lemma 1 we have

$$\int_{\Omega_{R/2}} (|\nabla u_N|^2 + |u_N|^{p+1})\, dx \leqq C$$

for some C depending on $R \in [1, N)$ but independent of N. By standard results we have that $\{u_N\}$ is bounded in $H^1(\Omega_R)$ and by diagonal extraction is follows that there exists u and a subsequence such that $u_N \to u$ in $H^1_{loc}(\Omega)$ weakly, in $L^{p+1}_{loc}(\Omega)$ weakly, in $L^2_{loc}(\Omega)$

strongly and a.e. in Ω. Using the monotonicity of the function $F(s)=|s|^{p-1}s$ we arrive at the conclusion.

Uniqueness. — Let u_1, u_2 be weak solutions of (P) such that $a(x)|u_i|^{p+1}\in L^1_{loc}(\Omega)$ and satisfy (3) for $\zeta=u_i\Theta_R$ for any $R>1$ $(i=1, 2)$. Let $v=u_1-u_2$. Since $v\in V_{loc}(\Omega)\cap L^{p+1}_{loc}(\Omega)$, arguing as in the proof of (5), we can take $\zeta=v\Theta_R$ in the integral identity (3) associated to u_i. We get

$$\sum_{i,j=1}^{n}\int_{\Omega_R} a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial (v\Theta_R)}{\partial x_i}dx+\int_{\Omega_R} A(x)|v|^{p+1}\Theta_R\,dx=0,$$

$$A(x)\equiv\begin{cases} a(x)\dfrac{[|u_1(x)|^{p-1}u_1(x)-|u_2(x)|^{p-1}u_2(x)]}{|u_1(x)-u_2(x)|^{p-1}(u_1(x)-u_2(x))} & \text{if } u_1(x)\neq u_2(x)\\ a_0 & \text{if } u_1(x)=u_2(x).\end{cases}$$

Using $p>1$ it is easy to see that $A(x)\geqq A_0$ a.e. on Ω, for some constant $A_0>0$. Then Lemma 1 can be applied and so estimates (6) and (7) hold by taking $f\equiv 0$ and replacing u by v, i.e.

$$\text{(8)}\qquad \int_{\Omega_{R/2}}|\nabla v|^2\,dx\leqq CR^{n-2-(4/(p-1))},$$

$$\text{(9)}\qquad \int_{\Omega_{R/2}}|v|^{p+1}\,dx\leqq CR^{n-(2(p+1)/(p-1))}.$$

Assume now $1<p<(n+2)/(n-2)$ or $n=2$. Then, from (8), making $R\to\infty$, we conclude that $\nabla v\equiv 0$ on Ω and using (9) we obtain that $v\equiv 0$. Let $p\geqq(n+2)/(n-2)$ and assume by contradiction that $v\not\equiv 0$ in Ω. Without lost of generality we can assume that there exists $\alpha>0$ and a ball $B_r(x_0)$ with $r>0$ and $x_0\in\Omega$ such that $v(x)>\alpha$ a.e. $x\in B_r(x_0)$. It is easy to prove that the function $w=(v-\alpha)_+$ satisfies the equality

$$\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}\frac{\partial w}{\partial x_j}\frac{\partial (w\Theta_R)}{\partial x_i}dx+\int_{\Omega} A(x)\frac{(w+\alpha)^p}{w^q}w^{q+1}\Theta_R\,dx=0$$

where $q<(n+2)/(n-2)$. Since $(w+\alpha)^p/w^q\geqq b>0$ for any $w>0$ and $q<p$, where b is a positive constant, from Lemma 1 we have

$$\int_{\Omega_{R/2}}|w|^{q+1}\,dx\leqq CR^{n-(2(q+1)/(q-1))}\quad\text{as } R\to+\infty.$$

Therefore $v\equiv 0$, which contradicts the assumption $w>0$ in $B_r(x_0)$. ■

Remark 1. — The first result where the existence and uniqueness of a solution of problem (P) (with $\Gamma_N=\varphi$) was given without growth conditions at infinity on f and solutions is due to Brezis [3]. The main difference with our existence result comes from the fact that we merely use an energy method which allows us to treat a greater class of functions f. On the other hand Brezis' uniqueness result uses an explicit supersolution and his method can not be applied to the case $\Gamma_N\neq\varphi$ (other generalizations of the Brezis' result are [1] and [5].

2. On the asymptotic behaviour of solutions at infinity. — We study the behaviour as $|x|\to+\infty$ of weak solutions of problem (P). As a matter of fact the relevant assumptions will be localized on the set $\Omega-\bar{B}_R$, for some $R'>0$. We assume

$$\text{(10)}\qquad u\in L^\infty(\Omega\cap\partial B_{R'}).$$

For simplicity in the statements we shall concentrate our attention in the case $f\equiv 0$ on $\Omega-\bar{B}_{R'}$, *i.e.* we shall deal with functions $u\in H^1_{loc}(\Omega-\bar{B}_{R'})$ such that

$a(x)|u|^{p+1} \in L^1_{loc}(\Omega - \bar{B}_{R'})$ and satisfy

(11) $$-Lu + a(x)|u|^{p-1}u = 0 \quad \text{in } \mathscr{D}'(\Omega - \bar{B}_{R'}).$$

We start with the case of Dirichlet boundary conditions "near the infinity", *i.e.*

(12) $$u = 0 \quad \text{on } \partial\Omega - \bar{B}_{R'}.$$

THEOREM 2. — *There exists a constant* $C > 0$ *such that*

$$|u(x)| \leq C|x|^{-2/(p-1)} \quad \textit{for a.e. } x \in \Omega - \bar{B}_{R'}.$$

Proof. — Let $N > R'$ and define $\Omega_N^{R'} = \{x \in \Omega : R' < |x| < N\}$. Let v_n be the weak solution of the problem

$$(P_N^R)\left\{\begin{array}{l} -Lv_N + a(x)|v_N|^{p-1}v_N = 0 \quad \text{in } \Omega_N^{R'} \\ v_N = u \quad \text{on } \Omega \cap \partial B_{R'}, \quad \text{and} \quad v_N = 0 \quad \text{on } (\partial\Omega \cap (B_N - B_{R'})) \cup (\Omega \cap \partial B_N). \end{array}\right.$$

Consider the function $V(x) = K|x|^{-2/(p-1)}$. It is easy to check (*see* e.g. Diaz [6], Chap. 1) that $-LV + a(x)|V|^{p-1}V \geq 0$ and that $V(x) \geq v_N(x)$ on $\Omega \cap \partial B_{R'}$ by assuming K large enough. Then, by the maximum principle we deduce

(13) $$|v_N(x)| \leq K|x|^{-2/(p-1)} \quad \text{a.e. } x \in \Omega_N^{R'}.$$

Letting $N \to +\infty$ the conclusion holds. ■

The following result shows how the "geometry" of Ω can have a crucial influence on the decay of u as $|x| \to \infty$.

THEOREM 3. — *Assume*

(14) $$\textit{diameter } (\Omega \cap \{x : |x| = N\}) \leq T, \qquad \forall N \geq R'$$

where the constant T *does not depend on* N. *Let u satisfying* (10), (11) *and* (12). *Then there exists* C *and* α, *positive constants, such that*

(15) $$|u(x)| \leq C e^{-\alpha|x|} \quad \textit{a.e. } x \in \Omega - B_{R'}.$$

Proof. — Let $N > R'$ and let v_N be the solution of $(P_N^{R'})$. By (13) we have $v_N \in L^\infty(\Omega_N^R)$. Taking $\zeta = v_N e^{\alpha|x|}$ as test function, $\alpha = \text{const.} > 0$, we obtain that

$$C_1 \int_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\,dx + a_0 \int_{\Omega_N^{R'}} |\nabla v_N|^{p+1} e^{\alpha|x|}\,dx \leq \frac{\alpha C_2}{2} \int_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\,dx$$
$$+ \frac{\alpha}{2} \int_{\Omega_N^{R'}} |v_N|^2 e^{\alpha|x|} + \int_{\{x \in \Omega : |x| = R'\}} \frac{\partial v_N}{\partial \nu} v_N e^{\alpha|x|}\,ds.$$

Using the Friedrich-Poincaré inequality we have

$$\int_{\Omega_N^{R'}} |v_N|^2 e^{\alpha|x|}\,dx \leq C_3 \int_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\,dx$$

according to assumption (14). Then, since $\{v_N\}$ is bounded in $H^1_{loc}(\Omega - \bar{B}_{R'})$, we obtain

$$\int_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\,dx + \int_{\Omega_N^{R'}} |v_N|^{p+1} e^{\alpha|x|}\,dx \leq C_4$$

for some $C_4 > 0$ independent of N, if we assume α small enough. Now, let $x_0 \in \Omega_N^{R'}$. Since v_N satisfies the boundary condition $v_N = 0$ on Γ_D, by the De Giorgi type theorem (*see* e.g. [8], Theorem 8.17) we have

$$\sup_{x \in B_{1/2}(x_0) \cap \Omega_N^{R'}} |v_N(x)| \leq C_5 \left[\int_{B_1(x_0) \cap \Omega_N^{R'}} |v_N|^{p+1}\,dx\right]^{1/(p+1)}$$
$$\leq C_6 \left[e^{-\alpha(|x_0|-1)} \int_{B_1(x_0) \cap \Omega_N^{R'}} |v_N|^{p+1} e^{\alpha|x|}\,dx\right]^{1/(p+1)} \leq C_7 \exp\left\{\frac{-\alpha}{(p+1)}|x_0|\right\}$$

Making $N \to \infty$ we get the conclusion. ■

Concerning the case of Neumann boundary conditions "near the infinity"

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega - B_{R'}, \tag{16}$$

similar arguments leads to the following result

THEOREM 4. — *Let u satisfying* (10), (11) *and* (16). *Then*

$$|u(x)| \leqq C|x|^{\beta}, \qquad a.e.\ x \in \Omega - \bar{B}_{R'}$$

where $\beta = [p(n-2)-(n+2)]/[p^2-1]$. *Moreover, if* $u \in C^{2,\mu}(\Omega - B_{R'})$ *for some* $\mu \in (0, 1)$ *then* $|u(x)| \to 0$ *as* $|x| \to \infty$.

The details of the proof can be found in [7].

Remark 2. — Section 2 generalizes in different ways several results by Kondratiev and Oleinik [9] related to the case of unbounded cylindrical domains $\Omega = \omega \times \mathbb{R}_+$ or $\Omega = \omega \times \mathbb{R}$ with ω an open bounded set of $\mathbb{R}^{N-1}$. When $L = \Delta$ and Ω is a cylinder the asymptotic behaviour of solutions have been studied from the point of view of dynamical systems since many years ago. Some results in this direction can be found in the article of Brada [2] where this author develops ideas already introduced in Veron [10], [11].

Note remise le 1er juillet 1992, acceptée le 7 juillet 1992.

REFERENCES

[1] F. BERNIS, Elliptic and Parabolic Semilinear Problems without Conditions at Infinity, *Arch. Rational Mech. Anal.*, 106, 3, 1989, pp. 217-241.

[2] A. BRADA, Comportement asymptotique de solutions d'équations elliptiques semi-lineaires dans un cylindre, *Thèse 3e cycle*, Univ. de Tours, 1987, and detailed article to appear.

[3] H. BREZIS, Semilinear equations in $\mathbb{R}^N$ without conditions at infinity, *Appl. Math. Optim.*, 12, 1984, pp. 271-282.

[4] H. BREZIS and F. E. BROWDER, Strongly nonlinear elliptic boundary value problems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 5, 1978, pp. 587-603.

[5] G. DÍAZ and R. LETELIER, Uniqueness for viscosity solutions of quasilinear elliptic equations in $\mathbb{R}^N$ without conditions at infinity, *Differential and Integral Equations* (to appear).

[6] J. I. DÍAZ, *Nonlinear partial differential equations and free boundaries*, Res. Notes in Math., No. 106, Pitman, London, 1985.

[7] J. I. DÍAZ and O. A. OLEINIK, article in preparation.

[8] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, Berlin, 1983.

[9] V. A. KONDRATIEV and O. A. OLEINIK, On asymptotic behaviour of solutions of some nonlinear elliptic equations in unbounded domains, in *Proc. conference dedicated to L. Nirenberg*, Trento, Italia, 1990.

[10] L. VERON, Comportement asymptotique des solutions d'équations elliptiques semi-linéaires dans $\mathbb{R}^N$, *Ann. Mat. Pura Appl.*, (4), 127, 1981, pp. 25-50.

[11] L. VERON, Équations d'évolution semi-linéaire du 1er ordre dans L^1, *Rev. Roumaine Math. Pures Appl.*, 27, No. 1, 1982.

J. I. D.: *Dpto de Matemática Aplicada, Univ. Complutense de Madrid*, 28040 *Madrid, España;*

O. A. O.: *Department of Mathematics and Mechanics, Moscow State University*, 119899 *Moscow, Russia.*